Mathematical
Inequalities
& Applications
Volume 10, Number 4 (2007), 805–809

SOME INEQUALITIES FOR UNIFORMLY LOCALLY UNIVALENT FUNCTIONS ON THE UNIT DISK

YONG CHAN KIM

(*communicated by H. Srivastava*)

Abstract. In this paper, we will consider some inequalities for a certain subclass $B(\lambda)$ of uniformly locally univalent holomorphic functions on the unit disk in terms of the norm of pre-Schwarzian derivative. We also investigate the relationships between the class $B(\lambda)$ and the Hardy space.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. The Hardy space $\mathscr{H}^p$ $(0 < p \leqslant \infty)$ consists of all functions holomorphic in $\mathbb{D}$ for which

$$\|f\|_p := \lim_{r\to 1^-} M_p(r,f) < \infty,$$

where

$$M_p(r,f) = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{1/p} & (0 < p < \infty) \\ \max\limits_{|z|\leqslant r} |f(z)| & (p = \infty). \end{cases}$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be holomorphic in $\mathbb{D}$ and let β be a real number. Flett [5] (see also [9]) defines the fractional integral of f of order β as $I^\beta f(z) = \sum_{n=0}^{\infty}(n+1)^{-\beta} a_n z^n$. If $\beta > 0$, then

$$I^\beta f(z) = \frac{1}{\Gamma(\beta)}\int_0^1 \left(\log\frac{1}{t}\right)^{\beta-1} f(tz)\, dt.$$

The fractional derivative $D^\beta f$ of f of order $\beta > 0$ is defined as $D^\beta f = I^{-\beta} f$.

Using the definition of fractional integral $I^\beta f$ Jung *et al.* [7] proved that if $\beta > 1$ and

$$\mathrm{Re}\,\{f'(z)\} > 0 \quad (z \in \mathbb{D}),$$

Mathematics subject classification (2000): 30C45, 30A32.
Key words and phrases: Hardy space, pre-Schwarzian derivative, uniformly locally univalent function.

then $I^\beta f \in \mathscr{H}^\infty$. Also, the following theorem is one of the extended versions of the Hardy-Littlewood theorem (*cf.* [4, Theorem 5.12]) on fractional integral.

THEOREM A. [Kim [8]] *If* $f \in \mathscr{H}^p$ *and* $f(z) = O(1-|z|)^{-\gamma}$ *with* $0 < \gamma \leqslant 1/p$, *then* $I^\beta f \in \mathscr{H}^q$ *with* $q = \gamma p/(\gamma - \beta)$ *where* $0 < \beta < \gamma$.

For a function f holomorphic on $\mathbb{D}$, the Bloch seminorm is given by

$$\|f\|_{\mathscr{B}} = \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)|,$$

and f is called a *Bloch function* when $\|f\|_{\mathscr{B}} < \infty$.

We will say that a holomorphic function f on the unit disk $\mathbb{D}$ is *uniformly locally univalent* if f is univalent on each hyperbolic disk $D(a,\rho) = \{z \in \mathbb{D}; |\frac{z-a}{1-az}| < \tanh\rho\}$ with radius ρ and center $a \in \mathbb{D}$ for a positive constant ρ.

The hyperbolic norm of the pre-Schwarzian derivative $T_f = f''/f'$ of a locally univalent function f on $\mathbb{D}$ is defined by

$$\|T_f\| = \sup_{z\in\mathbb{D}}(1 - |z|^2)|T_f(z)|.$$

This quantity can be regarded as the Bloch semi-norm of the function $\log f'$. We remark that the hyperbolic sup norm $\|T_f\|$ measures the deviation of the function from similarities. Also, it is known (cf. [15]) that a holomorphic function f on the unit disk is uniformly locally univalent if and only if the pre-Schwarzian derivative T_f of f is hyperbolically bounded, i.e., the norm $\|T_f\|$ is finite.

Let $\mathscr{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are holomorphic in $\mathbb{D}$. Also let $\mathscr{S}$ denote the subclass of $\mathscr{A}$ consisting of functions which are univalent. In this note, we may assume that a holomorphic function f on the unit disk is normalized so that $f(0) = 0$ and $f'(0) = 1$, i.e., $f \in \mathscr{A}$.

Let B denote the set of normalized uniformly locally univalent functions:

$$B = \{f \in \mathscr{A} \ : \ \|T_f\| < \infty\}.$$

The space B has a structure of non-separable complex Banach space under the Hornich operation ([14]).

For a non-negative real number λ we set

$$B(\lambda) = \{f \in \mathscr{A} \ : \ \|T_f\| \leqslant 2\lambda\}.$$

It is known that

$$\|T_f\| < 2 \Longrightarrow f \in \mathscr{H}^\infty. \tag{1}$$

In particular, a function in $B(1)$ need not be bounded. See [10] for further information.

The following theorem is significant in connection with univalent function theory.

THEOREM B. [Becker and Pommerenke [1], [2]] *The set $\mathscr{S}$ of normalized univalent holomorphic functions on the unit disk is contained in $B(3)$ and contains $B(\frac{1}{2})$. The result is sharp.*

Since $\|T_f\|$ is not invariant under Möbius transformations, it is often advantageous to consider the quantity

$$U_f(z) = \frac{1}{2}(1-|z|^2)T_f(z) - \overline{z},$$

because this satisfies the relation $U_{f\circ\omega} = U_f \circ \omega \cdot \omega'/|\omega'|$ for $\omega \in \mathrm{Aut}(\mathbb{D})$. The quantity

$$\mathrm{ord}(f) = \sup_{z\in\mathbb{D}} |U_f(z)|$$

is called the *order of function f* and extensively investigated by Pommerenke [11]. We note that

$$\frac{1}{2}\|T_f\| - 1 \leqslant \mathrm{ord}(f) \leqslant \frac{1}{2}\|T_f\| + 1. \tag{2}$$

Finally, we introduce two differential operators defined by Bonk *et al.* [3] as follows:

Suppose f is holomorpic in $\mathbb{D}$. Then D_jf $(j=1,2)$ is defined by

$$D_1f(z) = (1-|z|^2)f'(z),$$

$$D_2f(z) = (1-|z|^2)^2 f''(z) - 2\overline{z}(1-|z|^2)f'(z).$$

These differential operators have the invariant property

$$|D_j(S\circ f\circ T)| = |D_jf| \circ T \quad (j=1,2),$$

where S is any euclidean motion of $\mathbb{C}$ and T is any conformal automorphism of $\mathbb{D}$.

We also note that

$$\|f\|_{\mathscr{B}} = \sup_{z\in\mathbb{D}} |D_1f(z)|$$

and

$$U_f(z) = \frac{D_2f(z)}{2D_1f(z)}. \tag{3}$$

2. Inequalities and relationships between the class $\mathscr{B}(\lambda)$ and the Hardy space

Let λ be a non-negative real number. For a function $f \in \mathscr{B}(\lambda)$, Kim and Sugawa [10, Theorem 2.3] proved that

$$\left(\frac{1-|z|}{1+|z|}\right)^{\lambda} \leqslant |f'(z)| \leqslant \left(\frac{1+|z|}{1-|z|}\right)^{\lambda}. \tag{4}$$

From this fact, we see that every function in the class $B(1)$ is Bloch. Furthermore, using (2), (3) and (4), we obtain

THEOREM 2.1. *If $f \in B(1)$, then*

(i) $\|f\|_{\mathscr{B}} \leqslant 4$,

(ii) $\mathrm{ord}(f) \leqslant 2$,

(iii) $\sup_{z\in\mathbb{D}} |D_2f(z)| \leqslant 16$.

Next we consider relationships between the class $B(\lambda)$ and the Hardy space.

THEOREM C. [Pommerenke [13]] *Let β be a constant with $0 \leqslant \beta \leqslant 2$. If a univalent function $f \in \mathcal{S}$ satisfies that $f(z) = O(1-|z|)^{-\beta}$ as $|z| \to 1$, then the following holds.*

For $0 < p < \frac{1}{\beta}$, we have $f \in \mathcal{H}^p$. For $\frac{1}{\beta} < p$, we have

$$M_p(r,f) = O(1-r)^{1/p-\beta}.$$

Combining Theorem C with [10, Corollary 2.4], we have

THEOREM 2.2. *Let $f \in \mathcal{S}$ and let $\|T_f\| = 2\lambda$.*
If $\lambda < 1$, then $f \in \mathcal{H}^\infty$.
If $\lambda > 1$, then $f \in \mathcal{H}^p$ for any $0 < p < 1/(\lambda - 1)$.
If $\lambda = 1$, then $f \in BMOA$.

Proof. If $f \in B(\lambda)$ for $\lambda < 1$, from [10, Corollary 2.4] we see that f is bounded. If $\lambda > 1$ and $f \in \mathcal{S} \bigcap B(\lambda)$, Theorem B implies that $1 < \lambda \leqslant 3$. Since $f(z) = O(1-|z|)^{1-\lambda}$, it follows from Theorem C that

$$f \in \mathcal{H}^p \quad (0 < p < 1/(\lambda - 1)).$$

Finally, if $f \in B(1)$, then f is a Bloch function. Since $f \in \mathcal{S}$, $f \in BMOA$ if and only if f is Bloch (see [12]). $\square$

Note that $\mathcal{H}^\infty \subset BMOA \subset \bigcap_{0<p<\infty} \mathcal{H}^p$.

REMARK. Most of the above results can be extended to the case of p-valent, more generally, mean p-valent function with $p < \infty$ (see Hayman [6]).

Finally, using Theorem A we have

THEOREM 2.3. *Let $1 < \lambda \leqslant 1/p$. If $f \in B(\lambda)$ and $f' \in \mathcal{H}^p$, then*

$$f \in \mathcal{H}^q \qquad (q = \frac{\lambda p}{\lambda - 1}).$$

Proof. If $f \in B(\lambda)$, then $f'(z) = O(1-|z|)^{-\lambda}$. Since $f' \in \mathcal{H}^p$ for $1 < \lambda \leqslant 1/p$, it follows from Theorem A that

$$zI^1 f'(z) = \int_0^z f'(\zeta)\, d\zeta = f(z) \in \mathcal{H}^q \quad (q = \frac{\lambda p}{\lambda - 1}). \qquad \square$$

REMARK. (1) If f is Bloch and $f' \in \mathcal{H}^p \quad (0 < p < 1)$, then it is known ([8, Remark 2.6]) that

$$f \in \bigcap_{0<p<\infty} \mathcal{H}^p \tag{5}$$

So, in particular, (5) holds if $f \in B(1)$ and $f' \in \mathcal{H}^p \quad (0 < p < 1)$.

(2) If $f' \in \mathscr{H}^p$ for $0 < p < 1$, by the Hardy-Littlewood Theorem, it is well known that

$$f \in \mathscr{H}^q \qquad (q = \frac{p}{1-p}).$$

In theorem 2.3, using the condition $1 < \lambda \leqslant 1/p$ we get

$$\lambda p(1-p) = \lambda p - \lambda p^2 \geqslant \lambda p - p = p(\lambda - 1).$$

Hence we have

$$\frac{p}{1-p} \leqslant \frac{\lambda p}{\lambda - 1}.$$

REFERENCES

[1] J. BECKER, *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math. **255** (1972), 23–43.
[2] J. BECKER AND CH. POMMERENKE, *Schlichtheitskriterien und Jordangebiete*, J. Reine Angew. Math. **354** (1984), 74–94.
[3] M. BONK, D. MINDA AND H. YANAGIHARA, *Distortion theorems for Bloch functions*, Parcific J. Math. 179 (1997), 241–262.
[4] P. L. DUREN, *Theory of H^p Spaces*, Academic Press, New York and London, 1970.
[5] T. M. FLETT, *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl. 38 (1972), 746–765.
[6] W. K. HAYMAN, *Multivalent Functions, Second edition*, Cambridge University Press, London, 1994.
[7] I. B. JUNG, Y. C. KIM AND H. M. SRIVASTAVA, *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl. 176 (1993), 138–147.
[8] H. O. KIM, *Derivatives of Blaschke products*, Parcific J. Math. 114 (1984), 175–190.
[9] Y. C. KIM, S. H. LEE AND H. M. SRIVASTAVA, *Some properties of convolution operators in the class $\mathscr{P}_\alpha(\beta)$*, J. Math. Anal. Appl. 187 (1994), 498–512.
[10] Y. C. KIM AND S. SUGAWA, *Growth and coefficient estimates for uniformly locally univalent functions on the unit disk*, Rocky Mountain J. Math. 32 (2002), 179–200.
[11] CH. POMMERENKE, *Linear-invariante Familien analytischer Funktionen* I, II, Math. Ann. **155** (1964), 108–154, ibid. **156**(1964), 226–262.
[12] CH. POMMERENKE, *Schlichite Funktionen und analytische Funktionen von beschränkten mittler Oszillation*, Comment. Math. Helv. **52** (1977), 591–602.
[13] CH. POMMERENKE, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, 1992.
[14] S. YAMASHITA, *Banach spaces of locally schlicht functions with the Hornich operations*, Manuscripta Math. **16** (1975), 261–275.
[15] ______, *Almost locally univalent functions*, Monatsh. Math. **81** (1976), 235–240.
[16] ______, *Lectures on Locally Schlicht Functions*, Mathematical Seminar Notes, Tokyo Metropolitan Univ., 1977.

(Received May 4, 2006)

Yong Chan Kim
Department of Mathematics
College of Education
Yeungnam University
214-1 Daedong
Gyongsan 712-749
Korea
e-mail: kimyc@yu.ac.kr