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SOME NEW RETARDED VOLTERRA–FREDHOLM TYPE INTEGRAL INEQUALITIES WITH POWER NONLINEAR AND THEIR APPLICATIONS

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(*communicated by V. Lakshmikantham*)

Abstract. Some new explicit bounds on solutions to a class of new nonlinear retarded Volterra-Fredholm type integral inequalities are established, which can be used as effective tools in the study of certain integral equations. Applications examples are also indicated.

1. Introduction

In the study of ordinary differential equations and integral equations one often deal with certain integral inequalities. The Gronwall-Bellman inequality and its various linear and nonlinear generalizations are crucial in the discussion of in the existence, uniqueness, continuation, boundedness, oscillation and stability and other qualitative properties of solutions of differential and integral equations. The literature on such inequalities and their applications is vast; see [1, 8, 16, 20] and the references given therein.

To handle ordinary differential and integral equations with retardation, some delay Volterra-type integral inequalities are needed. During the past few years, some investigators have established some useful and interesting delay Volterra-type integral inequalities in order to achieve various goals; see [2, 4, 6, 7, 9-14, 22] and the references cited therein. Recently, in [21], Pachpatte has established the following one useful linear Volterra-Fredholm type integral inequality with retardation:

THEOREM 1.1. ([21]) *Let* $u(t), f(t) \in C(I, R_+)$, $a(t,s), b(t,s)$, $c(t,s) \in C(D, R_+)$, $a(t,s)$, $b(t,s)$ *are nondecreasing in* t *for each* $s \in I$, $h(t) \in C^1(I,I)$ *be nondecreasing with* $h(t) \leqslant t$ *on* $I, k \geqslant 0$ *be a constant, where* $I = [\alpha, \beta], R_+ = [0, \infty)$, $D = \{(t,s) \in I^2 : \alpha \leqslant s \leqslant t \leqslant \beta\}$ *and suppose that*

$$u(t) \leqslant k + \int_{h(\alpha)}^{h(t)} a(t,s)\left[f(s)u(s) + \int_{h(\alpha)}^{s} c(s,\sigma)u(\sigma)d\sigma\right]ds + \int_{h(\alpha)}^{h(\beta)} b(t,s)u(s)ds \quad (1.1),$$

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for $t \in I$. *If*

$$p(t) = \int_{h(\alpha)}^{h(\beta)} b(t,s)\exp(A(s))ds < 1,$$

for $t \in I$, *where*

$$A(t) = \int_{h(\alpha)}^{h(t)} a(t,\xi)\left[f(\xi) + \int_{h(\alpha)}^{\xi} c(\xi,\sigma)d\sigma\right]d\xi,$$

for $t \in I$, *then*

$$u(t) \leqslant \frac{k}{1-p(t)}\exp(A(t))$$

for $t \in I$.

When $a(t,s)f(s) = a(s)$, $c(t,s) = 0$, $b(t,s) = b(s)$, and $h(t) = t$ the result in Theorem will deduce the conclusion appeared in [1].

In this paper, we consider the explicit bounds on some general versions of (1.1) which the constant k on the right side of (1.1) is replaced by the function $l(t)$ and contain some power nonlinear terms respect to the unknown function $u(t)$ on the both side of (1.1). Our results can be used as handy and effective tools in the study the qualitative behavior of the solutions of certain retarded Volterra-Fredholm type integral equations. For illustrate this, some application examples are given. Our results also generalize some results in [7].

2. Retarded integral inequalities with power nonlinear

In what follows, R denotes the set of real numbers, $R_+ = [0,+\infty)$, $I = [t_0,T]$; $C^i(M,S)$ denotes the class of all i-times continuously differentiable functions defined on set M with range in the set $S(i = 1,2,\cdots)$ and $C^0(M,S) = C(M,S)$.

Before giving our main results, we need the following important lemma in the discussion of our proof.

LEMMA 2.1. ([6]) *Let* $a \geqslant 0, p \geqslant q \geqslant 0$ *and* $p \neq 0$, *then*

$$a^{\frac{q}{p}} \leqslant \frac{q}{p}K^{\frac{q-p}{p}}a + \frac{p-q}{p}K^{\frac{q}{p}}$$

for any $K > 0$.

THEOREM 2.1. *Let* $u(t)$, $l(t)$ *and* $f(t) \in C(I,R_+)$, $a(t,s)$, $b(t,s)$ *and* $c(t,s) \in C(D,R_+)$, $a(t,s)$, $b(t,s)$ *and* $c(t,s)$ *be nondecreasing in* t *for each* $s \in I$, $\alpha(t) \in C^1(I,I)$ *be nondecreasing with* $\alpha(t) \leqslant t$ *on* I, *where* $D = \{(t,s) \in I^2 : t_0 \leqslant s \leqslant t \leqslant T\}$. *If* $u(t)$ *satisfies*

$$u^p(t) \leqslant l(t) + \int_{\alpha(t_0)}^{\alpha(t)} a(t,s)\left[f(s)u^q(s)ds + \int_{\alpha(t_0)}^{s} b(s,\tau)u^r(\tau)d\tau\right]ds + \int_{\alpha(t_0)}^{\alpha(T)} c(t,s)u^n(s)ds \tag{2.1}$$

for $t \in I$, *where* $p \geqslant q \geqslant 0, p \geqslant r \geqslant 0, p \geqslant n \geqslant 0, p, q, r$ *and* n *are constants and*

$$\lambda_{rn}^{pq}(t) = \frac{n}{p} K^{\frac{n-p}{p}} \int_{\alpha(t_0)}^{\alpha(T)} c(t,s) \exp(A_{pqr}(s)) ds < 1, \tag{2.2}$$

then

$$u(t) \leqslant \left[l(t) + \frac{\bar{A}_{pqr}(t) + C_{pn}(t)}{1 - \lambda_{pqr}(t)} \exp\left(A_{pqr}(t)\right) \right]^{\frac{1}{p}} \tag{2.3}$$

for $t \in I$ *and any* $K_i > 0 \ \ (i = 1, 2, 3)$, *where*

$$A_{pqr}(t) = \int_{\alpha(t_0)}^{\alpha(t)} a(t,s) \left[\frac{q}{p} K_1^{\frac{q-p}{p}} f(s) + \frac{r}{p} K_2^{\frac{r-p}{p}} \int_{\alpha(t_0)}^{s} b(s,\tau) d\tau \right] ds, \tag{2.4}$$

$$\begin{aligned} \bar{A}_{pqr}(t) = \int_{\alpha(t_0)}^{\alpha(t)} a(t,s) \Bigg[& f(s) \left(\frac{q}{p} K_1^{\frac{q-p}{p}} l(s) + \frac{p-q}{p} K_1^{\frac{q}{p}} \right) \\ & + \int_{\alpha(t_0)}^{s} b(s,\tau) \left(\frac{r}{p} K_2^{\frac{r-p}{p}} l(\tau) + \frac{p-r}{p} K_2^{\frac{r}{p}} \right) d\tau \Bigg] ds \end{aligned} \tag{2.5}$$

and

$$C_{pn}(t) = \int_{\alpha(t_0)}^{\alpha(T)} c(t,s) \left[\frac{n}{p} K_3^{\frac{n-p}{p}} l(s) + \frac{p-n}{p} K_3^{\frac{n}{p}} \right] ds \tag{2.6}$$

for $t \in I$.

Proof. Define a function $v(t)$ by

$$v(t) = \int_{\alpha(t_0)}^{\alpha(t)} a(t,s) \left[f(s) u^q(s) ds + \int_{\alpha(t_0)}^{s} b(s,\tau) u^r(\tau) d\tau \right] ds + \int_{\alpha(t_0)}^{\alpha(T)} c(t,s) u^n(s) ds, \tag{2.7}$$

then

$$u^p(t) \leqslant l(t) + v(t),$$

or

$$u(t) \leqslant (l(t) + v(t))^{\frac{1}{p}}. \tag{2.8}$$

By Lemma 2.1 and (2.8), for any $K_i > 0 \ \ (i = 1, 2, 3)$, we have

$$u^q(t) \leqslant (l(t) + v(t))^{\frac{q}{p}} \leqslant \frac{q}{p} K_1^{\frac{q-p}{p}} (l(t) + v(t)) + \frac{p-q}{p} K_1^{\frac{q}{p}},$$
$$u^r(t) \leqslant (l(t) + v(t))^{\frac{r}{p}} \leqslant \frac{r}{p} K_2^{\frac{r-p}{p}} (l(t) + v(t)) + \frac{p-r}{p} K_2^{\frac{r}{p}}.$$

And

$$u^n(t) \leqslant (l(t) + v(t))^{\frac{n}{p}} \leqslant \frac{n}{p} K_3^{\frac{n-p}{p}} (l(t) + v(t)) + \frac{p-n}{p} K_3^{\frac{n}{p}}.$$

Substituting the last relations into (2.7) we get

$$
\begin{aligned}
v(t) \leqslant \int_{\alpha(t_0)}^{\alpha(t)} a(t,s) \Bigg[f(s) \left(\frac{q}{p} K_1^{\frac{q-p}{p}} (l(s)+v(s)) + \frac{p-q}{p} K_1^{\frac{q}{p}} \right) \\
+ \int_{\alpha(t_0)}^{s} b(s,\tau) \left(\frac{r}{p} K_2^{\frac{r-p}{p}} (l(\tau)+v(\tau)) + \frac{p-r}{p} K_2^{\frac{r}{p}} \right) d\tau \Bigg] ds \\
+ \int_{\alpha(t_0)}^{\alpha(T)} c(t,s) \left[\frac{n}{p} K_3^{\frac{n-p}{p}} (l(s)+v(s)) + \frac{p-n}{p} K_3^{\frac{n}{p}} \right] ds \\
\leqslant \bar{A}_{pqr}(t) + C_{pn}(t) + \int_{\alpha(t_0)}^{\alpha(t)} a(t,s) \left[\frac{q}{p} K_1^{\frac{q-p}{p}} f(s)v(s) + \frac{r}{p} K_2^{\frac{r-p}{p}} \int_{\alpha(t_0)}^{s} b(s,\tau)v(\tau)d\tau \right] ds \\
+ \frac{n}{p} K_3^{\frac{n-p}{p}} \int_{\alpha(t_0)}^{\alpha(T)} c(t,s)v(s)ds,
\end{aligned}
\tag{2.9}
$$

where $\bar{A}_{pqr}(t)$ and $C_{pn}(t)$ are defined as in (2.5) and (2.6) respectively. It is easy to see that $\bar{A}_{pqr}(t)$ and $C_{nr}(t)$ are nonnegative, continuous and nondecreasing for $t \in I$.

From the assumptions, we observe that $\alpha'(t) \geqslant 0$ for $t \in I$. Fixing $T' \in I$, then for $t_0 \leqslant t \leqslant T'$, from (2.9) we have

$$
\begin{aligned}
v(t) \leqslant \bar{A}_{pqr}(T') + C_{pn}(T') + \int_{\alpha(t_0)}^{\alpha(t)} a(T',s) \Bigg[\frac{q}{p} K_1^{\frac{q-p}{p}} f(s)v(s) \\
+ \frac{r}{p} K_2^{\frac{r-p}{p}} \int_{\alpha(t_0)}^{s} b(s,\tau)v(\tau)d\tau \Bigg] ds + \frac{n}{p} K_3^{\frac{n-p}{p}} \int_{\alpha(t_0)}^{\alpha(T)} c(T',s)v(s)ds.
\end{aligned}
\tag{2.10}
$$

Define a function $w(t), t \in [t_0, T']$ by the right hand side of (2.10). Then for $t \in [t_0.T'], w(t)$ is positive and nondecreasing in t,

$$
v(t) \leqslant w(t), \tag{2.11}
$$

$$
w(t_0) = \bar{A}_{pqr}(T') + C_{pn}(T') + \frac{n}{p} K_3^{\frac{n-p}{p}} \int_{\alpha(t_0)}^{\alpha(T)} c(T',s)v(s)ds \tag{2.12}
$$

and

$$
\begin{aligned}
w'(t) = a(T',\alpha(t)) \left[\frac{q}{p} K_1^{\frac{q-p}{p}} f(\alpha(t))v(\alpha(t)) + \frac{r}{p} K_2^{\frac{r-p}{p}} \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(s),\tau)v(\tau)d\tau \right] \alpha'(t) \\
\leqslant a(T',\alpha(t)) \left[\frac{q}{p} K_1^{\frac{q-p}{p}} f(\alpha(t))w(\alpha(t)) + \frac{r}{p} K_2^{\frac{r-p}{p}} \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t),\tau)w(\tau)d\tau \right] \alpha'(t).
\end{aligned}
\tag{2.13}
$$

From the last inequality we easily observe that

$$
\frac{w'(t)}{w(t)} \leqslant a(T',\alpha(t)) \left[\frac{q}{p} K_1^{\frac{q-p}{p}} f(\alpha(t)) + \frac{r}{p} K_2^{\frac{r-p}{p}} \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t),\tau)d\tau \right] \alpha'(t). \tag{2.14}
$$

By setting $t = s$ in (2.14) and integrating it with respect to s from t_0 to T' we obtain

$$w(T') \leqslant w(t_0)\exp\left(\int_{t_0}^{T'} a(T',\alpha(s))\left[\frac{q}{p}K_1^{\frac{q-p}{p}} f(\alpha(s)) + \frac{r}{p}K_2^{\frac{r-p}{p}}\int_{\alpha(t_0)}^{\alpha(s)} b(\alpha(s),\tau)d\tau\right]\alpha'(s)\right)ds. \tag{2.15}$$

Since T' is arbitrary, from (2.15) and (2.12) it has proved that

$$w(t) \leqslant w(t_0)\exp\left(\int_{t_0}^{t} a(t,\alpha(s))\left[\frac{q}{p}K_1^{\frac{q-p}{p}} f(\alpha(s)) + \frac{r}{p}K_2^{\frac{r-p}{p}}\int_{\alpha(t_0)}^{\alpha(s)} b(\alpha(s),\tau)d\tau\right]\alpha'(s)ds\right) \tag{2.16}$$

and

$$w(t_0) = \bar{A}_{pqr}(t) + C_{pn}(t) + \frac{n}{p}K_3^{\frac{n-p}{p}}\int_{\alpha(t_0)}^{\alpha(T)} c(t,s)v(s)ds. \tag{2.17}$$

By making the change of variable on the right hand of (2.16) and using $v(t) \leqslant w(t)$ for $t \in I$ we get

$$v(t) \leqslant w(t_0)\exp\left(A_{pqr}(t)\right) \tag{2.18}$$

for $t \in I$. Substituting (2.18) into (2.17) and combing with condition (2.2), it is easy to observe that

$$w(t_0) \leqslant \frac{\bar{A}_{pqr}(t) + C_{pn}(t)}{1 - \lambda_{rn}^{pq}(t)}. \tag{2.19}$$

Now the desired inequality in (2.3) follows by using (2.19) in (2.18) and combing with (2.8). □

When $p = 2$, $q = r = n = 1$ in Theorem 2.1 we get a Volterra-Fredholm-Ou-Iang type inequality as follows. About Ou-Iang type inequalities and its generalizations and applications, one can see [3, 5, 18–20, 24, 25].

COROLLARY 2.2. *Let $u(t)$, $f(s)$, $a(t,s)$, $b(t,s)$, $c(t,s)$ and $\alpha(t)$ are defined as in Theorem 2.1. If $u(t)$ satisfies*

$$u^2(t) \leqslant l(t) + \int_{\alpha(t_0)}^{\alpha(t)} a(t,s)\left[f(s)u(s)ds + \int_{\alpha(t_0)}^{s} b(s,\tau)u(\tau)d\tau\right]ds + \int_{\alpha(t_0)}^{\alpha(T)} c(t,s)u(s)ds \tag{2.20}$$

for $t \in I$ and

$$\lambda_{111}(t) = \frac{1}{2}K_3^{-\frac{1}{2}}\int_{\alpha(t_0)}^{\alpha(T)} c(t,s)\exp(A_{211}(s))ds < 1, \tag{2.21}$$

then

$$u(t) \leqslant \left[l(t) + \frac{\bar{A}_{211}(t) + B_{211}(t)}{1 - \lambda_{211}(t)}\exp\left(A_{211}(t)\right)\right]^{\frac{1}{2}} \tag{2.22}$$

for $t \in I$ *and any* $K_i > 0$ $(i = 1, 2, 3)$, *where*

$$A_{211}(t) = \frac{1}{2}\int_{\alpha(t_0)}^{\alpha(t)} a(t,s)\left[K_1^{-\frac{1}{2}} f(s) + K_2^{-\frac{1}{2}}\int_{\alpha(t_0)}^{s} b(s,\tau)d\tau\right] ds, \tag{2.23}$$

$$\bar{A}_{211}(t) = \frac{1}{2}\int_{\alpha(t_0)}^{\alpha(t)} a(t,s)\left[f(s)L_1(s) + \int_{\alpha(t_0)}^{s} b(s,\tau)L_2(\tau)d\tau\right] ds, \tag{2.24}$$

$$C_{21}(t) = \frac{1}{2}\int_{\alpha(t_0)}^{\alpha(T)} c(t,s)L_3(s)ds \tag{2.25}$$

and

$$L_i(t) = K_i^{-\frac{1}{2}} l(t) + K_i^{\frac{1}{2}}, i = 1, 2, 3$$

for $t \in I$.

When $p = q = r = n = 1$ we get an interesting result as following

COROLLARY 2.3. *Let* $u(t)$, $a(t,s)$, $b(t,s)$ *and* $\alpha(t)$ *are defined as in Theorem* 2.1. *If* $u(t)$ *satisfies*

$$u(t) \leqslant l(t) + \int_{\alpha(t_0)}^{\alpha(t)} a(t,s)\left[f(s)u(s)ds + \int_{\alpha(t_0)}^{s} b(s,\tau)u(\tau)d\tau\right] ds + \int_{\alpha(t_0)}^{\alpha(T)} c(t,s)u(s)ds \tag{2.26}$$

for $t \in I$ *and*

$$\lambda_{111}(t) = \int_{\alpha(t_0)}^{\alpha(T)} c(t,s)\exp(A_{111}(s))ds < 1, \tag{2.27}$$

then

$$u(t) \leqslant l(t) + \frac{\bar{A}_{111}(t) + C_{11}(t)}{1 - \lambda_{111}(t)} \exp\left(A_{111}(t)\right) \tag{2.28}$$

for $t \in I$, *where*

$$A_{111}(t) = \int_{\alpha(t_0)}^{\alpha(t)} a(t,s)\left[f(s) + \int_{\alpha(t_0)}^{s} b(s,\tau)d\tau\right] ds, \tag{2.29}$$

$$\bar{A}_{111}(t) = \int_{\alpha(t_0)}^{\alpha(t)} a(t,s)\left[f(s)l(s) + \int_{\alpha(t_0)}^{s} b(s,\tau)l(\tau)d\tau\right] ds \tag{2.30}$$

and

$$C_{11}(t) = \int_{\alpha(t_0)}^{\alpha(T)} c(t,s)l(s)ds \tag{2.31}$$

for $t \in I$.

REMARK 2.1. (i) When $l(t) \equiv k \geqslant 0$ (k is a constant), the inequality (2.26) has been studied in Theorem 1.1, but in this special case, under same conditions as in Theorem 1.1, a new estimate to the solution of (2.26) is established in (2.28), which is incomparable with the result given in Theorem 1.1.

(ii) Using the similar procedures of proof of Theorem 2.1, we can get a more general result as following.

THEOREM 2.4. *Let $u(t)$, $l(t)$ and $f_i(t) \in C(I,R_+)$, $a_i(t,s)$, $b_i(t,s)$ and $c_j(t,s) \in C(D,R_+)$, $a_i(t,s)$, $b_i(t,s)$ and $c_j(t,s)$ be nondecreasing in t for each $s \in I$, $\alpha_i(t)$ and $\beta_j(t) \in C^1(I,I)$ be nondecreasing with $\alpha_i(t), \beta_j(t) \leqslant t$ on I, where $D = \{(t,s) \in I^2 : t_0 \leqslant s \leqslant t \leqslant T\}$, $i = 1,2,\cdots,m_1$, $j = 1,2,\cdots,m_2$(m_1 and m_2 are some positive integers). If $u(t)$ satisfies*

$$u^p(t) \leqslant l(t) + \sum_{i=1}^{m_1} \int_{\alpha_i(t_0)}^{\alpha_i(t)} a_i(t,s) \left[f_i(s) u^{q_i}(s) ds + \int_{\alpha_i(t_0)}^{s} b_i(s,\tau) u^{r_i}(\tau) d\tau \right] ds + \sum_{j=1}^{m_2} \int_{\beta_j(t_0)}^{\beta_j(T)} c_j(t,s) u^{n_j}(s) ds \tag{2.32}$$

for $t \in I$, where $p \geqslant q_i \geqslant 0$, $p \geqslant r_i \geqslant 0$, $p \geqslant n_j \geqslant 0$, $K_{1i} > 0$, $K_{2i} > 0$, $K_{3j} > 0$, $p, q_i, r_i, n_j, K_{1i}, K_{2i}$ and $K_{3j}(i = 1,2,\cdots,m_1, j = 1,2,\cdots,m_2)$ are constants and

$$\lambda(t) = \sum_{j=1}^{m_2} \frac{n_j}{p} K_{2j}^{\frac{n_j - p}{p}} \int_{\beta_j(t_0)}^{\beta_j(T)} c_j(t,s) \exp(A(s)) ds < 1, \tag{2.33}$$

then

$$u(t) \leqslant \left[l(t) + \frac{\bar{A}(t) + C(t)}{1 - \lambda(t)} \exp\left(A(t)\right) \right]^{\frac{1}{p}} \tag{2.34}$$

for $t \in I$ and any $K > 0$, where

$$A(t) = \sum_{i=1}^{m_1} \int_{\alpha_i(t_0)}^{\alpha_i(t)} a_i(t,s) \left[\frac{q_i}{p} K_{1i}^{\frac{q_i - p}{p}} f_i(s) + \frac{r_i}{p} K_{2i}^{\frac{r_i - p}{p}} \int_{\alpha_i(t_0)}^{s} b_i(s,\tau) d\tau \right] ds, \tag{2.35}$$

$$\bar{A}(t) = \sum_{i=1}^{m_1} \int_{\alpha_i(t_0)}^{\alpha_i(t)} a_i(t,s) \left[f_i(s) \left(\frac{q_i}{p} K_{1i}^{\frac{q_i - p}{p}} l(s) + \frac{p - q_i}{p} K_{1i}^{\frac{q_i}{p}} \right) + \int_{\alpha_i(t_0)}^{s} b_i(s,\tau) \left(\frac{r_i}{p} K_{2i}^{\frac{r_i - p}{p}} l(\tau) + \frac{p - r_i}{p} K_{2i}^{\frac{r_i}{p}} \right) d\tau \right] ds \tag{2.36}$$

and

$$C(t) = \sum_{j=1}^{m_2} \int_{\beta_j(t_0)}^{\beta_j(T)} c_j(t,s) \left[\frac{n_j}{p} K_{3j}^{\frac{n_j - p}{p}} l(s) + \frac{p - n_j}{p} K_{3j}^{\frac{n_j}{p}} \right] ds \tag{2.37}$$

for $t \in I$.

REMARK 2.2. (i) When $m_1 = 2$, $p = q_1 = q_2 = 1$, $a_1(t,s) = a(s)$, $a_2(t,s) = b(s)$, $f_1(t) = f_2(t) = 1$, $\alpha_1(t) = t$, $b_1(t,s) = b_2(t,s) = 0$, $c_j(t,s) = 0$, $j = 1,2,\cdots,m_2$, from Theorem 2.4 we can get Theorem 2.1 given in [7];

(ii) When $m_1 = 2$, $p > 1$, $q_1 = q_2 = 1$, $a_1(t,s) = a(s)$, $a_2(t,s) = b(s)$, $f_1(t) = f_2(t) = 1$, $\alpha_1(t) = t$, $b_1(t,s) = b_2(t,s) = 0$, $c_j(t,s) = 0$, $j = 1,2,\cdots,m_2$, from Theorem 2.4(let $K_{11} = K_{12} = l(t)$) we can get Theorem 2.2 given in [7].

THEOREM 2.5. *Let $u(t)$ and $l(t) \in C(I,R_+), a(t,s)$ and $b(t,s) \in C(D,R_+)$, $a(t,s)$ and $b(t,s)$ be nondecreasing in t for each $s \in I$, $\alpha(t) \in C^1(I,I)$ be nondecreasing with $\alpha(t) \leqslant t$ on I, where $D = \{(t,s) \in I^2 : t_0 \leqslant s \leqslant t \leqslant T\}$. If $u(t)$ satisfies*

$$u^p(t) \leqslant l(t) + \int_{\alpha(t_0)}^{\alpha(t)} a(t,s)\left[f(s)u^q(s)ds + \int_{\alpha(t_0)}^{s} b(s,\tau)u^r(\tau)d\tau\right] ds + \int_{\alpha(t_0)}^{\alpha(T)} c(t,s)N(s,u(s))ds \tag{2.38}$$

for $t \in I$, where $p \geqslant q \geqslant 0$, $p \geqslant r \geqslant 0$, p,q and r are constants, $N,M \in (R_+^2, R_+)$ satisfying

$$0 \leqslant N(t,x) - N(t,y) \leqslant M(t,y)(x-y), \tag{2.39}$$

and

$$\lambda_{pqr}(t) = \int_{\alpha(t_0)}^{\alpha(T)} c(t,s)M\left(s, \frac{p-1}{p} + \frac{1}{p}l(s)\right)\exp(A_{pqr}(s))ds < 1, \tag{2.40}$$

then

$$u(t) \leqslant \left[l(t) + \frac{\bar{A}_{pqr}(t) + \bar{C}_p}{1-\lambda_{pqr}(t)}\exp\left(A_{pqr}(t)\right)\right]^{\frac{1}{p}} \tag{2.41}$$

for $t \in I$ and any $K_i > 0$ $(i = 1,2)$, where $A_{pqr}(t)$ and $\bar{A}_{pqr}(t)$ are defined as in (2.4) and (2.5), respectively, and

$$\bar{C}_p = \int_{\alpha(t_0)}^{\alpha(T)} N\left(s, \frac{p-1}{p} + \frac{1}{p}l(s)\right) ds. \tag{2.42}$$

Proof. Define a function $\bar{v}(t)$ by

$$\bar{v}(t) = \int_{\alpha(t_0)}^{\alpha(t)} a(t,s)\left[f(s)u^q(s)ds + \int_{\alpha(t_0)}^{s} b(s,\tau)u^r(\tau)d\tau\right] ds + \int_{\alpha(t_0)}^{\alpha(T)} c(t,s)N(s,u(s))ds, \tag{2.43}$$

then

$$u^p(t) \leqslant l(t) + \bar{v}(t)$$

or

$$u(t) \leqslant (l(t) + \bar{v}(t))^{\frac{1}{p}}. \tag{2.44}$$

By Lemma 2.1, for any $K > 0$, we have

$$u(t) \leqslant (l(t) + \bar{v}(t))^{\frac{1}{p}} \leqslant \frac{1}{p}(l(t) + \bar{v}(t)) + \frac{p-1}{p},$$

$$u^q(t) \leqslant (l(t) + \bar{v}(t))^{\frac{q}{p}} \leqslant \frac{q}{p}K_1^{\frac{q-p}{p}}(l(t) + \bar{v}(t)) + \frac{p-q}{p}K_1^{\frac{q}{p}}$$

and

$$u^r(t) \leqslant (l(t) + v(t))^{\frac{r}{p}} \leqslant \frac{r}{p}K_2^{\frac{r-p}{p}}(l(t) + \bar{v}(t)) + \frac{p-r}{p}K_2^{\frac{r}{p}}$$

Substituting the last relations into(2.43)and using (2.39), it follows that

$$\begin{aligned}
\bar{v}(t) \leqslant & \int_{\alpha(t_0)}^{\alpha(t)} a(t,s)\Bigg[f(s)\left(\frac{q}{p}K_1^{\frac{q-p}{p}}(l(s)+\bar{v}(s))+\frac{p-q}{p}K_1^{\frac{q}{p}}\right) \\
& +\int_{\alpha(t_0)}^{s} b(s,\tau)\left(\frac{r}{p}K_2^{\frac{r-p}{p}}(l(\tau)+\bar{v}(\tau))+\frac{p-r}{p}K_2^{\frac{r}{p}}\right)d\tau\Bigg]ds \\
& +\int_{\alpha(t_0)}^{\alpha(T)} N\left(s,\frac{p-1}{p}+\frac{1}{p}(l(s)+\bar{v}(s))\right)ds-\int_{\alpha(t_0)}^{\alpha(T)} N\left(s,\frac{p-1}{p}+\frac{1}{p}l(s)\right)ds \\
& +\int_{\alpha(t_0)}^{\alpha(T)} N\left(s,\frac{p-1}{p}+\frac{1}{p}l(s)\right)ds \\
\leqslant & \bar{A}_{pqr}(t)+\bar{C}_p(t)+\int_{\alpha(t_0)}^{\alpha(t)} a(t,s)\left[\frac{q}{p}K_1^{\frac{q-p}{p}}f(s)\bar{v}(s)+\frac{r}{p}K_2^{\frac{r-p}{p}}\int_{\alpha(t_0)}^{s} b(s,\tau)\bar{v}(\tau)d\tau\right]ds \\
& +\int_{\alpha(t_0)}^{\alpha(T)} c(t,s)M\left(s,\frac{p-1}{p}+\frac{1}{p}l(s)\right)\bar{v}(s)ds,
\end{aligned} \tag{2.45}$$

where $\bar{C}_p$ is defined in (2.42). Obviously, $\bar{A}_{pqr}(t)$ and $\bar{C}_p$ are nonnegative, continuous and nondecreasing for $t \in I$. Taking similar procedure from (2.9) to (2.19) in the proof of Theorem 2.1 to (2.45), we can get the desired inequality (2.41). □

REMARK 2.3. As Theorem 2.4, using similar arguments in the proof of Theorem 2.5, we can get a more general version of (2.38), but for space–saving, the details are omitted here.

3. Applications

Consider retarded Volterra-Fredholm integral equations of the form

$$\begin{aligned}
x^p(t) = l(t) + \int_{t_0}^{t} F\Big(t,s,x(s-h(s)),\int_{t_0}^{s} G(s,\tau,x(\tau-h(\tau)))d\tau\Big)ds \\
+\int_{t_0}^{T} H\Big(t,s,x(s-h(s))\Big)ds
\end{aligned} \tag{3.1}$$

for $t \in I$, where $x,l \in C(I,R), h \in C^1(I,I)$ be nonincreasing with $t-h(t) \geqslant t_0, h(t_0)=0, t-h(t) \in C^1(I,I), h'(t) < 1, F \in C(D\times R^2,R), G,H \in C(D\times R,R)$ and p is a constant. As pointed out in [23], Volterra-Fredholm type integral equations arises frequently in many applied areas which include engineering, mechanics, potential theory, electrostatics, etc.

When $p=1$, $F(t,s,x,y)=\lambda_1 K_1(t,s)x^q$, $H(t,s,x)=\lambda_2 K_2(t,s)x^r$, $h(t)\equiv 0$, (3.1) becomes

$$x(t) = f(t) + \lambda_1\int_{t_0}^{t} K_1(t,s)x^q(s)ds + \lambda_2\int_{t_0}^{T} K_2(t,s)x^r(s)ds, \quad t \in I. \tag{3.1$_0$}$$

Under some suitable conditions, Yalcinbas [26] used a Taylor expansion approach method to get the solution of $(3.1)_0$ which is expressed in the form

$$x(t)=\sum_{n=0}^{N}\frac{1}{n!}x^{(n)}(c)(t-c)^n,\ t,c\in I.$$

Here, we apply our results to study the boundedness, uniqueness, and continuous dependence of the solutions of (3.1).

THEOREM 3.1. *Assume that the functions* F,G *and* H *in* (3.1) *satisfies the conditions*

$$|F(t,s,x,y)|\leqslant a(t,s)[f(s)|x|^q+|y|], \tag{3.2}$$

$$|G(t,s,x)|\leqslant b(t,s)|x|^r \tag{3.3}$$

and

$$|H(t,s,x)|\leqslant c(t,s)|x|^n, \tag{3.4}$$

where functions $a(t,s)$, $b(t,s)$, $c(t,s)$ *and* $f(t)$, *constants* p,q,r *and* n *are as in Theorem* 2.1, *and let* $M^*=\max_{t\in I}\{1/(1-h'(t))\}$. *If*

$$\lambda_1^*(t)=\frac{n}{p}K_3^{\frac{n-p}{p}}\int_{t_0}^{T-h(T)}M^*c(t,\xi+h(s))\exp(A^*(\xi))d\xi<1, \tag{3.5}$$

for $t\in I$,*where*

$$A_1^*(t)=\int_{t_0}^{t-h(t)}M^*a(t,\xi+h(s))\left[f(\xi+h(s))+\int_{t_0}^{\xi}M^*b(\xi+h(s),\sigma+h(\tau))d\sigma\right]d\xi \tag{3.6}$$

$$\begin{aligned}\overline{A_1^*}(t)=\int_{t_0}^{t-h(t)}M^*a(t,\xi+h(s))\Bigg[&\left(\frac{q}{p}K_1^{\frac{q-p}{p}}|l(\xi)|+\frac{p-q}{p}K_1^{\frac{q}{p}}\right)\\&+\int_{t_0}^{\xi}M^*b(\xi+h(s),\sigma+h(\tau))\left(\frac{r}{p}K_2^{\frac{r-p}{p}}|l(\sigma)|+\frac{p-r}{p}K_2^{\frac{r}{p}}\right)d\sigma\Bigg]d\xi\end{aligned} \tag{3.7}$$

and

$$C_1^*(t)=\int_{t_0}^{T-h(T)}M^*c(t,\xi+h(s))\left[\frac{n}{p}K_3^{\frac{n-p}{p}}|l(\xi)|+\frac{p-n}{p}K_3^{\frac{n}{p}}\right]d\xi \tag{3.8}$$

for t,s,τ *in* I *and any* $K_i>0$ $(i=1,2,3)$. *If* $x(t)$ *is a solution of equation* (3.1) *on* I, *then*

$$|x(t)|\leqslant\left[|l(t)|+\frac{\overline{A_1^*}(t)+C_1^*(t)}{1-\lambda_1^*(t)}\exp\left(A_1^*(t)\right)\right]^{\frac{1}{p}} \tag{3.9}$$

for $t\in I$.

Proof. Let $x(t)$ be a solution of (3.1). Using conditions (3.2)-(3.4) in (3.1) and making the change of variable, we have

$$
\begin{aligned}
|x(t)|^p \leqslant\ & |l(t)| + \int_{t_0}^{t} a(t,s)\left[f(s)|x(s-h(s))|^q + \int_{t_0}^{s} b(s,\tau)|x(\tau-h(\tau))|^r d\tau\right] ds \\
& + \int_{t_0}^{T} c(t,s)|x(s-h(s))|^n ds \\
\leqslant\ & |l(t)| + \int_{t_0}^{t} a(t,s)\left[f(s)|x(s-h(s))|^q + \int_{t_0}^{s-h(s)} M^* b(s,\sigma+h(\tau))|x(\sigma)|^r d\sigma\right] ds \\
& + \int_{t_0}^{T-h(T)} M^* c(t,\xi+h(s))|x(\xi)|^n d\xi \\
\leqslant\ & |l(t)| + \int_{t_0}^{t-h(t)} M^* a(t,\xi+h(s))\Bigg[f(\xi+h(s))|x(\xi)|^q \\
& \qquad + \int_{t_0}^{\xi} M^* b(\xi+h(s),\sigma+h(\tau))|x(\sigma)|^r d\sigma\Bigg] d\xi \\
& \qquad + \int_{t_0}^{T-h(T)} M^* c(t,\xi+h(s))|x(\xi)|^n d\xi
\end{aligned}
\tag{3.10}
$$

for t,s,τ in I. Now a suitable application of Theorem 2.1 to (3.10) yields the desired estimate in (3.9). □

THEOREM 3.2. *Assume that the functions F,G and H in equation (3.1) satisfy the conditions*

$$|F(t,s,x,y) - F(t,s,\overline{x},\overline{y})| \leqslant a(t,s)\left(f(s)|x^p - \overline{x}^p| + |y-\overline{y}|\right), \tag{3.11}$$

$$|G(t,s,x) - G(t,s,\overline{x})| \leqslant b(t,s)|x^p - \overline{x}^p|, \tag{3.12}$$

$$|H(t,s,x,y) - H(t,s,\overline{x},\overline{y})| \leqslant c(t,s)|x^p - \overline{x}^p| \tag{3.13}$$

and

$$\lambda_2^*(t) = \int_{t_0}^{T-h(T)} c(t,\xi+h(s))\exp(A_2^*(\xi))d\xi < 1,$$

where

$$A_2^*(t) = \int_{t_0}^{t-h(t)} M^* a(t,\xi+h(s))\left[f(\xi+h(s)) + \int_{t_0}^{\xi} M^* b(\xi+h(s),\sigma+h(\tau))d\sigma\right] d\xi,$$

functions a,b,c and f are defined as in Theorem 3.1, $p>0$ is a constant, then (3.1) has at most one positive solution on I.

Proof. Let $x(t)$ and $\overline{x}(t)$ be two solutions of (3.1) on I, using the conditions (3.11)–(3.13) to (3.1) we have

$$\begin{aligned}|x^p(t)-\overline{x}^p(t)| \leqslant \int_{t_0}^{t} a(t,s)\Big[f(s)|x^p(s-h(s))-\overline{x}^p(s-h(s))| \\ +\int_{t_0}^{s} b(s,\tau)|x^p(\tau-h(\tau))-\overline{x}^p(\tau-h(\tau))|d\tau\Big]ds \\ +\int_{t_0}^{T} c(t,s)[|x^p(s-h(s))-\overline{x}^p(s-h(s))|ds\end{aligned} \tag{3.14}$$

Now making a change of variables on the right side of (3.14) and taking the similar procedure as in the proof of Theorem 3.1 we have

$$\begin{aligned}|x^p(t)-\overline{x}^p(t)| \leqslant \int_{t_0}^{t-h(t)} M^* a(t,\xi+h(s))\Big[f(\xi+h(s))|x^p(\xi)-\overline{x}^p(\xi)| \\ +\int_{t_0}^{\xi} M^* b(\xi+h(s),\sigma+h(\tau))|x^p(\sigma)-\overline{x}^p(\sigma)|d\sigma\Big]d\xi \\ +\int_{t_0}^{T-h(T)} M^* c(t,\xi+h(s))|x^p(\xi)-\overline{x}^p(\xi)|d\xi\end{aligned} \tag{3.15}$$

where M^* is defined as in Theorem 3.1. A suitable application of Corollary 2.3 to the function $|x^p(t)-\overline{x}^p(t)|$ in (3.15) yields that

$$|x^p(t)-\overline{x}^p(t)| \leqslant 0$$

for $t \in I$. Hence $x=\overline{x}$ on I. $\square$

The following theorem investigate the continuous dependence of the solutions of (3.1) on the functions F, G and H. For this we consider the following variation of (3.1):

$$\begin{aligned}x^p(t) = \overline{l}(t) + \int_{t_0}^{t} \overline{F}\Big(t,s,x(s-h(s)),\int_{t_0}^{s}\overline{G}(s,\tau,x(\tau-h(\tau)))d\tau\Big)ds \\ +\int_{t_0}^{T}\overline{H}\Big(t,s,x(s-h(s))\Big)ds\end{aligned} \tag{3.1*}$$

for $t \in I$, where $\overline{F} \in C(D\times R^2,R), \overline{G},\overline{H} \in C(D\times R,R)$ and $p>0$ is a constant.

THEOREM 3.3. *Consider* (3.1) *and* (3.1^*). *If*
(i)

$$|F(t,s,x_1,y_1)-F(t,s,x_2,y_2)| \leqslant a(t,s)[f(s)|x_1^p-x_2^p|+|y_1-y_2|],$$
$$|G(t,s,x_1)-G(t,s,x_2)| \leqslant b(t,s)|x_1^p-x_2^p|,$$

and

$$|H(t,s,x_1)-H(t,s,x_2)| \leqslant c(t,s)|x_1^p-x_2^p|;$$

(ii)

$$|l(t) - \bar{l}(t)| \leqslant \frac{\varepsilon}{4};$$

(iii)

$$\lambda_2^*(t) = \int_{t_0}^{T-h(T)} c(t, \xi + h(s)) \exp(A_2^*(\xi)) d\xi < 1;$$

(iv) *for all solutions* $\overline{x}$ *of* $(3.1*)$,

$$\int_{t_0}^{t} \left| F\left(t, s, \overline{x}(s-h(s)), \int_{t_0}^{s} \overline{G}(s, \tau, \overline{x}(\tau - h(\tau)))d\tau\right) \right.$$
$$\left. - \overline{F}\left(t, s, \overline{x}(s-h(s)), \int_{t_0}^{s} \overline{G}(s, \tau, \overline{x}(\tau - h(\tau)))d\tau\right) \right| ds \leqslant \frac{\varepsilon}{4},$$

$$\int_{t_0}^{t} a(t,s) \left(\int_{t_0}^{s} \left| G(s, \tau, \overline{x}(\tau - h(\tau)))d\tau) - \overline{G}(s, \tau, \overline{x}(\tau - h(\tau)))\right| d\tau \right) ds \leqslant \frac{\varepsilon}{4},$$

and

$$\int_{t_0}^{T} \left| H\left(t, s, \overline{x}(s-h(s))\right) - \overline{H}\left(t, s, \overline{x}(s-h(s))\right) \right| ds \leqslant \frac{\varepsilon}{4},$$

for all $s, t \in I$ *and* $x_1, x_2, y_1, y_2 \in R$*,where* $\varepsilon > 0$ *is an arbitrary constant, then*

$$|x^p(t) - \overline{x}^p(t)| \leqslant \varepsilon \left(1 + \frac{A_2^*(t) + C_3^*(t)}{1 - \lambda_2^*(t)} \exp\left(A_2^*(t)\right) \right) \tag{3.16}$$

for $t \in I$*, where*

$$C_3^*(t) = \int_{t_0}^{T-h(T)} M^* c(t, \xi + h(s)) d\xi \tag{3.17}$$

$A_2^*(t)$ *and* $\lambda_2^*(t)$ *are defined as in Theorem* 3.2 *for* $t \in I$*.*

Proof. Let $x(t)$ and $\overline{x}(t)$ be the solutions of (3.1) and $(3.1*)$, respectively. Then $x(t)$ satisfies (3.1) and $\overline{x}(t)$ satisfies $(3.1*)$. Hence

$$|x^p(t) - \overline{x}^p(t)| \leqslant |l(t) - \bar{l}(t)| + \int_{t_0}^{t} \left| F\left(t, s, x(s-h(s)), \int_{t_0}^{s} G(\tau, \overline{x}(\tau - h(\tau)))d\tau\right) \right.$$
$$\left. - \overline{F}\left(t, s, \overline{x}(s-h(s)), \int_{t_0}^{s} \overline{G}(\tau, \overline{x}(\tau - h(\tau)))d\tau\right) \right| ds$$
$$+ \int_{t_0}^{T} |H(s, x(s-h(s))) - \overline{H}(s, \overline{x}(s-h(s)))| ds$$
$$\leqslant \frac{\varepsilon}{4} + \int_{t_0}^{t} |F(t, s, x(s-h(s)), \int_{t_0}^{s} G(\tau, \overline{x}(\tau - h(\tau)))d\tau)$$
$$- F(t, s, \overline{x}(s-h(s)), \int_{t_0}^{s} \overline{G}(\tau, \overline{x}(\tau - h(\tau)))d\tau)| ds$$

$$
\begin{aligned}
&+\int_{t_0}^{t}|F(t,s,\overline{x}(s-h(s)),\int_{t_0}^{s}\overline{G}(\tau,\overline{x}(\tau-h(\tau)))d\tau)\\
&-\overline{F}(t,s,\overline{x}(s-h(s)),\int_{t_0}^{s}\overline{G}(\tau,\overline{x}(\tau-h(\tau)))d\tau)|ds\\
&+\int_{t_0}^{T}|H(t,s,x(s-h(s)))-H(t,s,\overline{x}(s-h(s)))|ds\\
&+\int_{t_0}^{T}|H(t,s,\overline{x}(s-h(s)))-\overline{H}(t,s,\overline{x}(s-h(s)))|ds\\
\leqslant\ &\frac{\varepsilon}{2}+\int_{t_0}^{t}a(t,s)\Big[f(s)|x^p(s-h(s))-\overline{x}^p(s-h(s))|\\
&+\int_{t_0}^{s}\big|G(s,\tau,x(\tau-h(\tau)))d\tau)-\overline{G}(s,\tau,\overline{x}(\tau-h(\tau)))\big|\,d\tau)\Big]\,ds\\
\leqslant\ &\frac{3\varepsilon}{4}+\int_{t_0}^{t}a(t,s)\Big[f(s)|x^p(s-h(s))-\overline{x}^p(s-h(s))|\\
&+\int_{t_0}^{s}|G(s,\tau,x(\tau-h(\tau)))d\tau)-G(s,\tau,\overline{x}(\tau-h(\tau)))|\,d\tau\\
&+\int_{t_0}^{s}\big|G(s,\tau,\overline{x}(\tau-h(\tau)))d\tau)-\overline{G}(s,\tau,\overline{x}(\tau-h(\tau)))\big|\,d\tau)\Big]\,ds\\
&+\int_{t_0}^{T}c(t,s)|x^p(s-h(s))-\overline{x}^p(s-h(s))|ds\\
\leqslant\ &\frac{3\varepsilon}{4}+\int_{t_0}^{t}a(t,s)\Big[f(s)|x^p(s-h(s))-\overline{x}^p(s-h(s))|\\
&+\int_{t_0}^{s}b(s,\tau)|x^p(\tau-h(\tau))-\overline{x}^p(\tau-h(\tau))|d\tau\Big]ds\\
&+\int_{t_0}^{T}c(t,s)|x^p(s-h(s))-\overline{x}^p(s-h(s))|ds\\
&+\int_{t_0}^{t}a(t,s)\left(\int_{t_0}^{s}\big|G(s,\tau,\overline{x}(\tau-h(\tau)))d\tau)-\overline{G}(s,\tau,\overline{x}(\tau-h(\tau)))\big|\,d\tau\right)ds\\
\leqslant\ &\varepsilon+\int_{t_0}^{t}a(t,s)\Big[f(s)|x^p(s-h(s))-\overline{x}^p(s-h(s))|\\
&+\int_{t_0}^{s}b(s,\tau)|x^p(\tau-h(\tau))-\overline{x}^p(\tau-h(\tau))|d\tau\Big]ds\\
&+\int_{t_0}^{T}c(t,s)|x^p(s-h(s))-\overline{x}^p(s-h(s))|ds
\end{aligned}
$$

by assumptions (i) - (iv).

By making a change of variable on the right of the last inequality and taking the similar procedure as in the proofs of Theorem 3.1 we have

$$\begin{aligned}|x^p(t)-\overline{x}^p(t)| \leqslant \varepsilon &+ \int_{t_0}^{t-h(t)} M^* a(t,\xi+h(s))\Big[f(\xi+h(s))|x^p(\xi)-\overline{x}^p(\xi)| \\ &+ \int_{t_0}^{\xi} M^* b(\xi+h(s),\sigma+h(\tau))|x^p(\sigma)-\overline{x}^p(\sigma)|d\sigma\Big] d\xi \\ &+ \int_{t_0}^{T-h(T)} M^* c(t,\xi+h(s))|x^p(\xi)-\overline{x}^p(\xi)|d\xi\end{aligned} \tag{3.18}$$

for t,s,τ in I. Now a suitable application of Corollary 2.3 to (3.18) yields the desired estimate in (3.16).

Evidently, if function $\dfrac{A_2^*(t)+C_3^*(t)}{1-\lambda_2^*(t)}\exp\left(A_2^*(t)\right)$ is bounded on I, so

$$|x^p(t)-\overline{x}^p(t)| \leqslant \varepsilon K$$

for some $K>0$ and $t\in I$. Hence x^p depends continuously on F,G and H. □

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