Mathematical
Inequalities
& Applications
Volume 10, Number 4 (2007), 755–760

ON THE TWO–STEP PROJECTION METHODS AND APPLICATIONS TO VARIATIONAL INEQUALITIES

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(*communicated by R. U. Verma*)

Abstract. The purpose of this paper is, by using the two-step projection methods, to study the convergence analysis and the approximation solvability of the system of nonlinear variational inequalities in the setting of Hilbert spaces. The results presented in this paper generalize and improve the corresponding results in Verma [4].

1. Introduction and preliminaries

Projection and projection type method play an important role in the numerical solution of variational inequality theory based on their convergence analysis.

Recently, Verma [4] introduced the general two-step model for projection methods and then applied it to the approximation solvability for a class of strongly monotone and Lipschitz nonlinear variational inequalities in Hilbert spaces.

The purpose of this paper is, by using the two-step projection methods, to study further the convergence analysis and the approximation solvability for a class of systems of nonlinear variational inequalities in the setting of Hilbert spaces. The results presented in this paper generalize and improve the corresponding results in Verma [3], [4].

Throughout this paper, we assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, K is a closed convex subset of H and $T : K \to H$ is a mapping.

We consider a system of two nonlinear variational inequality (in short, SNVI) problems as follows:

Find $x^*, y^* \in K$ such that

$$\begin{cases} \langle \rho T(y^*) + x^* - y^*, x - x^* \rangle \geqslant 0, & \forall\, x \in K, \quad \rho > 0, \\ \langle \eta T(x^*) + y^* - x^*, x - y^* \rangle \geqslant 0, & \forall\, x \in K, \quad \eta > 0. \end{cases} \tag{A}$$

The SNVI problem (A) is equivalent to the following projection problem:

$$\begin{cases} x^* = P_K[y^* - \rho T(y^*)], & \rho > 0, \\ y^* = P_K[x^* - \eta T(x^*)], & \eta > 0, \end{cases} \tag{B}$$

Mathematics subject classification (2000): 47H05, 47H09, 49M05.

Key words and phrases: two-step projection method, system of nonlinear variational inequalities, r-strongly monotone mapping, μ-Lipschitz mapping.

The first author was supported by the Natural Science Foundation of Yibin University (2005-Z3) and the second author was supported by the Korea Research Foundation Grant (No. KRF-2004-041-C00033).

Paper MIA-10-70

where P_K is the projection of H onto K.

Special cases of the SNVI problem (A) are following:

(1) If $\eta = 0$, then, from the SNVI problem (A), we have the following nonlinear variational inequality (in short, NVI) problem:

Find $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle \geqslant 0, \quad \forall\, x \in K. \tag{C}$$

(2) If K is a closed convex cone of H, then the SNVI problem (A) is equivalent to a system of nonlinear complementarity problems (in short, SNC):

Find $x^*, y^* \in K$ such that

$$\begin{cases} T(x^*) \in K^*, \quad T(y^*) \in K^*, \\ \langle \rho T(y^*) + x^* - y^*, x^* \rangle = 0, \quad \rho > 0, \\ \langle \eta T(y^*) + y^* - x^*, y^* \rangle = 0, \quad \eta > 0, \end{cases} \tag{D}$$

where K^* is a polar cone of K defined by

$$K^* = \{f \in H : \langle f, x \rangle \geqslant 0 \text{ for all } x \in K\}.$$

In order to give the main results, we first recall some definitions, notations and lemmas:

DEFINITION 1.1. Let $T : H \to H$ be a mapping.

(1) T is said to be monotone if, for any $x, y \in H$,

$$\langle Tx - Ty, x - y \rangle \geqslant 0.$$

(2) T is said to be r-strongly monotone if, for any $x, y \in H$, there exists a constant $r > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geqslant r\|x - y\|^2.$$

(3) T is said to be L-Lipschitz if, for any $x, y \in H$, there exists a constant $L \geqslant 1$ such that

$$\|Tx - Ty\| \leqslant L\|x - y\|.$$

LEMMA 1.1. ([1]) *For any given* $z \in H$ *and* $x \in K$, $x = P_K(z)$ *if and only if* $\langle x - z, y - x \rangle \geqslant 0$ *for all* $y \in K$.

LEMMA 1.2. ([2]) *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *be three nonnegative sequences satisfying the following conditions:*

$$a_{n+1} \leqslant (1 - t_n)a_n + b_n + c_n, \quad n \geqslant 0,$$

where $t_n \in (0, 1)$ *for all* $n \geqslant 0$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = \circ(t_n)$ *and* $\sum_{n=0}^{\infty} c_n < \infty$. *Then* $a_n \to 0$ *as* $n \to \infty$.

2. The two-step projection methods

This section is devoted to study the general two-step models for projection methods and its special forms are applied to study the convergence analysis for the approximation solvability of the SNVI problem (A).

Algorithm 2.1. For an arbitrarily chosen initial point $x_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ in K generated by

$$\begin{cases} x_{n+1} = (1-\alpha_n-\gamma_n)x_n + \alpha_n P_K[y_n - \rho T(y_n)] + \gamma_n u_n, & n \geqslant 0, \\ y_n = (1-\beta_n-\delta_n)x_n + \beta_n P_K[x_n - \eta T(x_n)] + \delta_n v_n, & n \geqslant 0, \end{cases}$$

where P_K is the projection of H onto K, $\rho, \eta > 0$ are constants, $\{u_n\}$, $\{v_n\}$ are bounded sequences in K and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

$$0 \leqslant \alpha_n + \gamma_n \leqslant 1, \quad 0 \leqslant \beta_n + \delta_n \leqslant 1, \quad n \geqslant 0.$$

If $\beta_n = 0$ and $\delta_n = 0$ for all $n \geqslant 0$, then, from Algorithm 2.1, we have the following:

Algorithm 2.2. For an arbitrarily initial point $x_0 \in K$, compute the sequence $\{x_n\}$ in K generated by

$$x_{n+1} = (1-\alpha_n-\gamma_n)x_n + \alpha_n P_K[x_n - \rho T(x_n)] + \gamma_n u_n, \quad n \geqslant 0,$$

where $0 \leqslant \alpha_n + \gamma_n \leqslant 1$ for all $n \geqslant 0$ and $\{u_n\}$ is a bounded sequence in K.

If $\beta_n = 1$ and $\delta_n = 0$ for all $n \geqslant 0$, then, from Algorithm 2.1, we have the following:

Algorithm 2.3. For an arbitrarily chosen initial point $x_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ in K generated by

$$\begin{cases} x_{n+1} = (1-\alpha_n-\gamma_n)x_n + \alpha_n P_K[y_n - \rho T(y_n)] + \gamma_n u_n, & n \geqslant 0, \\ y_n = P_K[x_n - \eta T(x_n)], & n \geqslant 0, \end{cases}$$

where $0 \leqslant \alpha_n + \gamma_n \leqslant 1$ for all $n \geqslant 0$ and $\{u_n\}$ is a bounded sequence in K.

REMARK 2.1. If $\gamma_n = 0$ and $\delta_n = 0$ for all $n \geqslant 0$, then, from Algorithm 2.1, we have Algorithm 2.1 in Verma [4].

3. The main results

Now, we present, based on Algorithm 2.1, the approximation solvability of the SNVI problem (A) involving r-strongly monotone and μ-Lipschitz continuous mapping in Hilbert spaces.

THEOREM 3.1. *Let H be a real Hilbert space and K be a nonempty closed convex subset of H. Let $T : K \to H$ be r-strongly monotone and μ-Lipschitz continuous. Suppose that $x^*, y^* \in K$ form a solution of the SNVI problem (A). Let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by Algorithm 2.1. If the following conditions are satisfied:*

(i) $0 \leqslant \alpha_n + \gamma_n \leqslant 1$ *and* $0 \leqslant \beta_n + \delta_n \leqslant 1$ *for all* $n \geqslant 0$,

(ii) $\beta_n \to 1$, $\delta_n \to 0$, $\sum_{n=0}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $0 < \rho < \frac{2r}{\mu^2}$ *and* $0 < \eta < \frac{2r}{\mu^2}$,

then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* x^* *and* y^*, *respectively.*

Proof. Since x^*, y^* form a solution of the SNVI problem (A), we have

$$x^* = P_K[y^* - \rho T(y^*)], \quad \rho > 0,$$
$$y^* = P_K[x^* - \eta T(x^*)], \quad \eta > 0.$$

Applying Algorithm 2.1, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n-\gamma_n)(x_n - x^*) \\ &\quad + \alpha_n\{P_K[y_n - \rho T(y_n)] - P_K[y^* - \rho T(y^*)]\} + \gamma_n(u_n - x^*)\| \\ &\leqslant (1-\alpha_n-\gamma_n)\|x_n - x^*\| \\ &\quad + \alpha_n\|P_K[y_n - \rho T(y_n)] - P_K[y^* - \rho T(y^*)]\| + \gamma_n\|u_n - x^*\| \\ &\leqslant (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|y_n - y^* - \rho[T(y_n)] - T(y^*)]\| + M\gamma_n, \quad n \geqslant 0, \end{aligned} \tag{3.1}$$

where

$$M = \max\{\sup_{n\geqslant 0}\|u_n - x^*\|, \sup_{n\geqslant 0}\|v_n - y^*\|, \|x^* - y^*\|\} < \infty.$$

Since T is r-strongly monotone and μ-Lipschitz continuous, we have

$$\begin{aligned} &\|y_n - y^* - \rho[T(y_n) - T(y^*)]\|^2 \\ &\quad = \|y_n - y^*\|^2 - 2\rho\langle T(y_n) - T(y^*), y_n - y^*\rangle + \rho^2\|T(y_n) - T(y^*)\|^2 \\ &\quad \leqslant \|y_n - y^*\|^2 - 2\rho r\|y_n - y^*\|^2 + \rho^2\mu^2\|y_n - y^*\|^2 \\ &\quad = (1 - 2\rho r + (\rho\mu)^2)\|y_n - y^*\|^2 \\ &\quad = \theta^2\|y_n - y^*\|^2, \quad n \geqslant 0, \end{aligned} \tag{3.2}$$

where $\theta = \sqrt{1 - 2\rho r + (\rho\mu)^2} < 1$ by the condition (iii).

Substituting (3.2) into (3.1) and simplifying the resultant result , we have

$$\|x_{n+1} - x^*\| \leqslant (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta\|y_n - y^*\| + M\gamma_n, \quad n \geqslant 0. \tag{3.3}$$

Similarly, we have

$$\begin{aligned} \|y_n - y^*\| &= \|(1-\beta_n-\delta_n)(x_n - y^*) + \beta_n\{P_K[x_n - \eta T(x_n)] \\ &\quad - P_K[x^* - \eta T(x^*)]\} + \delta_n(v_n - y^*)\| \\ &\leqslant (1-\beta_n-\delta_n)\|x_n - y^*\| + \beta_n\|P_K[x_n - \eta T(x_n)] \\ &\quad - P_K[x^* - \eta T(x^*)]\| + \delta_n\|v_n - y^*\| \\ &\leqslant (1-\beta_n)\|x_n - y^*\| + \beta_n\|x_n - x^* - \eta[T(x_n) - T(x^*)]\| + M\delta_n \\ &\leqslant (1-\beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^* - \eta[T(x_n) - T(x^*)]\| \\ &\quad + (1-\beta_n)\|x^* - y^*\| + M\delta_n \\ &\leqslant (1-\beta_n)\|x_n - x^*\| + \beta_n\sqrt{1 - 2\eta r + (\eta\mu)^2}\|x_n - x^*\| + M[(1-\beta_n) + \delta_n] \\ &= (1-\beta_n)\|x_n - y^*\| + \beta_n\sigma\|x_n - x^*\| + M[(1-\beta_n) + \delta_n] \\ &\leqslant \|x_n - x^*\| + M[(1-\beta_n) + \delta_n], \quad n \geqslant 0, \end{aligned} \tag{3.4}$$

where

$$\sigma = \sqrt{1 - 2\eta r + (\eta\mu)^2} < 1.$$

From (3.3) and (3.4), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\leqslant \|(1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta\{x_n - x^*\| \\ &\quad + M[(1-\beta_n)+\delta_n]\} + M\gamma_n \\ &\leqslant [1-\alpha_n(1-\theta)]\|x_n - x^*\| \\ &\quad + \alpha_n M[(1-\beta_n)+\delta_n] + M\gamma_n, \quad n \geqslant 0. \end{aligned} \tag{3.5}$$

Taking

$$\begin{aligned} a_n &= \|x_n - x^*\|, \quad t_n = (1-\theta)\alpha_n, \quad n \geqslant 0, \\ c_n &= M\gamma_n, \quad b_n = \alpha_n M[(1-\beta_n)+\delta_n], \quad n \geqslant 0, \end{aligned}$$

in Lemma 1.2, by the conditions (ii) and (iii), we have $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = \circ(t_n)$, $\sum_{n=0}^{\infty} c_n < \infty$. This implies that all the conditions in Lemma 1.2 are satisfied. Therefore, we have

$$\|x_n - x^*\| \to 0$$

as $n \to \infty$. Again, from (3.4), we have

$$\|y_n - y^*\| \to 0$$

as $n \to \infty$. This completes the proof.

REMARK 3.1. (1) Theorem 3.1 generalizes and improves the main result in Verma [4]. Especially, in the cases that $\gamma_n = \delta_n = 0$ for all $n \geqslant 0$, Theorem 3.1 correct some important mistakes appeared in the proof of Theorem 3.1 in [4].

(2) Taking

$$\begin{aligned} \alpha_n &= \frac{1}{n+1}, \quad \beta_n = \frac{n^2+1}{n^2+2}, \quad n \geqslant 0, \\ \gamma_n &= \frac{n-1}{n^3}, \quad \delta_n = \frac{1}{n^2+2}, \quad n \geqslant 0, \end{aligned}$$

then the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ satisfy the conditions (i) and (ii) in Theorem 3.1.

In Theorem 3.1, taking $\beta_n = 1$ and $\delta_n = 0$ for all $n \geqslant 0$, then, from Theorem 3.1, we have the following:

THEOREM 3.2. *Let H be a real Hilbert space, K be a nonempty closed convex subset of H and $T : K \to H$ be a r-strongly monotone and μ-Lipschitz continuous mapping. Let $x^*, y^* \in K$ form a solution of the SNVI problem (A) and $\{x_n\}$, $\{y_n\}$ be the sequences generated by Algorithm 2.3. If the following conditions are satisfied:*

(i) $0 \leqslant \alpha_n + \gamma_n \leqslant 1$ for all $n \geqslant 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} \gamma_n < \infty$,

(ii) $0 < \rho < \frac{2r}{\mu^2}$ and $0 < \eta < \frac{2r}{\mu^2}$,

then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to x^* and y^*, respectively.

THEOREM 3.3. *Let H, K and T be the same as in Theorem 3.1. Let $x^* \in K$ be a solution of the NVI problem (C) and $\{x_n\}$ be a sequence generated by Algorithm 2.2. If the following conditions are satisfied:*

(i) $0 \leqslant \alpha_n + \gamma_n \leqslant 1$ *for all* $n \geqslant 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} \gamma_n < \infty$,

(ii) $0 < \rho < \frac{2r}{\mu^2}$,

then the sequences $\{x_n\}$ converges strongly to x^*.

Proof. Taking $\eta = 0$ and $\delta_n = 0$ for all $n \geqslant 0$ in Theorem 3.1, the conclusion of Theorem 3.3 can be obtained from Theorem 3.1 immediately.

REFERENCES

[1] S. S. CHANG, *Variational Inequality and Complementarity Problem Theory with Applications*, Shanghai Sci. and Tech. Literature Publishers, Shanghai, 1991 (in Chinese).

[2] L. S. LIU, *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl., **194**, (1995), 114–127.

[3] RAM U. VERMA, *Generalized class of partial relaxed monotonicity and its connections*, Advan. in Nonlinear Variat. Inequal., **7**, (2) (2004), 155–164.

[4] RAM U. VERMA, *Generalized convergence analysis for two-step projection methods and applications to variational problems*, Appl. Math. Lett., **18**, (2005), 1286–1292.

(Received January 6, 2006)

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