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CIRCULAR INTERLACING WITH RECIPROCAL POLYNOMIALS

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Abstract. The purpose of this paper is to show that all zeros of the reciprocal polynomial

$$P_m(z) = \sum_{k=0}^{m} A_k z^k \quad (z \in \mathbb{C})$$

of degree $m \geqslant 2$ with real coefficients $A_k \in \mathbb{R}$ (i.e. $A_m \neq 0$ and $A_k = A_{m-k}$ for all $k = 0, \dots, \left[\frac{m}{2}\right]$) are on the unit circle, if there is a $B \in \mathbb{R}$ such that $A_m B \geqslant 0, |A_m| \geqslant |B|$ and

$$|A_m + B| \geqslant \sum_{k=1}^{m-1} |A_k + B - A_m|$$

holds.

If the inequality is strict then the zeros of P_m have the form $e^{\pm u_j}$ $(j = 1, \dots, \left[\frac{m}{2}\right])$ where

$$\frac{2(j-1)\pi}{m} < u_j < \frac{2j\pi}{m} \quad (j = 1, \dots, \left[\frac{m}{2}\right])$$

and they are simple (for odd m, in addition to these zeros, $-1 = e^{-i\pi}$ is a zero too).

This implies that the polynomial P_m (with $A_m > 0$) and $z^{2m} - 1$ satisfy the circular interlacing condition.

If in the inequality (for the coefficients) equality holds, then double zeros may arise, we discuss how this can happen.

1. Introduction

Recently J. Mckee and C. Smyth [6] proved that there are Salem numbers of every trace. One essential part of the proof was a novel construction of polynomials of specified negative trace, using pairs of polynomials whose zeros interlace on the unit circle.

A pair of relatively prime polynomials P and Q are said to satisfy the *circular interlacing condition* if they both have real coefficients, positive leading term, and all their zeros lie on the unit circle, and interlace there. The last condition means that the zeros of P and Q can be written as $e^{i\alpha_j}$ and $e^{i\beta_j}$ $j = 1, \dots, m$ respectively where

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < \alpha_1.$$

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The importance of interlacing polynomials is shown by the following

PROPOSITION 1. (J. Mckee and C. Smyth [6]) *Suppose that the polynomials P and Q satisfy the circular interlacing condition, have integer coefficients, and that P is monic (and thus cyclotomic). Then*

(a) if $P(1)=0$, or $Q(1)=0$ and $2P(1)-Q'(1)<0$, then $(z^2-1)P(z)-zQ(z)$ is the minimal polynomial of a Salem number (or perhaps a reciprocal Pisot number), possibly multiplied by a cyclotomic polynomial. [Note: one of $P(1)$ and $Q(1)$ is always zero.]

(b) always $(z^2-z-1)Q(z)-zQ(z)$ is the minimal polynomial of a Pisot number.

The starting point of this paper was different. We were looking for sufficient conditions, more general than the previous ones ([3, 9, 4]), which ensure that all zeros of reciprocal (or self-inversive) polynomials are on the unit circle. The interlacing is a byproduct, which in the light of the paper [6] seems to be important.

The first author [3] proved that all zeros of the reciprocal polynomial

$$P_m(z)=\sum_{k=0}^{m} A_k z^k \quad (z\in\mathbb{C})$$

of degree $m\geqslant 2$ with real coefficients $A_k\in\mathbb{R}$ (i.e. $A_m\neq 0$ and $A_k=A_{m-k}$ for all $k=0,\dots,\left[\frac{m}{2}\right]$) are on the unit circle, if

$$|A_m|\geqslant \sum_{k=1}^{m-1}|A_k-A_m| \tag{1}$$

holds, moreover the zeros are located quite regularly.

A. Schinzel [9] generalized this result for self-inversive polynomials. He proved that all zeros of the polynomial

$$P_m(z)=\sum_{k=0}^{m} A_k z^k \in\mathbb{C}[z]$$

satisfying

$$A_k\in\mathbb{C}, A_m\neq 0, A_{m-k}=\varepsilon\overline{A}_k \quad (k=0,\dots,m) \text{ with a fixed } \varepsilon\in\mathbb{C}, |\varepsilon|=1 \tag{2}$$

are on the unit circle if

$$|A_m|\geqslant \inf_{\substack{c,d\in\mathbb{C}\\ |d|=1}}\sum_{k=0}^{m}\left|cA_k-d^{m-k}A_m\right|. \tag{3}$$

If the inequality is strict the zeros are simple. Polynomials satisfying (3) are called *self-inversive* see e.g. [1, 2, 7].

The authors proved [4] that if the coefficients of a self-inversive polynomial P_m satisfy the inequality

$$|A_m|\geqslant \frac{1}{2}\sum_{k=1}^{m-1}|A_k| \tag{4}$$

then all zeros of P_m are on the unit circle. Moreover, they found the approximate location of the zeros.

Here we find a common generalization of the conditions (1), (4) in case of reciprocal polynomials. Also the approximate location of the zeros is given and the case of multiple zeros is studied.

Our basic tool is the Chebyshev transformation of semi-reciprocal polynomials. Concerning this transformation we refer to [3].

2. The main result and its proof

MAIN THEOREM. *All zeros of the reciprocal polynomial* $P_m(z) = \sum k = 0^m A_k z^k$ $(z \in \mathbb{C})$ *of degree* $m \geqslant 2$ *with real coefficients* $A_k \in \mathbb{R}$ *(i.e.* $A_m \neq 0$ *and* $A_k = A_{m-k}$ *for all* $k = 0, \ldots, \left[\frac{m}{2}\right]$ *) are on the unit circle, if*

$$|A_m + B| \geqslant \sum_{k=1}^{m-1} |A_k + B - A_m| \tag{5}$$

holds with some $B \in \mathbb{R}$ *satisfying*

$$A_m B \geqslant 0,\ |A_m| \geqslant |B|. \tag{6}$$

Moreover if the inequality (5) *is strict and* $m = 2n$ *is even then the zeros of* P_m *have the form* $e^{\pm u_j}$ $(j = 1, \ldots, n)$ *where*

$$\frac{2(j-1)\pi}{2n} < u_j < \frac{2j\pi}{2n} \quad (j = 1, \ldots, n) \tag{7}$$

and they are simple.

If the inequality (5) *is strict and* $m = 2n + 1$ *is odd then the zeros of* P_m *are* $-1 = e^{i\pi}$ *and* $e^{\pm u_j}$ $(j = 1, \ldots, n)$ *where*

$$\frac{2(j-1)\pi}{2n+1} < u_j < \frac{2j\pi}{2n+1} \quad (j = 1, \ldots, n) \tag{8}$$

and they are simple.

REMARK 1. With $B = 0$ we obtain from (5) the condition (1) of the first author [3], while with $B = A_m$ we obtain condition (4) of [4]. In the second case our result for the location of zeros gives the same as in [4], in the first case it differs from that of [3].

REMARK 2. If in (5) strict inequality holds then, by (7), (8) the polynomials P_m (with $A_m > 0$) and $Q_m(z) := z^m - 1$ satisfy the *circular interlacing property.*

REMARK 3. (added in proof) It is easy to check that in case of $A_m > 0$ (5) holds with some $B \in \mathbb{R}$ satisfying $A_m \geqslant B \geqslant 0$ if and only if one of the inequalities

$$\begin{aligned}
A_m &\geqslant \sum_{k=1}^{m-1} |A_k - A_m|, \\
2A_m &\geqslant \sum_{k=1}^{m-1} |A_k|, \\
2A_m - A_i &\geqslant \sum_{k=1}^{m-1} |A_k - A_i| \quad \text{where } i = 1, \ldots, m-1 \text{ is such that } A_m > A_i > 0,
\end{aligned}$$

is satisfied, that is if (5) holds with $B=0$, $B=A_m$, $B=A_m-A_i$ $(i=1,\dots,m-1$, $A_m>A_i>0)$ respectively. In this way one can get rid of the inconvenient existence condition (concerning B) in the main theorem.

Proof. We show that all zeros of the Chebyshev transform $\mathcal{T}P_m$ of P_m are in the interval $[-2,2]$.

With the notation $v_j(z)=z^j+z^{j-1}+\cdots+1=\dfrac{z^{j+1}-1}{z-1}$, $e_j(z)=z^j$, $w_j(z)=z^j+1$ $(j=0,1,\dots)$, $a_k=A_k+B-A_m$ $(k=0,\dots,m)$ we have for even $m=2n$

$$P_{2n}(z)=(A_m-B)v_{2n}(z)+\sum_{k=0}^{n-1}a_ke_k(z)\cdot w_{2n-2k}(z)+a_ne_n(z),$$

hence by the linearity of the Chebyshev transform

$$\mathcal{T}P_{2n}(x)=(A_m-B)\mathcal{T}v_{2n}(x)+\sum_{k=0}^{n-1}a_k\mathcal{T}(e_k\cdot w_{2n-2k})(x)+a_n\mathcal{T}(e_n)(x).$$

The Chebyshev transforms we need here have been calculated in [3]. Thus we have

$$\mathcal{T}P_{2n}(x)=(A_m-B)\left[U_n\left(\frac{x}{2}\right)+U_{n-1}\left(\frac{x}{2}\right)\right]+\sum_{k=0}^{n-1}2a_kT_{n-k}\left(\frac{x}{2}\right)+a_nT_0\left(\frac{x}{2}\right)$$

where T_n and U_n are the nth Chebyshev polynomial of the first and second kind, defined by $T_n(\cos x)=\cos nx$ $(n=0,1,\dots)$ and $U_n(\cos x)=\frac{\sin(n+1)x}{\sin x}$ $(n=-1,0,1,\dots)$ respectively (see for example in [8]).

For odd $m=2n+1$ we have $P_{2n+1}(z)=(z+1)\tilde{P}_{2n}(z)$ with

$$\tilde{P}_{2n}(z)=(A_m-B)\tilde{v}_{2n}(z)+\sum_{k=0}^{n}a_ke_k(z)\tilde{w}_{2n-2k}(z)$$

where

$$\tilde{v}_{2n}(z)=\frac{v_{2n+1}(z)}{z+1}=z^{2n}+z^{2n-2}+\cdots+z^2+1=v_n(z^2),$$
$$\tilde{w}_{2n-2k}(z)=\frac{w_{2n+1-2k}(z)}{z+1}=\frac{z^{2n+1-2k}+1}{z+1}\ (k=0,\dots,n).$$

Taking the Chebyshev transforms of these functions from [3] we get

$$\begin{aligned}\mathcal{T}\tilde{P}_{2n}(x)&=(A_m-B)\mathcal{T}\tilde{v}_{2n}(x)+\sum_{k=0}^{n}a_k\mathcal{T}(e_k\cdot\tilde{w}_{2n-2k})(x)\\&=(A_m-B)U_n\left(\frac{x}{2}\right)+\sum_{k=0}^{n}a_k\left[U_{n-k}\left(\frac{x}{2}\right)-U_{n-k-1}\left(\frac{x}{2}\right)\right].\end{aligned}$$

From now on *we treat odd and even m's separately.*

In the proof *we may assume* that $A_m > 0, A_m \geqslant B \geqslant 0$ (otherwise, if $A_m < 0$, we multiply P_m by -1 and replace B by $-B$).

Case 1. $m = 2n$ is even. Let

$$x_j = 2\cos y_j \quad \text{with} \quad y_j = \frac{2j\pi}{m} = \frac{j\pi}{n} \quad (j = 0, \dots, n)$$

then

$$U_n\left(\frac{x_j}{2}\right) + U_{n-1}\left(\frac{x_j}{2}\right) = \frac{\sin(n+1)y_j}{\sin y_j} + \frac{\sin ny_j}{\sin y_j} = \frac{\sin\frac{2n+1}{2}y_j}{\sin\frac{1}{2}y_j} = \frac{\sin\left(j\pi + \frac{j\pi}{2n}\right)}{\sin\frac{j\pi}{2n}}$$
$$= \frac{\sin j\pi}{\sin\frac{j\pi}{2n}}\cos\frac{j\pi}{2n} + \cos j\pi = 2n\,\delta_{j0} + (-1)^j$$
$$T_{n-k}\left(\frac{x_j}{2}\right) = \cos(n-k)y_j = \cos(n-k)\frac{j\pi}{n} = \cos\left(j\pi - \frac{jk\pi}{n}\right) = (-1)^j\cos\frac{jk\pi}{n}$$

where δ_{ij} denotes Kronecker's symbol ($\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$).

Thus, for $j = 0, \dots, n$ we have

$$\mathcal{T}P_{2n}(x_j) = (-1)^j\left[2n(A_m - B)\delta_{j0} + (A_m + B) + \sum_{k=1}^{n-1} 2a_k\cos\frac{jk\pi}{n} + a_n\cos j\pi\right] \quad (9)$$

where $a_k = A_k + B - A_m\,(k = 1, \dots, 2n)$.

Assume that *strict inequality holds in* (5). We claim that

$$\operatorname{sgn}\mathcal{T}P_{2n}(x_j) = (-1)^j \quad (j = 0, \dots, n). \quad (10)$$

To prove this we observe that by (5) and by $a_k = a_{m-k}$

$$A_m + B > \sum_{k=1}^{m-1}|a_k| = \sum_{k=1}^{n-1}2|a_k| + |a_n| \geqslant \sum_{k=1}^{n-1}2\left|a_k\cos\frac{jk\pi}{n}\right| + |a_n\cos j\pi|$$

therefore the expression E_j in bracket in (9) can be estimated as

$$E_j \geqslant A_m + B - \sum_{k=1}^{n-1}2\left|a_k\cos\frac{jk\pi}{n}\right| - |a_n\cos j\pi| > 0.$$

Hence there is a zero α_j of the Chebyshev transform $\mathcal{T}P_{2n}$ in each interval $]x_j, x_{j+1}[$ $(j = 0, \dots, n-1)$ and by Lemma 1 of [3] the corresponding zeros of P_m are $e^{\pm iu_j}$ where $\alpha_j = 2\cos u_j$, $u_j \in [0, \pi]$ proving (7) and showing that the zeros are simple.

Assume now that *equality holds in* (5) i.e.

$$A_m + B = \sum_{k=1}^{m-1}|A_k + B - A_m|.$$

By $A_m + B > 0$ *there is a* $k_0 \in \{1, \dots, m-1\}$ *such that* $A_{k_0} + B - A_m \neq 0$. Let

$$A_{k_0}^{[l]} = A_{k_0} - \frac{1}{l} \operatorname{sgn}(A_{k_0} + B - A_m),$$
$$A_{m-k_0}^{[l]} = A_{m-k_0} - \frac{1}{l} \operatorname{sgn}(A_{m-k_0} + B - A_m).$$

For $l \in \mathbb{N}$ large enough, say $l \geqslant l_0$, we have

$$|A_{k_0}^{[l]} + B - A_m| = |A_{k_0} + B - A_m| - \frac{1}{l} < |A_{k_0} + B - A_m|,$$
$$|A_{m-k_0}^{[l]} + B - A_m| = |A_{m-k_0} + B - A_m| - \frac{1}{l} < |A_{m-k_0} + B - A_m|.$$

Denoting by $P_m^{[l]}$ $(l \geqslant l_0)$ the polynomial obtained from P_m by replacing its coefficients A_{k_0}, A_{m-k_0} by $A_{k_0}^{[l]}, A_{m-k_0}^{[l]}$ respectively the condition (5) holds with strict inequality for $P_m^{[l]}$ thus all of its zeros are on the unit circle. On the other hand the zeros of $P_m^{[l]}$ tend to the zeros of P_m as $l \to \infty$ (see [5] Theorem (1,4)) we conclude that the zeros of the latter are on the unit circle too. In this case in (7) some inequality signs may have to be replaced by equality signs and P_m may have multiple zeros.

Case 2. $m = 2n+1$ is odd. Let

$$x_j = 2\cos y_j \text{ with } y_j = \frac{2j\pi}{m} = \frac{2j\pi}{2n+1} \quad (j = 0, \dots, n)$$

then for all $j = 0, \dots, n$

$$U_n\left(\frac{x_j}{2}\right) = \frac{\sin(n+1)y_j}{\sin y_j} = \frac{\sin\left(\frac{2n+1}{2}y_j + \frac{1}{2}y_j\right)}{2\sin\frac{1}{2}y_j\cos\frac{1}{2}y_j} = \frac{1}{2}\left(\frac{\sin\frac{2n+1}{2}y_j}{\sin\frac{1}{2}y_j} + \frac{\cos\frac{2n+1}{2}y_j}{\cos\frac{1}{2}y_j}\right)$$
$$= \frac{1}{2}\left(\frac{\sin j\pi}{\sin\frac{j\pi}{2n+1}} + \frac{\cos j\pi}{\cos\frac{j\pi}{2n+1}}\right) = \frac{1}{2}\left((2n+1)\delta_{j0} + \frac{(-1)^j}{\cos\frac{j\pi}{2n+1}}\right),$$
$$U_{n-k}\left(\frac{x_j}{2}\right) - U_{n-k-1}\left(\frac{x_j}{2}\right) = \frac{\sin(n-k+1)y_j}{\sin y_j} - \frac{\sin(n-k)y_j}{\sin y_j} = \frac{\cos\frac{2(n-k)+1}{2}y_j}{\cos\frac{1}{2}y_j}$$
$$= \frac{\cos\left(j\pi - \frac{2kj\pi}{2n+1}\right)}{\cos\frac{j\pi}{2n+1}} = \frac{(-1)^j\cos\frac{2kj\pi}{2n+1}}{\cos\frac{j\pi}{2n+1}}.$$

Thus, for $j = 0, \dots, n$ we have

$$\mathcal{T}\tilde{P}_{2n}(x_j) = \frac{(-1)^j}{2\cos\frac{j\pi}{2n+1}}\left[(2n+1)(A_m - B)\delta_{j0} + (A_m + B) + \sum_{k=1}^{n} 2a_k\cos\frac{2jk\pi}{2n+1}\right]. \tag{11}$$

Assume that *strict inequality holds in* (5). We claim that

$$\operatorname{sgn}(\mathcal{T}\tilde{P}_{2n}(x_j)) = (-1)^j \quad (j = 0, 1, \dots, n). \tag{12}$$

To justify this observe that by (5)

$$A_m + B > \sum_{k=1}^{n} 2|a_k| \geqslant \sum_{k=1}^{n} 2\left|a_k \cos\frac{2jk\pi}{2n+1}\right|.$$

therefore the expression $\tilde{E}_j$ in bracket in (11) can be estimated as

$$\tilde{E}_j \geqslant A_m + B - \sum_{k=1}^{n} 2\left|a_k \cos\frac{2jk\pi}{2n+1}\right| > 0.$$

Hence there is a zero α_j of the Chebyshev transform $\mathcal{T}\tilde{P}_{2n}$ in each interval $]x_j, x_{j+1}[$ $(j = 0, \dots, n-1)$ and by Lemma 1 of [3] the corresponding zeros of $\tilde{P}_{2n}$ are $e^{\pm iu_j}$ where $\alpha_j = 2\cos u_j$, $u_j \in [0, \pi]$ proving (8) and showing that the zeros are simple.

Assume now that *equality holds in* (5). We can complete the proof analogously to the case 1, by obtaining the zeros of $\tilde{P}_{2n}$ as limits $l \to \infty$ of zeros of a sequence of polynomials $\tilde{P}_{2n}^{[l]}$, where the (modified) coefficients of the corresponding sequence $P_{2n+1}^{[l]}$ satisfy (5) with strict inequality. Again in this case in (8) some inequality signs may have to be replaced by equality signs and P_m may have multiple zeros. □

3. Multiple zeros

From the proof of the Main Theorem it turns out that multiple zeros are possible only if in (5) equality stands.

In the sequel *suppose that equality holds in* (5), and for the sake of definiteness consider the case of *even* $m = 2n$. As we have seen in the proof, the zeros of P_{2n} are $e^{\pm iu_j}$ $(j = 1, \dots, n)$ where u_j are limits of $u_j^{[l]}$ as $l \to \infty$, $e^{\pm iu_j^{[l]}}$ $(j = 1, \dots, n)$ being the zeros of $P_{2n}^{[l]}$ satisfying

$$\frac{(j-1)\pi}{n} < u_j^{[l]} < \frac{j\pi}{n} \quad (j = 1, \dots, n; l \geqslant l_0).$$

Taking the limit we get

$$0 \leqslant u_1 \leqslant \frac{\pi}{n} \leqslant \cdots \leqslant \frac{(j-1)\pi}{n} \leqslant u_j \leqslant \frac{j\pi}{n} \leqslant \cdots \leqslant u_n \leqslant \pi. \tag{13}$$

For each $j = 1, \dots, n$ in the inequality $\frac{(j-1)\pi}{n} \leqslant u_j \leqslant \frac{j\pi}{n}$ only at one side can be equality, at the other side strict inequality must hold.

If $0 = u_1$ $(u_n = \pi)$ then by Lemma 1 of [3] $e^{i0} = 1 (e^{i\pi} = -1)$ is a double zero of P_{2n}.

If in two consecutive (double) inequalities of (13) we have equalities in the middle and inequalities at the sides, i.e. we have

$$\frac{(j-1)\pi}{n} < u_j = \frac{j\pi}{n} = u_{j+1} < \frac{(j+1)\pi}{n}$$

for some $j = 1, \dots, n-1$ then $2\cos\dfrac{j\pi}{n}$ is a double zero of $\mathcal{T}P_{2n}$ and by Lemma 1 of [3] $e^{\pm\frac{ij\pi}{n}}$ are double zeros of P_{2n}.

Thus *the multiple zeros of* P_{2n} *can be at most double zeros and these double zeros can only be the numbers* $e^{\pm\frac{ij\pi}{n}}$ $(j = 0, \dots, n)$.

Suppose that $A_{2n} > 0, A_{2n} \geqslant B \geqslant 0$ *and equality holds in* (5).

In order that $e^{\frac{ij\pi}{n}}$ *for* $j = 0, n$ *be a double zero of* P_{2n} *it is necessary and sufficient that* $A_{2n} + B = \sum_{k=1}^{2n} |a_k|$, $\mathcal{T}P_{2n}(2\cos\frac{j\pi}{n}) = 0$, *or by* (9)

$$\begin{cases} A_{2n} + B - \displaystyle\sum_{k=1}^{n-1} 2|a_k| - |a_n| = 0 \\ 2n(A_{2n} - B)\delta_{j0} + (A_{2n} + B) + \displaystyle\sum_{k=1}^{n-1} 2a_k \cos\frac{jk\pi}{n} + a_n \cos j\pi = 0 \end{cases} \tag{14}$$

hold.

In order that $e^{\frac{\pm ij\pi}{n}}$ *for* $j = 1, \dots, n-1$ *be double zeros of* P_{2n} *it is necessary and sufficient that* $A_{2n} + B = \sum_{k=1}^{2n} |a_k|$, $\mathcal{T}P_{2n}(2\cos\frac{j\pi}{n}) = 0$, $(\mathcal{T}P_{2n})'(2\cos\frac{j\pi}{n}) = 0$, *or by* (9)

$$\begin{cases} A_{2n} + B - \displaystyle\sum_{k=1}^{n-1} 2|a_k| - |a_n| = 0 \\ A_{2n} + B + \displaystyle\sum_{k=1}^{n-1} 2a_k \cos\frac{jk\pi}{n} + a_n \cos j\pi = 0 \\ \dfrac{(A_{2n} - B)n}{2\sin^2\frac{j\pi}{2n}} + \displaystyle\sum_{k=1}^{n-1} 2a_k(n-k)\frac{\sin\frac{kj\pi}{n}}{\sin\frac{j\pi}{n}} = 0 \end{cases} \tag{15}$$

hold.

Rewriting the condition $(\mathcal{T}P_{2n})'(2\cos\frac{j\pi}{n}) = 0$ as the third line of (15) requires some calculations.

First we differentiate $\mathcal{T}P_{2n}$ and use the identity $T_n' = nU_{n-1}$ to obtain

$$(\mathcal{T}P_{2n})'(x) = \frac{A_{2n} - B}{2}\left[U_n'\left(\frac{x}{2}\right) + U_{n-1}'\left(\frac{x}{2}\right)\right] + \sum_{k=0}^{n-1} a_k(n-k)U_{n-k-1}\left(\frac{x}{2}\right).$$

With $x = \cos\vartheta$ we have after some simplifications

$$U_n'(x) = \frac{d}{d\vartheta}\frac{\sin(n+1)\vartheta}{\sin\vartheta}\frac{d\vartheta}{dx} = \frac{n\sin(n+2)\vartheta - (n+2)\sin n\vartheta}{-2\sin^3\vartheta}$$

and

$$U_n'(x) + U_{n-1}'(x) = \frac{2n\sin\vartheta\,(\cos n\vartheta + \cos(n+1)\vartheta) - 2\sin n\vartheta(1+\cos\vartheta)}{-2\sin^3\vartheta}.$$

where for $\sin\vartheta = 0$ the right hand side is defined by its limit.

We get, after some trigonometric manipulations, that

$$U_n'\left(\cos\frac{j\pi}{n}\right) + U_{n-1}'\left(\cos\frac{j\pi}{n}\right) = \frac{(-1)^{j+1}n\left(1+\cos\frac{j\pi}{n}\right)}{\sin^2\frac{j\pi}{n}} = \frac{(-1)^{j+1}n}{2\sin^2\frac{j\pi}{2n}}$$

and finally

$$(\mathcal{T}P_{2n})'\left(2\cos\frac{j\pi}{n}\right) = (-1)^{j+1}\frac{1}{2}\left[\frac{(A_{2n}-B)n}{2\sin^2\frac{j\pi}{2n}} + \sum_{k=1}^{n-1}2a_k(n-k)\frac{\sin\frac{kj\pi}{n}}{\sin\frac{j\pi}{n}}\right]$$

the formula we used in (15).

From the systems of equations (14), (15) one can easily find conditions for the coefficients of P_{2n} which ensure that $e^{\pm\frac{ij\pi}{n}}$ are double zeros of it. The case of odd degree can be dealt with similarly.

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