Mathematical
Inequalities
& Applications
Volume 10, Number 4 (2007), 799–804

COMPLETE MONOTONICITY OF THE LOGARITHMIC MEAN

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(*communicated by K. Stolarsky*)

Abstract. In the paper, the logarithmic mean is proved to be completely monotonic and an open problem about the logarithmically complete monotonicity of the extended mean values is posed. As two remarks, some errors appeared in [7, 21, 34] are corrected.

1. Introduction

Recall [12, 35] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geqslant 0 \tag{1}$$

for $x \in I$ and $n \geqslant 0$. Recall [2] that if $f^{(k)}(x)$ for some nonnegative integer k is completely monotonic on an interval I, but $f^{(k-1)}(x)$ is not completely monotonic on I, then $f(x)$ is called a completely monotonic function of k-th order on an interval I. Recall also [19, 22, 24] that a function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k [\ln f(x)]^{(k)} \geqslant 0 \tag{2}$$

for $k \in \mathbb{N}$ on I. It has been proved in [3, 19, 22] and other references that a logarithmically completely monotonic function on an interval I is also completely monotonic on I. The logarithmically completely monotonic functions have close relationships with both the completely monotonic functions and Stieltjes transforms. For detailed information, please refer to [3, 12, 25, 35] and the references therein.

For two positive numbers a and b, the logarithmic mean $L(a,b)$ is defined [33] by

$$L(a,b) = \begin{cases} \dfrac{b-a}{\ln b - \ln a}, & a \neq b; \\ a, & a = b. \end{cases} \tag{3}$$

Mathematics subject classification (2000): 26A48, 26A51.

Key words and phrases: Completely monotonic function, logarithmically completely monotonic function, logarithmic mean, Stolarsky's mean values, extended mean values.

The second author was partially supported by NSF (with Grant Number 2007110004) of the Department of Education of Henan Province, China.

This is one of the most important means of two positive variables. See [4, 8, 13, 18, 20, 21] and the list of references therein. This mean has been researched or mentioned on at least 13 pages in [4]. See the index "Logarithmic mean" on [4, p. 532]. However, no any complete monotonicity on mean values is founded in the authoritative book [4].

In 2005, the logarithmically complete monotonicity for ratios of mean values was obtained in [6].

The main aim of this paper is to prove the complete monotonicity of the logarithmic mean L. Our main result is as follows.

THEOREM 1. *The logarithmic mean*

$$L_{s,t}(x) = L(x+s, x+t) \tag{4}$$

is a completely monotonic function of first order in $x > -\min\{s,t\}$ *for* $s,t \in \mathbb{R}$ *with* $s \neq t$.

As by-product of the proof of Theorem 1, the following logarithmically completely monotonic property of the function

$$g_{s,t;u}(x) = (x+s)^{1-u}(x+t)^u \tag{5}$$

for $s,t \in \mathbb{R}$ with $s \neq t$ and $u \in (0,1)$ is deduced.

COROLLARY 1. *Let* $s,t \in \mathbb{R}$ *with* $s \neq t$ *and* $u \in (0,1)$. *Then the function* $g_{s,t;u}(x)$ *defined by* (5) *is a completely monotonic function of first order in* $x > -\min\{s,t\}$. *More strongly, the function*

$$\frac{\partial g_{s,t;u}(x)}{\partial x} = \left(\frac{x+t}{x+s}\right)^u \left[1 + \frac{u(s-t)}{x+t}\right] \tag{6}$$

is logarithmically completely monotonic in $x > -\min\{s,t\}$.

REMARK 1. The Stolarsky's mean values or extended mean values $E(r,s;x,y)$ was defined first in [33] by

$$\begin{aligned}
E(r,s;x,y) &= \left(\frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r}\right)^{1/(s-r)}, && rs(r-s)(x-y) \neq 0;\\
E(r,0;x,y) &= \left(\frac{1}{r} \cdot \frac{y^r - x^r}{\ln y - \ln x}\right)^{1/r}, && r(x-y) \neq 0;\\
E(r,r;x,y) &= \frac{1}{e^{1/r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r - y^r)}, && r(x-y) \neq 0;\\
E(0,0;x,y) &= \sqrt{xy}, && x \neq y;\\
E(r,s;x,x) &= x, && x = y;
\end{aligned}$$

where x and y are positive numbers and $r,s \in \mathbb{R}$. Its monotonicity, Schur-convexity, logarithmic convexity, comparison, generalizations, applications and history have been investigated in many articles such as [4, 5, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 23, 26, 27, 28, 29, 30, 32, 36, 37, 38] and the references therein, especially the book [4] and the expository paper [18].

For $x,y>0$ and $r,s\in\mathbb{R}$, let

$$E^{[1]}_{r,s;x,y}(w)=E(r+w,s+w;x,y) \tag{7}$$

with $w\in\mathbb{R}$, and let

$$E^{[2]}_{r,s;x,y}(w)=E(r,s;x+w,y+w) \tag{8}$$

and

$$E^{[3]}_{r,s;x,y}(w)=E(r+w,s+w;x+w,y+w) \tag{9}$$

with $w>-\min\{x,y\}$. Motivated by Theorem 1, it is natural to pose an open problem: What about the (logarithmically) complete monotonicity of the functions $E^{[i]}_{r,s;x,y}(w)$ in w for $1\leqslant i\leqslant 3$?

REMARK 2. In [31] it was conjectured that the function

$$\left(\frac{\frac{1}{n}\sum_{i=1}^{n}i^r}{\frac{1}{n+1}\sum_{i=1}^{n+1}i^r}\right)^{1/r} \tag{10}$$

is decreasing with $r\in(-\infty,\infty)$. It is worthwhile to point out that this conjecture was mistakenly claimed to be verified in [34], since there exists a fatal and simple error in the proof of Lemma 3 in [34]. This corrects the *Addendum* in [21, p. 562]. Up to now, this conjecture is still keep open, to the best of our knowledge.

REMARK 3. In a private e-mail on 20 August 2007, Dr. A. Witkowski pointed out that the necessary condition in Lemma 1 cited in [21, p. 561] from [7, p. 395] is not valid, his counterexample is

$$\phi(x)=x^2+x\sin x.$$

Consequently, Lemma 1 in [21, p. 561] should be restated as follows: *Let the second derivative of $\phi(x)$ be continuous on $(-\infty,\infty)$ with $\phi(0)=0$ and let*

$$g(x)=\begin{cases}\dfrac{\phi(x)}{x}, & x\neq 0;\\ \phi'(0), & x=0.\end{cases} \tag{11}$$

If $\phi(x)$ is (strictly) convex, then $g(x)$ is (strictly) increasing with $x\in(-\infty,\infty)$. Luckily, this does not influence the correctness of the main results in [21], since the necessary condition of Lemma 1 in [21, p. 561] has no use in [21].

2. Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1. In [4, p. 386], an integral representation of the logarithmic mean $L(a,b)$ for positive numbers a and b is given:

$$L(a,b)=\int_0^1 a^{1-u}b^u\,\mathrm{d}u. \tag{12}$$

From this, it follows easily that

$$L_{s,t}(x) = \int_0^1 g_{s,t;u}(x)\,\mathrm{d}u \tag{13}$$

and

$$\frac{\mathrm{d}L_{s,t}(x)}{\mathrm{d}x} = \int_0^1 \left(\frac{x+t}{x+s}\right)^u \frac{x+(1-u)t+us}{x+t}\,\mathrm{d}u > 0. \tag{14}$$

This means that the function $L_{s,t}(x)$ is increasing, and then it is not completely monotonic in $x > -\min\{s,t\}$.

In [1, p. 230, 5.1.32], it is listed that

$$\ln\frac{b}{a} = \int_0^\infty \frac{e^{-au}-e^{-bu}}{u}\,\mathrm{d}u. \tag{15}$$

Taking the logarithm of the integrand in (14) and utilizing (15) yields

$$\begin{aligned}
\ln\frac{\partial g_{s,t;u}(x)}{\partial x} &= u\ln\frac{x+t}{x+s} + \ln\frac{x+(1-u)t+us}{x+t}\\
&= u\int_0^\infty \frac{e^{-(x+s)v}-e^{-(x+t)v}}{v}\,\mathrm{d}v + \int_0^\infty \frac{e^{-(x+t)v}-e^{-[x+(1-u)t+us]v}}{v}\,\mathrm{d}v\\
&= \int_0^\infty \frac{ue^{-(x+s)v}+(1-u)e^{-(x+t)v}-e^{-[x+(1-u)t+us]v}}{v}\,\mathrm{d}v.
\end{aligned}$$

Employing the well known Jensen's inequality [4, p. 31, Theorem 12] for convex functions and considering that the function e^{-x} is convex gives

$$q_{s,t;u;v}(x) \triangleq ue^{-(x+s)v}+(1-u)e^{-(x+t)v}-e^{-[x+(1-u)t+us]v} > 0. \tag{16}$$

Hence, for positive integers $m \in \mathbb{N}$,

$$(-1)^m\frac{\partial^m}{\partial x^m}\left\{\ln\frac{\partial g_{s,t;u}(x)}{\partial x}\right\} = \int_0^\infty v^{m-1}q_{s,t;u;v}(x)\,\mathrm{d}v > 0. \tag{17}$$

This implies that the function $\frac{\partial g_{s,t;u}(x)}{\partial x}$ is logarithmically completely monotonic in $x > -\min\{s,t\}$. Further, since a logarithmically completely monotonic function is also completely monotonic (see [3, 12, 19, 22, 24, 25] and the references therein), the function $\frac{\partial g_{s,t;u}(x)}{\partial x}$ is completely monotonic in $x > -\min\{s,t\}$. Therefore, the function

$$\frac{\mathrm{d}L_{s,t}(x)}{\mathrm{d}x} = \int_0^1 \frac{\partial g_{s,t;u}(x)}{\partial x}\,\mathrm{d}u \tag{18}$$

is completely monotonic in $x > -\min\{s,t\}$. Theorem 1 is proved. □

Proof of Corollary 1. This follows from the proof of Theorem 1 directly. □

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(Received May 5, 2007)

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