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ON BOUNDS OF MATRIX EIGENVALUES

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Abstract. In this paper, we give the estimates both of upper and lower bound of eigenvalues of a simple matrix. The estimates are shaper than the known results.

1. Introduction

As is well known, the eigenvalues of a matrix play an important role in solving linear systems [1, 3, 5], especially in the perturbation problems [2, 6]. The purpose of this note is to give a specific estimate of the eigenvalue.

Let $A = (a_{ij})$ be an $n \times n$ complex matrix with conjugate transpose A^*, $\overline{A}$ denote the conjugate, and tr$A$ represent the trace of matrix A. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A, then

$$\sum_{i=1}^{n} |\lambda_i|^2 \leqslant \|A\|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2 = \mathrm{tr}(AA^*),$$

where $\|A\|$ denotes the Frobenius norm of A. Let

$$\Re_A = \frac{A + A^*}{2},$$

$$\Im_A = \frac{A - A^*}{2i},$$

we call $\Re_A$ the Hermitian real part and $\Im_A$ the Hermitian imaginary part of A. Let

$$q_A = \|A\|^2 - \frac{|\mathrm{tr}(A)|^2}{n},$$

$$\Delta_A = \frac{\|AA^* - A^*A\|^2}{2}.$$

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2. Main theorem

THEOREM 2.1. *Suppose* λ *is an eigenvalue of the* $n \times n$ *complex matrix* A *with geometric multiplicity* t, *then*

$$\left|\lambda - \frac{\operatorname{tr}(A)}{n}\right| \leqslant \sqrt{\frac{n-t}{(2n-t)t}}\sqrt{\frac{n-t}{n}q_A + \sqrt{q_A^2 - \frac{(2n-t)t}{n^2}\Delta_A}} \tag{2.1}$$

THEOREM 2.2. *Suppose* $\lambda_{\Re_A}$, $\lambda_{\Im_A}$ *are the eigenvalues of the* $n \times n$ *complex matrices* $\Re_A$ *and* $\Im_A$ *with geometric multiplicity* t, *respectively, then*

$$\left|\lambda_{\Re_A} - \frac{\operatorname{tr}(\Re_A)}{n}\right| \leqslant \sqrt{\frac{n-t}{nt}q_{\Re_A}}, \tag{2.2}$$

$$\left|\lambda_{\Im_A} - \frac{\operatorname{tr}(\Im_A)}{n}\right| \leqslant \sqrt{\frac{n-t}{nt}q_{\Im_A}}. \tag{2.3}$$

3. Proof of theorem

Before we give the proof of Theorems 2.1 and 2.2, we present some lemmas.

LEMMA 3.1. [4] *Let* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *be the eigenvalues of* A, *then*

$$\sum_{j=1}^{n} |\lambda_j|^2 \leqslant \sqrt{\|A\|^4 - \Delta_A}.$$

LEMMA 3.2. *Let* $n \times n$ *matrix* A, $\operatorname{rank}(A)$ *represent the rank of* A, *then*

$$|\operatorname{tr}(A)|^2 \leqslant \operatorname{rank}(A)\sqrt{\|A\|^4 - \Delta_A}.$$

Proof. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A, $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Suppose the number of nonzero eigenvalues is k, then without loss of generality, we can let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the nonzero eigenvalues of A. It is easy obtained that

$$k \leqslant \operatorname{rank}(A).$$

Now, let $R = \Lambda + M$ is a schur triangular form of A, i.e., $A = U^*RU$, U is unitary orthogonal, Λ is diagonal and M is upper triangular. From Lemma 3.1, we have

$$\sum_{j=1}^{k} |\lambda_j|^2 \leqslant \sqrt{\|A\|^4 - \Delta_A}.$$

Then

$$|\operatorname{tr}(A)|^2 = \left|\sum_{j=1}^{k} \lambda_j\right|^2 \leqslant k\sum_{j=1}^{k} |\lambda_j|^2 \leqslant \operatorname{rank}(A)\sum_{j=1}^{k} |\lambda_j|^2 \leqslant \operatorname{rank}(A)\sqrt{\|A\|^4 - \Delta_A}.$$

This shows the validity of conclusion. □

Now we give the proof of Theorem 2.1.

Proof. Let $M = \lambda I - A$, where I is $n \times n$ the identity matrix, λ is t multiple eigenvalue of A, then we have

$$\operatorname{rank}(M) = \operatorname{rank}(\lambda I - A) \leqslant n - t,$$

and the following equality

$$\Delta_M = \frac{\|(\lambda I - A)(\overline{\lambda} I - A^*) - (\overline{\lambda} I - A^*)(\lambda I - A)\|}{2} = \Delta_A.$$

From Lemma 3.2, we have

$$\begin{aligned} |\operatorname{tr}(M)|^2 &\leqslant \operatorname{rank}(M)\sqrt{\|M\|^4 - \Delta_M} \\ &\leqslant (n-t)\sqrt{\|M\|^4 - \Delta_M} \leqslant (n-t)\sqrt{\|\lambda I - A\|^4 - \Delta_A}. \end{aligned} \tag{3.1}$$

In addition, after direct manipulations,

$$\begin{aligned} |\operatorname{tr}(\lambda I - A)|^2 &= \operatorname{tr}(\lambda I - A)\operatorname{tr}(\overline{\lambda} I - A^*) \\ &= n^2|\lambda|^2 - n\lambda \operatorname{tr}(A^*) - n\overline{\lambda}\operatorname{tr}(A) + |\operatorname{tr}(A)|^2 = n\sigma + |\operatorname{tr} A|^2, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \|\lambda I - A\|^4 &= (\operatorname{tr}(\lambda I - A)\operatorname{tr}(\lambda I - A)^*)^2 \\ &= (n|\lambda|^2 - \lambda \operatorname{tr}(A^*) - \overline{\lambda}\operatorname{tr}(A) + \|A\|^2)^2 = (\sigma + \|A\|^2)^2, \end{aligned} \tag{3.3}$$

where $\sigma = n|\lambda|^2 - \lambda \operatorname{tr}(A^*) - \overline{\lambda}\operatorname{tr}(A)$.

Eliminating the σ from the formulae (3.2) and (3.3), we obtain

$$\|\lambda I - A\|^4 = \left(\frac{|\operatorname{tr}(\lambda I - A)|^2 - |\operatorname{tr}(A)|^2}{n} + \|A\|^2\right)^2. \tag{3.4}$$

Let $s = \left|\lambda - \frac{\operatorname{tr}(A)}{n}\right|^2$, $q_A = \|A\|^2 - \frac{|\operatorname{tr}(A)|^2}{n}$, then

$$|\operatorname{tr}(\lambda I - A)|^2 = n^2 s, \qquad \|\lambda I - A\|^4 = (ns + q_A)^2,$$

and by reformulating (3.1), we have

$$n^2 s \leqslant (n-t)\sqrt{(ns + q_A)^2 - \Delta_A},$$

after direct computations,

$$s = \left|\lambda - \frac{\operatorname{tr}(A)}{n}\right|^2 \leqslant \frac{n-t}{(2n-t)t}\left(\frac{n-t}{n} q_A + \sqrt{q_A^2 - \frac{(2n-t)t}{n^2}\Delta_A}\right).$$

The result follows immediately. $\square$

For Theorem 2.2, we note that the following inequality

$$\sqrt{\frac{n-t}{(2n-t)t}}\sqrt{\frac{n-t}{n} q_A + \sqrt{q_A^2 - \frac{(2n-t)t}{n^2}\Delta_A}} \leqslant \sqrt{\frac{n-t}{nt}} q_A,$$

and the equality holds if and only if $\Delta_A = 0$, that is to say, A is normal, i.e., $AA^* = A^*A$. Considering $\Re_A$ and $\Im_A$ are both normal matrices, then from Theorem 2.1, we know the validity of Theorem 2.2.

REMARK. For the bounds estimate of largest moduli eigenvalue $|\lambda|_{\max}$ of matrix A, in [7, 8], the following inequality was given

$$\frac{|\mathrm{tr}(A)|}{n} \leqslant |\lambda|_{\max} \leqslant \frac{|\mathrm{tr}(A)|}{n} + \sqrt{\frac{n-1}{n} q_A}. \tag{3.5}$$

We can see that the estimates (2.1), (2.2), (2.3) are sharper than (3.5) in some extent. That is to say, the results presented in this paper improve the known conclusions in [7, 8] partially, and can be taken as some supplements for known conclusions [5, 7, 8], especially for the the upper bounds estimate of matrix eigenvalues.

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