Mathematical
Inequalities
& Applications
Volume 10, Number 4 (2007), 733–744

ON THE ZEROS OF A CLASS OF POLYNOMIALS

W. M. SHAH AND A. LIMAN

(*communicated by Th. M. Rassias*)

Abstract. In this paper we prove some results concerning the distribution of the zeros of a polynomial in the complex plane. Our results not only contain some known generalizations of Eneström-Kakeya theorem but also a variety of interesting results can be deduced from them by a fairly uniform procedure.

1. Introduction

If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that $a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_1 \geqslant a_0 > 0$, then according to a well-known result in the theory of distribution of the zeros of polynomials, due to Eneström and Kakeya (for reference, see[11, p. 136 or 12, p. 272]), all the zeros of $P(z)$ lie in $|z| \leqslant 1$.

We may apply this result to the polynomial $P(tz)$ to obtain the following more general result.

THEOREM A. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that*

$$t^n a_n \geqslant t^{n-1} a_{n-1} \geqslant \cdots \geqslant t a_1 \geqslant a_0 > 0,$$

then all the zeros of $P(z)$ lie in $|z| \leqslant t$.

In the literature [1-12] there exist some extensions and generalizations of Eneström-Kakeya theorem. Govil and Rahman [8] extended this theorem to the polynomials with complex coefficients. As a refinement of the result of Govil and Rahman, Govil and Jain [7] proved the following:

THEOREM B. *Let $P(z) = \sum_{j=0}^{n} a_j z^j \not\equiv 0$ be a polynomial with complex coefficients such that*

$$|\arg\ a_k - \beta| \leqslant \alpha \leqslant \frac{\pi}{2}, \quad k = 0, 1, 2, \cdots, n$$

for some β and

$$|a_n| \geqslant |a_{n-1}| \geqslant \cdots \geqslant |a_1| \geqslant |a_0|,$$

Mathematics subject classification (2000): 30C10, 30C15.
Key words and phrases: Polynomials, moduli of the zeros, Eneström-Kakeya theorem.

then $P(z)$ has all its zeros in the ring-shaped region given by

$$R_3 \leqslant |z| \leqslant R_2.$$

Here

$$R_2 = \frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1}{|a_n|}\right\}^{\frac{1}{2}}$$

$$R_3 = \frac{1}{2M_2^2}\left[- R_2^2|b|(M_2 - |a_0|) + \{4|a_0|R_2^2M_2^3 + R_2^4|b|^2(M_2 - |a_0|^2)\}^{\frac{1}{2}}\right],$$

where

$$M_1 = |a_n|R_2,$$
$$M_2 = |a_n|R_2^2\left[R + R_2 - \frac{|a_0|}{|a_n|}(\cos\alpha + i\sin\alpha)\right],$$
$$c = |a_n - a_{n-1}|,$$
$$b = a_1 - a_0$$

and

$$R = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{k=0}^{n-1}|a_k|.$$

Aziz and Mohammad [2] used Schewarz's lemma and proved the following generalization of Eneström-Kakeya theorem.

THEOREM C. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real coefficients. If $t_1 > t_2 \geqslant 0$ can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geqslant 0,\ r = 1, 2, \cdots, n+1\ ,\ (a_{-1} = a_{n+1} = 0),$$

then all the zeros of $P(z)$ lie in $|z| \leqslant t_1$.

Recently Gardner and Govil [6] considered a larger class of polynomials and proved the following more general result.

THEOREM D. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If* $\mathrm{Re}(a_j) = \alpha_j$ *and* $\mathrm{Im}(a_j) = \beta_j$, *for $j = 0, 1, 2, \cdots, n$, $a_n \neq 0$ and for some k and r and for some $t \geqslant 0$,*

$$\alpha_0 \leqslant t\alpha_1 \leqslant t^2\alpha_2 \leqslant \cdots \leqslant t^k\alpha_k \geqslant t^{k+1}\alpha_{k+1} \geqslant \cdots \geqslant t^n\alpha_n$$

and

$$\beta_0 \leqslant t\beta_1 \leqslant t^2\beta_2 \leqslant \cdots \leqslant t^r\beta_r \geqslant t^{r+1}\beta_{r+1} \geqslant \cdots \geqslant t^n\beta_n$$

then $P(z)$ has all its zeros in $R_1 \leqslant |z| \leqslant R_2$, where

$$R_1 = \min\{(t|a_0|/2(t^k\alpha_k + t^r\beta_r) - (\alpha_0 + \beta_0) - t^n(\alpha_n + \beta_n - |a_n|), t\}$$

and

$$R_2 = \max\left\{(|a_0|t^{n+1} - t^{n-1}(\alpha_0+\beta_0) - t(\alpha_r+\beta_r) + (t^2+1)(t^{n-k-1}\alpha_k + t^{n-r-1}\beta_r) + (t^2-1)\left(\sum_{j=1}^{k+1} t^{n-j-1}\alpha_j + \sum_{j=1}^{r-1} t^{n-j-1}\beta_j\right) + (1-t^2)\left(\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=r+1}^{n-1} t^{n-j-1}\beta_j\right)\Big/|a_n|, \frac{1}{t}\right\}.$$

In this paper, as inspired by the Theorems C and D above, we make use of a generalized form of Schewarz's Lemma and prove some more general results in the distribution of the zeros of polynomials. These results include not only the above theorems as special cases, but also lead to a standard development of interesting generalizations of some well-known results by a fairly uniform procedure.

THEOREM 1. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If $t_1 \neq 0$ and t_2 are real numbers with $t_1 \geqslant t_2 \geqslant 0$, such that*

$$\max_{|z|=R}\left|\sum_{j=0}^{n+1}(a_j t_1 t_2 + a_{j-1}(t_1-t_2) - a_{j-2})z^{n-j+2}\right| \leqslant M_1 \tag{1}$$

$$\max_{|z|=R}\left|\sum_{j=1}^{n+2}(a_j t_1 t_2 + a_{j-1}(t_1-t_2) - a_{j-2})z^{j}\right| \leqslant M_2, \tag{2}$$

where $a_{-2} = a_{-1} = 0 = a_{n+1} = a_{n+2}$ and R is any positive real number. Then all the zeros of $P(z)$ lie in the ring-shaped region

$$\min(r_2, R) \leqslant |z| \leqslant \max(r_1, \frac{1}{R}),$$

where

$$r_1 = \frac{2M_1^2}{\{R^4|(t_1-t_2)a_n - a_{n-1}|^2(M_1-|a_n|)^2 + 4|a_n|R^2M_1^3\}^{\frac{1}{2}} - |(t_1-t_2)a_n - a_{n-1}|(M_1-|a_n|)R^2} \tag{3}$$

$$r_2 = \frac{1}{2M_2^2}\Big[\{R^4|a_1t_1t_2 + a_0(t_1-t_2)|^2(M_2 - |a_0|t_1t_2)^2 + 4M_2^3R^2|a_0|t_1t_2\}^{\frac{1}{2}} - R^2(M_2 - |a_0|t_1t_2)|a_1t_1t_2 + a_0(t_1-t_2)|\Big]. \tag{4}$$

REMARK 1. Theorems A, B, C and many other such generalizations of Eneström-Kakeya theorem can be easily deduced from Theorem 1 by a suitable choice of R, t_1 and t_2. As an example, we show Theorem C is a special case of Theorem 1. For this, suppose that Theorem 1 satisfies the hypothesis of Theorem C. We have from (1) for $|z| = R = \frac{1}{t_1}$

$$\left|\sum_{j=0}^{n+1}(a_jt_1t_2+a_{j-1}(t_1-t_2)-a_{j-2})z^{n-j+2}\right| \leqslant \sum_{j=0}^{n+1}|(a_jt_1t_2+a_{j-1}(t_1-t_2)-a_{j-2})|\frac{1}{t_1^{n-j+2}}$$
$$= a_n = M_1(say).$$

It can be easily verified that

$$\begin{aligned} r_1 &= \frac{2M_1^2}{-|(t_1-t_2)a_n-a_{n-1}|(M_1-|a_n|)R^2+\{R^4|(t_1-t_2)a_n-a_{n-1}|^2(M_1-|a_n|)^2+4|a_n|R^2M_1^3\}^{\frac{1}{2}}} \\ &= \frac{|(t_1-t_2)a_n-a_{n-1}|(M_1-|a_n|)R^2+\{R^4|(t_1-t_2)a_n-a_{n-1}|^2(M_1-|a_n|)^2+4|a_n|R^2M_1^3\}^{\frac{1}{2}}}{2|a_n|M_1R^2} \\ &= \frac{|(t_1-t_2)a_n-a_{n-1}|}{2}\left\{\frac{1}{|a_n|}-\frac{1}{M_1}\right\}+\left[\frac{|(t_1-t_2)a_n-a_{n-1}|^2}{4}\left\{\frac{1}{|a_n|}-\frac{1}{M_1}\right\}^2+\frac{M_1}{|a_n|R^2}\right]^{\frac{1}{2}}. \end{aligned} \tag{5}$$

Now, using value of M_1 in (5), we get $r_1 = t_1$. This is precisely the conclusion of Theorem C due to Aziz and Mohammad [1].

Next, we use Theorem 1 to prove the following result, which includes some well-known extensions of Eneström-Kakeya theorem due to Dewan and Bidkham [5], Govil and Jain [7], Aziz and Mohammad [1], and also includes a generalization of a result due to Aziz and Shah [3].

THEOREM 2. *Let* $P(z) = \sum_{j=0}^{n} a_jz^j$ *be a polynomial of degree* n. *If for some* $t_1 \neq 0$ *and* $t_1 \geqslant t_2 \geqslant 0$, *such that*

$$\max_{|z|=R}\left|\sum_{j=0}^{n+1}(a_jt_1t_2 + a_{j-1}(t_1 - t_2) - a_{j-2})z^{n-j+1}\right| \leqslant M_3 \tag{6}$$

$$\max_{|z|=R}\left|\sum_{j=1}^{n+2}(a_jt_1t_2 + a_{j-1}(t_1 - t_2) - a_{j-2})z^{j-1}\right| \leqslant M_4, \tag{7}$$

where R is any positive real number, then all the zeros of $P(z)$ *lie in*

$$\min\left\{\frac{t_1t_2|a_0|}{M_4}, R\right\} \leqslant |z| \leqslant \max\left\{\frac{M_3}{|a_n|}, \frac{1}{R}\right\}. \tag{8}$$

The following corollary immediately follows from Theorem 2, if we take $t_2 = 0$.

COROLLARY 1. *Let* $P(z) = \sum_{j=0}^{n} a_jz^j$ *be a polynomial of degree* n. *If for some* $t > 0$,

$$\max_{|z|=R}|ta_0z^n + (ta_1 - a_0)z^{n-1} + \cdots + (ta_n - a_{n-1})| \leqslant M_3', \tag{9}$$

where R is any positive real number,then all the zeros of $P(z)$ lie in

$$|z| \leqslant \max\left\{\frac{M_3'}{|a_n|}, \frac{1}{R}\right\}. \tag{10}$$

This result was also independently proved by Aziz and Shah [3]. If we assume $ta_j - a_{j-1} \geqslant 0,$ for all $j = 1, 2, \cdots, n,$ in Corollary 1, we get the following:

COROLLARY 2. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n and for some $t > 0,$ $ta_j - a_{j-1} \geqslant 0,$ for all $j = 1, 2, \cdots, n,$ then all the zeros of $P(z)$ lie in*

$$|z| \leqslant \max\left[\frac{1}{|a_n|}\left\{|a_0|t^{n+1} - a_0 t^{n-1} + ta_n + (t^2 - 1)\sum_{j=1}^{n-1} t^{n-j-1} a_j\right\}, \frac{1}{t}\right].$$

For $t = 1$, Corollary 2 reduces to the following:

COROLLARY 3. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients, $a_n \neq 0,$ satisfying $a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_1 \geqslant a_0$ then all the zeros of $P(z)$ lie in*

$$|z| \leqslant \max\left\{\frac{a_n - a_0 + |a_0|}{|a_n|}, 1\right\}.$$

REMARK 2. Since $\frac{a_n - a_0 + |a_0|}{|a_n|} \geqslant 1,$ Corollary 3 reduces to the result of Joyal, Labelle and Rahman [10]. Also for $a_0 > 0,$ it reduces to Eneström - Kakeya theorem.

REMARK 3. Since in the proof of Theorem 2, we consider $F(z) = (t_2 + z)(t_1 - z)P(z)$ and for $t_2 = 0,$ $F(z)$ has a zero at origin. Therefore, inequality (8), for $t_2 = 0$ cannot provide any improvement to the bound obtained by Aziz and Shah [3]. In this case, it is natural to ask that what can be said about the modulii of the zeros of $P(z)$ analogous to Theorem 2. In reply to this question, we have been able to prove the following:

THEOREM 3. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some $t > 0$*

$$\max_{|z|=R} |-a_n z^n + (ta_n - a_{n-1})z^{n-1} + \cdots + (ta_1 - a_0)| \leqslant M_4, \tag{11}$$

where R is any positive real number. Then all the zeros of $P(z)$ lie in

$$|z| < \min\left\{\frac{t|a_0|}{M_4|}, R\right\}. \tag{12}$$

If we take $R = t$, the following Corollary is immediate.

COROLLARY 4. *Let $P(z)$ is a polynomial of degree n. If for some positive real number t,*

$$a_n t^n \geqslant a_{n-1} t^{n-1} \geqslant \cdots \geqslant a_1 t \geqslant a_0,$$

then $P(z)$ does not vanish in

$$|z| \leqslant \min\left\{\frac{t|a_0|}{t^n(|a_n| - a_n) - a_0}, t\right\}. \tag{13}$$

Combining Corollary 3 and Corollary 4 for $t = 1$ and noting that

$$\max\left\{\frac{a_n - a_0 + |a_0|}{|a_n|}, 1\right\} = \frac{a_n - a_0 + |a_0|}{|a_n|}$$

and

$$\min\left\{\frac{|a_0|}{|a_n| + a_n - a_0}, 1\right\} = \frac{|a_0|}{|a_n| + a_n - a_0},$$

we get the following generalization as well as the improvement of Eneström-Kakeya theorem.

THEOREM 4. *Suppose $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n. If $a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_1 \geqslant a_0$, then all the zeros of $P(z)$ lie in the ring-shaped region*

$$\frac{|a_0|}{|a_n| + a_n - a_0} \leqslant |z| \leqslant \frac{|a_0| - a_0 + a_n}{|a_n|}. \tag{14}$$

The result is best possible and equality holds for the polynomial

$$P(z) = z^n + z^{n-1} + \cdots + z + 1.$$

While seeking an extension of Theorem D analogous to Theorem 4, we have been able to obtain the following result, which immediately follows on combining Corollary 1 and Theorem 3. The theorem not only shall contain Theorem D and many other such results as special cases, but also is a refinement of Theorem 2 due to Aziz and Shah [3].

THEOREM 5. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some $t > 0$,*

$$\max_{|z|=R} |ta_0 z^n + (ta_1 - a_0)z^{n-1} + \cdots + (ta_n - a_{n-1})| \leqslant M'$$

and

$$\max_{|z|=R} |-a_n z^n + (ta_n - a_{n-1})z^{n-1} + \cdots + (ta_1 - a_0)| \leqslant M'',$$

where R is any positive real number. Then all the zeros of $P(z)$ lie in the ring-shaped region

$$\min\left\{\frac{t|a_0|}{M''}, R\right\} \leqslant |z| \leqslant \max\left\{\frac{M'}{|a_n|}, \frac{1}{R}\right\}. \tag{15}$$

As mentioned above many well-known generalizations of Eneström- Kakeya theorem follow from Theorem 5 by a fairly uniform procedure. Here, for example, we show that the theorem of Gardner and Govil [6] can be deduced from Theorem 5 by simple calculations.

For this, assuming the hypothesis of Theorem D, and taking $a_j = \alpha_j + i\beta_j$, $j = 0,1,2,\cdots,n$ and $R = t$ in Theorem 5, we have for $|z| = t$,

$$
\begin{aligned}
|ta_0z^n + (ta_1 - a_0)z^{n-1} + \cdots + (ta_n - a_{n-1})| &\leqslant |a_0|t^{n+1} + \sum_{j=1}^{n} |ta_j - a_{j-1}|t^{n-j} \\
&\leqslant |a_0|t^{n+1} + \sum_{j=1}^{n} |t\alpha_j - \alpha_{j-1}|t^{n-j} + \sum_{j=0}^{n} |t\beta_j - \beta_{j-1}|t^{n-j} \\
&= t^{n+1}|a_0| - t^{n-1}(\alpha_0 - \beta_0) - t(\alpha_n - \beta_n) + (t^2+1)(t^{n-k-1}\alpha_k + t^{n-r-1}\beta_r) \\
&\quad + (t^2-1)\left(\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{r-1} t^{n-r-1}\beta_j\right) \\
&\quad + (1-t^2)\left(\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=r+1}^{n-1} t^{n-r-1}\beta_r\right) \\
&= M'.
\end{aligned}
$$

Also

$$
\begin{aligned}
&|-a_nz^n + (ta_n - a_{n-1})z^{n-1} + \cdots + (ta_1 - a_0)| \\
&\qquad \leqslant |a_n|t^n + \sum_{j=1}^{n} |ta_j - a_{j-1}|t^{j-1} \\
&\qquad \leqslant |a_n|t^n + \sum_{j=1}^{n} |t\alpha_j - \alpha_{j-1}|t^{j-1} + \sum_{j=1}^{n} |t\beta_j - \beta_{j-1}|t^{j-1} \\
&\qquad = 2(t^k|\alpha_k| + t^r\beta_r) - (\alpha_0 + \beta_0) - t^n(\alpha_n + \beta_n - |a_n|) \\
&\qquad = M''.
\end{aligned}
$$

Using these observations in Theorem 5, we get the conclusion of Theorem D.

2. Lemmas

For the proofs of these theorems, we need the following lemmas. The first lemma is due to Govil, Rahman and Schmeisser [9]

LEMMA 1. *If $f(z)$ is analytic in $|z| \leqslant 1$, $f(0) = a$ where $|a| < 1$, $f'(0) = b$, $|f(z)| \leqslant 1$ on $|z| \leqslant 1$, then for $|z| \leqslant 1$,*

$$
|f(z)| \leqslant \frac{(1-|a|)|z|^2 + |b||z| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |b||z| + (1-|a|)}.
$$

The estimate is sharp.

From Lemma 1, one can easily deduce the following:

LEMMA 2. *If $f(z)$ is analytic in $|z| \leqslant R$, $f(0) = 0$, $f'(0) = b$, $|f(z)| \leqslant M$ for $|z| = R$, then*

$$|f(z)| \leqslant \frac{M|z|}{R^2}\frac{M|z| + R^2|b|}{M + |b||z|} \text{ for } |z| \leqslant R.$$

3. Proofs of the Theorems

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned} F(z) &= (t_2 + z)(t_1 - z)P(z) \\ &= -a_n z^{n+2} + (a_n(t_1 - t_2) - a_{n-1})z^{n+1} \\ &\quad + (a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2})z^n \\ &\quad + \cdots + (a_1 t_1 t_2 + a_0(t_1 - t_2))z + a_0 t_1 t_2. \end{aligned} \tag{16}$$

Let

$$\begin{aligned} G(z) &= z^{n+2}F(1/z) \\ &= -a_n + (a_n(t_1 - t_2) - a_{n-1})z \\ &\quad + (a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2})z^2 \\ &\quad + \cdots + (a_1 t_1 t_2 + a_0(t_1 - t_2)z^{n+1} + a_0 t_1 t_2 z^{n+2}, \end{aligned}$$

so that

$$|G(z)| \geqslant |a_n| - |H(z)|, \tag{17}$$

where

$$H(z) = (a_n(t_1 - t_2) - a_{n-1})z + (a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2})z^2 + \cdots + a_0 t_1 t_2 z^{n+2}.$$

Clearly $H(0) = 0$ and $H'(0) = a_n(t_1 - t_2) - a_{n-1}$. Since by (1) $|H(z)| \leqslant M_1$ for $|z| = R$, therefore it follows by Lemma 2, that

$$|H(z)| \leqslant \frac{M_1|z|}{R^2}\frac{M_1|z| + R^2|a_n(t_1 - t_2) - a_{n-1}|}{M_1 + |a_n(t_1 - t_2) - a_{n-1}||z|} \; for \; |z| \leqslant R.$$

Using this in (17), we get for $|z| \leqslant R$.

$$\begin{aligned} |G(z)| &\geqslant |a_n| - \frac{M_1|z|}{R^2}\frac{M_1|z| + R^2|a_n(t_1 - t_2) - a_{n-1}|}{M_1 + |a_n(t_1 - t_2) - a_{n-1}||z|} \\ &= \frac{-M_1^2|z|^2 + R^2|a_n(t_1 - t_2) - a_{n-1}|(|a_n| - M_1)|z| + |a_n|R^2M_1}{R^2(M_1 + |a_n(t_1 - t_2) - a_{n-1}||z|)} \\ &> 0, \end{aligned}$$

if

$$M_1^2|z|^2 - R^2|a_n(t_1 - t_2) - a_{n-1}|(|a_n| - M_1)|z| - |a_n|R^2M_1 < 0.$$

Thus $|G(z)| > 0$, if

$$|z| < \frac{1}{2M_1^2}\Big[-R^2|a_n(t_1-t_2)-a_{n-1}|(|a_n|-M_1) + \{R^4|(t_1-t_2)a_n - a_{n-1}|^2(M_1-|a_n|)^2 + 4|a_n|R^2M_1^3|a_n|R^2\}^{\frac{1}{2}}\Big] = \frac{1}{r_1}.$$

Consequently, all the zeros of $G(z)$ lie in $|z| \geqslant \min\left(\frac{1}{r_1}, R\right)$. Since $F(z) = z^{n+2}G(1/z)$, it follows that all the zeros of $F(z)$ and hence all the zeros of $P(z)$ lie in

$$|z| \leqslant \max\left(r_1, \frac{1}{R}\right). \tag{18}$$

Again from (16),we have

$$|F(z)| \geqslant |a_0|t_1t_2 - |T(z)| \tag{19}$$

where

$$T(z) = -a_nz^{n+2} + (a_n(t_1-t_2)-a_{n-1})z^{n+1} + \cdots + (a_1t_1t_2 + a_0(t_1-t_2))z.$$

Clearly $T(0) = 0$ and $T'(0) = a_1t_1t_2 + a_0(t_1-t_2)$. Since by (2), $|T(z)| \leqslant M_2$ for $|z| = R$, therefore it follows by Lemma 2, that

$$|T(z)| \leqslant \frac{M_2|z|\Big((M_2|z| + R^2|a_1t_1t_2 + a_0(t_1-t_2)|\Big)}{R^2\Big(M_2 + |a_1t_1t_2 + a_0(t_1-t_2)||z|\Big)}, \text{ for } |z| \leqslant R.$$

Using this in (19), we get for $|z| \leqslant R$,

$$\begin{aligned}|F(z)| &\geqslant |a_0|t_1t_2 - \frac{M_2|z|\Big(M_2|z| + R^2|a_1t_1t_2 + a_0(t_1-t_2)|\Big)}{R^2\Big(M_2 + |a_1t_1t_2 + a_0(t_1-t_2)||z|\Big)}\\ &= \frac{-M_2^2|z|^2 - R^2(M_2 - |a_0|t_1t_2)|a_1t_1t_2 + a_0(t_1-t_2)||z| + |a_0|t_1t_2R^2M_2}{R^2\Big(M_2 + |a_1t_1t_2 + a_0(t_1-t_2)||z|\Big)}\\ &> 0,\end{aligned}$$

if

$$M_2^2|z|^2 + R^2(M_2 - |a_0|t_1t_2)|a_1t_1t_2 + a_0(t_1-t_2)||z| - |a_0|t_1t_2R^2M_2 < 0.$$

Thus $|F(z)|>0$, if

$$|z|<\frac{1}{2M_2^2}\Bigg[-R^2(M_2-|a_0|)t_1t_2+a_0(t_1-t_2)| \\ +\{R^4(M_2-|a_0|t_1t_2)^2|a_1t_1t_2+a_0(t_1-t_2)|^2+4M_2^3R^2|a_0|t_1t_2\}^{\frac{1}{2}}\Bigg] \\ =r_2.$$

This shows that all the zeros of $F(z)$ and hence of the polynomial $P(z)$ lie in

$$|z|\geqslant \min(r_2,R). \tag{20}$$

Combining (18) and (20), the desired result follows.

Proof of Theorem 2. From (1) and (6), we have

$$\max_{|z|=R}\left|\sum_{j=0}^{n+1}(a_jt_1t_2+a_{j-1}(t_1-t_2)-a_{j-2})z^{n-j+2}\right|\leqslant M_3R=M_1\ (say).$$

Replacing M_1 by M_3R in (3), we get from Theorem 1,

$$r_1=\frac{2M_3^2}{-|(t_1-t_2)a_n-a_{n-1}|(M_3R-|a_n|)+\left\{|(t_1-t_2)a_n-a_{n-1}|^2(M_3R-|a_n|)^2+4|a_n|R^2M_3^3\right\}^{\frac{1}{2}}}. \tag{21}$$

Now suppose that $M_3R\geqslant|a_n|$, then $M_3R-|a_n|\geqslant 0$. Since $|(t_1-t_2)a_n-a_{n-1}|\leqslant M_3$, therefore,we have

$$|(t_1-t_2)a_n-a_{n-1}|(M_3R-|a_n|)\leqslant M_3(M_3R-|a_n|).$$

Equivalently

$$M_3|a_n|+|(t_1-t_2)a_n-a_{n-1}|(M_3R-|a_n|)\leqslant M_3^2R.$$

From this we easily conclude that

$$2M_3|a_n|\leqslant -|(t_1-t_2)a_n-a_{n-1}|(M_3R-|a_n|) \\ +\left\{|(t_1-t_2)a_n-a_{n-1})|^2(M_3R-|a_n|)^2+4M_3^3|a_n|R\right\}^{\frac{1}{2}}.$$

This with the help of (21) implies

$$r_1\leqslant\frac{M_3}{|a_n|}.$$

Hence, it follows by Theorem 1, that all the zeros of $P(z)$ lie in the circle

$$|z|\leqslant\frac{M_3}{|a_n|},\ if\ |a_n|\leqslant M_3R. \tag{22}$$

Now if $|a_n| \geqslant M_3R,$ then from (6), it clearly follow, that

$$\left|\sum_{j=0}^{n+1}(a_jt_1t_2 + a_{j-1}(t_1 - t_2) - a_{j-2})z^{n-j+2}\right| \leqslant |a_n| \ for \ |z| = R.$$

Using Rouche's theorem, it follows that the polynomial

$$G(z) = -a_n + \sum_{j=0}^{n+1}(a_jt_1t_2 + a_{j-1}(t_1 - t_2) - a_{j-2})z^{n-j+2}$$

does not vanish in $|z| < R$. This means that the polynomial $F(z) = z^{n+2}G(1/z)$ does not vanish in $|z| > \frac{1}{R}$. Since every zero of $P(z)$ is also a zero of $F(z)$, we conclude that all the zeros of $P(z)$ lie in the circle

$$|z| \leqslant \frac{1}{R}, \ if \ |a_n| > M_3R. \tag{23}$$

From (22) and (23), we conclude that all the zeros of $P(z)$ lie in

$$|z| \leqslant \max\left\{\frac{M_3}{|a_n|}, \frac{1}{R}\right\}. \tag{24}$$

Now, making use of (2) and (7) and proceeding similarly as above one can easily prove that all the zeros of $P(z)$ lie in

$$|z| \geqslant \min\left\{\frac{t_1t_2|a_0|}{M_4}, R\right\}. \tag{25}$$

Combining (24) and (25), the proof of Theorem 2 is complete.

Proof of Theorem 3. The proof of Theorem 3 follows on the same lines as the proof of Theorem 2. We omit the details.

Proof of Corollary 2. We have from (9) for $R = t$ and the fact $ta_j - a_{j-1} \geqslant 0$ for $j = 1, 2, \cdots, n$

$$\begin{aligned}\max_{|z|=R} |ta_0z^n &+ (ta_1 - a_0)z^{n-1} + (ta_2 - a_1)z^{n-2} \\ &+ \cdots + (ta_{n-1} - a_{n-2})z + (ta_n - a_{n-1})| \\ &\leqslant |a_0|t^{n+1} + |(ta_1 - a_0)|t^{n-1} + |(ta_2 - a_1)|t^{n-2} \\ &\quad + \cdots + |(ta_{n-1} - a_{n-2})|t + |(ta_n - a_{n-1})| \\ &= |a_0|t^{n+1} + t^na_1 - a_0t^{n-1} + t^{n-1}a_2 - a_1t^{n-2} \\ &\quad + \cdots + t^2a_{n-1} - ta_{n-2} + ta_n - a_{n-1} \\ &= |a_0|t^{n+1} - a_0t^{n-1} + a_nt + (t^2 - 1)\sum_{j=1}^{n-1} a_jt^{n-j-1} \\ &= M^{'} \text{ (say)}.\end{aligned}$$

Using this in (10), the desired result follows.

REFERENCES

[1] A. AZIZ, Q. G. MOHAMMAD, *Zero-free regions for polynomials and some generalizations of Eneström-Kakeya theorem*, Cand. Math. Bull., **27**, (1984), 265–272.

[2] A. AZIZ, Q. G. MOHAMMAD, *On the zeros of a certain class of polynomials and related analytic functions*, J. Math. Anal. Appl., **75**, (1980), 495–502.

[3] A. AZIZ, W. M. SHAH, *On the zeros of polynomials and related analytic functions*, Glasnik Mate., **33**, (1998), 173–184.

[4] G. T. CARGO, O. SHISHA, *Zeros of polynomials and fractional order differences of their coefficients*, J. Math. Anal. Appl., **7**, (1963), 176–182.

[5] K. K. DEWAN, M. BIDKHAM, *On the Eneström-Kakeya theorem*, J. Math. Anal. Appl., **180**, (1993), 29–36.

[6] R. B. GARDNER, N. K. GOVIL, *On the location of the zeros of a polynomial*, J. Approx. Theory, **76**, (1994), 286–292.

[7] N. K. GOVIL, V. K. JAIN, *On the Eneström-Kakeya theorem II*, J. Approx Theory, **22**, (1978), 1–10.

[8] N. K. GOVIL, Q. I. RAHMAN, *On the Eneström-Kakeya theorem II*, Tohoku Math. J., **20**, (1968), 126–136.

[9] N. K. GOVIL, Q. I. RAHMAN AND G. SCHMEISSER, *On the derivative of a polynomial*, Illinois, Math. Jour., **23**, (1979), 319–329.

[10] A. JOYAL, G. LABELLE AND Q. I. RAHMAN, *On the location of zeros of polynomials*, Canad. Math. Bull., **10**, (1967), 53–63.

[11] M. MARDEN, *Geometry of Polynomials*, IInd Ed. Math, Surveys 3, Amer. Math. Soc., Providence, RI, (1966).

[12] G. V. MILOVANOVIC, D. S. MITRINOVIC AND TH. M. RASSIAS, *Topics in Polynomials, Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore (1994).

(Received October 24, 2003)

W. M. Shah
P. G. Department of Mathematics
Baramulla College
Kashmir 193101
India
e-mail: wmshah@rediffmail.com

A. Liman
Department of Mathematics
National Institute of Technology
Kashmir 190006
India
e-mail: abliman22@yahoo.com