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SOME INEQUALITIES INVOLVING GENERALIZED BESSEL FUNCTIONS

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Dedicated to my wife Kati

(*communicated by J. Sándor*)

Abstract. Let u_p be the generalized and normalized Bessel function depending on parameters b, c, p and let $\lambda_p(x) = u_p(x^2)$, $x \in \mathbb{R}$. In this paper we extend to the function λ_p some well-known classical inequalities like Mahajan's inequality, Mitrinović's inequality, improvements of Jordan's inequality, Redheffer's inequality, using an adequate integral representation of the function λ_p and the monotone form of l'Hospital's rule. Moreover we prove that the integral

$$\varsigma_p(x) = \int_0^x \lambda_p(t)\,\mathrm{dt}$$

is sub-additive (super-additive) under certain conditions on parameters b, c, p.

1. Introduction and preliminaries

The sine and cosine functions are particular cases of normalized Bessel functions, while the hyperbolic sine and hyperbolic cosine functions are particular cases of normalized modified Bessel functions. Thus it is natural to generalize some formulas and inequalities involving these elementary functions to normalized Bessel functions and normalized modified Bessel functions, respectively. Recently, the author extended some well known inequalities, like Lazarević's inequality, Turán-type inequality [8], Kober's inequality [8] to the function λ_p, defined bellow. In this paper our aim is to continue this investigation: we extend some well-known classical inequalities, like Mahajan's inequality, Mitrinović's inequality, Jordan's inequality, Redheffer's inequality to the function λ_p, using an adequate integral representation of the function λ_p.

This paper is organized as follows. In this section we give the definition and some basic facts about the function λ_p in the question. In section 2 we extend Mahajan's inequality using a result of L. Lorch and M. E. Muldoon [16]. On the other hand we offer the hyperbolic counterpart of Mitrinović's inequality [18, p. 240]. In section 3 we use the Redheffer inequality to obtain a new lower bound for the function λ_p. In section

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4 we extend Cusa's inequality to generalized and normalized Bessel functions and we obtain refinements of some known inequalities established in [8]. In section 5 we obtain new bounds for the function λ_p by extending three improvements of the well-known Jordan's inequality. Finally, in section 6 we prove that the integral

$$\varsigma_p(x) = \int_0^x \lambda_p(t)\,\mathrm{dt}$$

is sub-additive (super-additive) under certain conditions on parameters b, c, p. This result extends the recent result of S. Koumandos [15], who among other things proved that the sine integral is sub-additive on $[0,\infty)$.

For the definition of the function λ_p let us recall some basic facts. The generalized Bessel function of the first kind v_p is defined [7] as a particular solution of the differential equation

$$x^2y''(x) + bxy'(x) + \left[cx^2 - p^2 + (1-b)p\right]y(x) = 0,$$

where $b,p,c \in \mathbb{R}$, and v_p has the infinite series representation

$$v_p(x) = \sum_{n\geqslant 0} \frac{(-1)^n c^n}{n!\Gamma\left(p+n+\frac{b+1}{2}\right)} \cdot \left(\frac{x}{2}\right)^{2n+p} \text{ for all } x \in \mathbb{R}.$$

This function permits us to study the classical Bessel function J_p and the modified Bessel function I_p together. For $c = 1$ ($c = -1$ respectively) and $b = 1$ the function v_p reduces to the function J_p (I_p respectively). Now the generalized and normalized Bessel function of the first kind is defined [7] as follows

$$u_p(x) = 2^p\Gamma(\kappa)\cdot x^{-p/2}v_p(x^{1/2}) = \sum_{n\geqslant 0} \frac{(-c/4)^n}{(\kappa)_n}\frac{x^n}{n!} \text{ for all } x \in \mathbb{R}, \qquad (1)$$

where $\kappa := p + (b+1)/2 \neq 0, -1, -2, \ldots$, and $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the well known Pochhammer symbol defined in terms of Euler's Γ-function. This function is related to an obvious transform of the hypergeometric function ${}_0F_1$, i.e. $u_p(x) = {}_0F_1(\kappa, -cx/4)$ and satisfies the following differential equation

$$xy''(x) + \kappa y'(x) + (c/4)y(x) = 0.$$

For properties of the function u_p, such as differentiation formula, integral representation, lower and upper bounds, and interesting functional inequalities we refer to the papers [5, 6, 7, 9]. Let us consider the function λ_p defined by

$$\lambda_p(x) = u_p(x^2) = \sum_{n\geqslant 0} \frac{(-c/4)^n}{(\kappa)_n}\frac{x^{2n}}{n!} \text{ for all } x \in \mathbb{R}. \qquad (2)$$

For $c = 1$ ($c = -1$ respectively) and $b = 1$ this function reduces to the function $\mathscr{J}_p$ ($\mathscr{I}_p$ respectively) defined by

$$\mathscr{J}_p(x) = 2^p\Gamma(p+1)x^{-p}J_p(x), \quad \mathscr{I}_p(x) = 2^p\Gamma(p+1)x^{-p}I_p(x).$$

For later use it is worth mentioning that in particular (for $p=-1/2$ and $p=1/2$, respectively) the functions $\mathscr{J}_p$ and $\mathscr{I}_p$ reduces to some elementary functions, like

$$\mathscr{J}_{-1/2}(x)=\cos x,\ \ \mathscr{I}_{-1/2}(x)=\cosh x,\ \ \mathscr{J}_{1/2}(x)=\frac{\sin x}{x},\ \ \mathscr{I}_{1/2}(x)=\frac{\sinh x}{x}.$$

2. Mitrinovic's, Mahajan's inequality

In 1979. A. Mahajan [17] extended a result of D. S. Mitrinović [18, p. 240] by proving that if $p>-1$, then

$$(x+1)\mathscr{J}_p\left(\frac{\pi}{x+1}\right)-x\mathscr{J}_p\left(\frac{\pi}{x}\right)>1 \text{ for all } x>\pi\frac{\pi+\sqrt{\pi^2+32(p+2)}}{16(p+2)}. \tag{3}$$

D. S. Mitrinović has proved the case $p=-1/2$ of (3), i.e.

$$(x+1)\cos\left(\frac{\pi}{x+1}\right)-x\cos\left(\frac{\pi}{x}\right)>1, \tag{4}$$

but just for $x\geqslant\sqrt{3}\simeq 1.732050808\ldots$, while A. Mahajan's generalization yields a better interval of validity for (4), namely $x>1.407014637\ldots$. In 1987 L. Lorch and M. E. Muldoon [16] proved that the largest interval of validity for (4) is $(1,\infty)$ and for (3) is (x_1,∞) where x_1 is the largest root of $\varphi_1(x+1)=\varphi_1(x)$ and $\varphi_1(x)=x[\mathscr{J}_p(\pi/x)-1]$. In the next theorem our aim is to extend inequality (3) for the function λ_p. In the case of $c>0$ we use the method of L. Lorch and M. E. Muldoon [16] mutatis mutandis, while for $c<0$ we give a different proof. As we can see the analogous of inequality (4) for the hyperbolic cosine function holds just if $x\in(-1,0)$, otherwise the inequality is reversed.

THEOREM 5. *If* $\kappa,c>0$ *then*

$$(x+1)\lambda_p\left(\frac{\pi}{x+1}\right)-x\lambda_p\left(\frac{\pi}{x}\right)>1 \tag{6}$$

holds for all $x<-x_2-1$ *or* $x>x_2$, *where* x_2 *is the largest root of* $\varphi_2(x+1)=\varphi_2(x)$ *and* $\varphi_2(x)=x[\lambda_p(\pi/x)-1]$. *Moreover if* $\kappa>0$ *and* $c<0$, *then* (6) *holds for all* $x\in(-1,0)$, *while if* $x<-1$ *or* $x>0$, *then* (6) *is reversed.*

Proof. Let us consider the function $\varphi_3:\mathbb{R}\setminus\{-1,0\}\to\mathbb{R}$ defined by

$$\varphi_3(x)=(x+1)\lambda_p\left(\frac{\pi}{x+1}\right)-x\lambda_p\left(\frac{\pi}{x}\right)-1.$$

It is easy to verify that $\varphi_3(x-1/2)=\varphi_3(-x-1/2)$ for all $x\neq-1,0$. Thus clearly the graph of the function φ_3 is symmetric with respect to the straight line $x=-1/2$. Now let us distinguish the cases when $c>0$ and $c<0$.

Suppose that $c>0$. It is known from part (iv) of Proposition 2.16 [7] that

$$[x^{-p}v_p(x)]'=-c\cdot x^{-p}v_{p+1}(x) \tag{7}$$

holds for all $\kappa > 0$ and $c, x \in \mathbb{R}$. Observe that using (1) and (2) we easily obtain $\lambda_p(x) = 2^p\Gamma(\kappa)x^{-p}v_p(x)$. Thus from (7) one has

$$\lambda_p'(x) = -c \cdot 2^p\Gamma(\kappa)x^{-p}v_{p+1}(x),$$

$$\lambda_p''(x) = -c \cdot 2^p\Gamma(\kappa)x^{-p-1}[v_{p+1}(x) - cxv_{p+2}(x)],$$

so λ_p is a (decreasing and) concave function for all sufficiently small positive x, say $x \in (0, \alpha)$. Due to L. Lorch and M. E. Muldoon [16] we know that if the function f is concave on $(0, \beta]$, then

$$\frac{f(r)}{r} - \frac{f(s)}{s} > \left(\frac{1}{r} - \frac{1}{s}\right) f(0) \text{ for all } 0 < r < s \leqslant \beta. \tag{8}$$

Moreover if f is continuous, then this inequality remains true for certain r and s, one or both possibly greater than β, provided that for every $s > \beta$ we restrict our attention to those values of r less than the smallest value of r for which (8) becomes an equality. Thus from (8) we obtain that

$$\frac{\lambda_p(r)}{r} - \frac{\lambda_p(s)}{s} > \left(\frac{1}{r} - \frac{1}{s}\right) \lambda_p(0) \text{ for all } 0 < r < s < \alpha.$$

Putting $r = \pi/(x+1)$ and $s = \pi/x$, we obtain $\varphi_3(x) > 0$ for all $x > x_2$. Clearly by the symmetry of the function φ_3 we have $\varphi_3(x) > 0$ for all $x < -x_2 - 1$.

Now assume that $c < 0$. If $x \in (-1, 0)$, then

$$\begin{aligned}
\varphi_3(x) =&(x+1) \cdot \sum_{n\geqslant 0} b_n \left(\frac{\pi}{x+1}\right)^{2n} - x \cdot \sum_{n\geqslant 0} b_n \left(\frac{\pi}{x}\right)^{2n} - 1 \\
=&(x+1) \cdot \sum_{n\geqslant 1} b_n \left(\frac{\pi}{x+1}\right)^{2n} - x \cdot \sum_{n\geqslant 1} b_n \left(\frac{\pi}{x}\right)^{2n} \\
=&\sum_{n\geqslant 1} b_n \pi^{2n} \left[\frac{1}{(x+1)^{2n-1}} - \frac{1}{x^{2n-1}}\right] > 0.
\end{aligned}$$

From the symmetry of the function φ_3 it is enough to show that $\varphi_3(x) < 0$ for all $x > 0$. By Proposition 2.17, [7] we know that the generalized and normalized Bessel function of the first kind of order p satisfies

$$(4\kappa)u_p'(x) = (-c)u_{p+1}(x), \tag{9}$$

for all $x \in \mathbb{R}$ and $\kappa \neq 0, -1, -2, \ldots$. From l'Hospital's rule and (9) it is easy to verify that

$$\begin{aligned}
\lim_{x\to\infty} \varphi_3(x) &= \lim_{x\to\infty} x \left[\lambda_p\left(\frac{\pi}{x+1}\right) - \lambda_p\left(\frac{\pi}{x}\right)\right] \\
&= \lim_{x\to\infty} \pi \left[\frac{x^2}{(x+1)^2}\lambda_p'\left(\frac{\pi}{x+1}\right) - \lambda_p'\left(\frac{\pi}{x}\right)\right] \\
&= \lim_{x\to\infty} 2\pi\left(-\frac{c}{4\kappa}\right)\left[\frac{x^2}{(x+1)^3}\lambda_{p+1}\left(\frac{\pi}{x+1}\right) - \frac{1}{x}\lambda_{p+1}\left(\frac{\pi}{x}\right)\right] = 0.
\end{aligned}$$

In what follows we want to prove that the function φ_3 is strictly concave on $(0,\infty)$. Thus using the above mentioned limit (the graph of the function φ_3 is tangent to the x-axis at infinity) we may conclude that φ_3 is strictly negative, which completes the proof. For this let us consider the function $\varphi_4(x) = x^3\lambda_p''(x)$. Since from (9)

$$\varphi_4'(x) = 2b_1(p)x^2[3\lambda_{p+1}(x) + 12b_1(p+1)x^2\lambda_{p+2}(x) + 4b_1(p+1)b_1(p+2)x^4\lambda_{p+3}(x)],$$

so $\varphi_4'(x) > 0$ for all real $x > 0$, it follows that φ_4 is strictly increasing (here $b_1(p) = (-c)/(4\kappa)$). Finally

$$\varphi_3''(x) = \frac{\pi^2}{(x+1)^3}\lambda_p''\left(\frac{\pi}{x+1}\right) - \frac{\pi^2}{x^3}\lambda_p''\left(\frac{\pi}{x}\right),$$

hence the required result follows. □

REMARK 2.8. Taking $b = 1$ and $c = -1$ in Theorem 5 we get the hyperbolic counterpart of inequalities (3) and (4), namely

$$(x+1)\mathscr{I}_p\left(\frac{\pi}{x+1}\right) - x\mathscr{I}_p\left(\frac{\pi}{x}\right) > 1 \text{ for all } x \in (-1,0),\ p > -1,$$

and in particular for $p = -1/2$

$$(x+1)\cosh\left(\frac{\pi}{x+1}\right) - x\cosh\left(\frac{\pi}{x}\right) > 1.$$

When $x < -1$ or $x > 0$ both inequalities are reversed.

3. Redheffer's inequality

In 1969 R. Redheffer [23] established the following well-known inequality

$$\mathscr{J}_{1/2}(x) = \frac{\sin x}{x} \geqslant \frac{\pi^2 - x^2}{\pi^2 + x^2} \text{ for all } x \in \mathbb{R}. \tag{10}$$

Throughout this paper, it should be understood that functions such as $(\sin x)/x$, which have removable singularities at $x = 0$, have had these singularities removed in statements like (10). Recall that in 2004 E. Neuman [19, Theorem 2.2] using Gauss-Gegenbauer quadrature formula proved that if $p > -1/2$, then for all $|x| \leqslant \pi/2$

$$\frac{1}{3(p+1)}\left[2p+1+(p+2)\cos\left(\sqrt{\frac{3}{2(p+2)}}x\right)\right] \geqslant \mathscr{J}_p(x) \geqslant \cos\left(\frac{x}{\sqrt{2(p+1)}}\right). \tag{11}$$

Note that clearly when $p = -1/2$ we have equality in (11). Observe that $\mathscr{J}_{\kappa-1}(x\sqrt{c}) = \lambda_p(x)$, thus changing in the previous inequality p with $\kappa - 1$ and x with $x\sqrt{c}$, we deduce that [9, Theorem 4.2] if $c \in [0,1]$ and $\kappa \geqslant 1/2$, then for all $|x| \leqslant \pi/2$

$$\frac{1}{3\kappa}\left[2\kappa - 1 + (\kappa+1)\cos\left(\sqrt{\frac{3c}{2(\kappa+1)}}x\right)\right] \geqslant \lambda_p(x) \geqslant \cos\left(\sqrt{\frac{c}{2\kappa}}x\right). \tag{12}$$

Using (10) we can found an other lower bound for the function λ_p, which is valid for all real numbers. Here the key tool is the Sonine integral formula [27, p. 373], which expresses any Bessel function in terms of an integral involving a Bessel function of lower order.

THEOREM 13. *If* $c \geqslant 0$ *and* $\kappa \geqslant 3/2$, *then for all* $x \in \mathbb{R}$

$$\lambda_p(x) \geqslant \frac{\pi^2 - cx^2}{\pi^2 + cx^2}.$$

Proof. From the Sonine integral formula [27, p. 373] for Bessel functions

$$J_{q+p+1}(x) = \frac{x^{p+1}}{2^p\Gamma(p+1)} \int_0^{\pi/2} J_q(x\sin\theta)\sin^{q+1}\theta\cos^{2p+1}\theta\,\mathrm{d}\theta,\ p,q > -1,\ x \in \mathbb{R}$$

we obtain immediately the following formula which will be useful in the sequel

$$\mathscr{J}_{q+p+1}(x) = \frac{2}{B(p+1,q+1)} \int_0^{\pi/2} \mathscr{J}_q(x\sin\theta)\sin^{2q+1}\theta\cos^{2p+1}\theta\,\mathrm{d}\theta,\ p,q > -1, \tag{14}$$

where $x \in \mathbb{R}$ and $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ is the well-known Euler's beta function. Changing in (14) p with $p - 1/2$ and taking $q = 1/2$ one has for all $p > -1/2$, $x \in \mathbb{R}$

$$\mathscr{J}_{p+1}(x) = \frac{2}{B\left(p+\frac{1}{2},\frac{3}{2}\right)} \int_0^{\pi/2} \mathscr{J}_{1/2}(x\sin\theta)\sin^2\theta\cos^{2p}\theta\,\mathrm{d}\theta\,. \tag{15}$$

Using (10) it follows that for all $\theta \in [0,\pi/2]$ and $x \in \mathbb{R}$

$$\mathscr{J}_{1/2}(x\sin\theta)\sin^2\theta\cos^{2p} \geqslant \frac{\pi^2 - x^2\sin^2\theta}{\pi^2 + x^2\sin^2\theta}\sin^2\theta\cos^{2p} \geqslant \frac{\pi^2 - x^2}{\pi^2 + x^2}\sin^2\theta\cos^{2p},$$

thus using (15), and (10) again we obtain

$$\mathscr{J}_{p+1}(x) \geqslant \frac{\pi^2 - x^2}{\pi^2 + x^2} \text{ for all } x \in \mathbb{R} \text{ and } p \geqslant -1/2.$$

Finally using again $\mathscr{J}_{\kappa-1}(x\sqrt{c}) = \lambda_p(x)$ and changing in the previous inequality p with $\kappa - 2$ and x with $x\sqrt{c}$, we get the required result. □

REMARK 3.7. Recently C. P. Chen, J. W. Zhao and F. Qi [11] by using mathematical induction and infinite product representation of cosine function established the following Redheffer-type inequality

$$\mathscr{J}_{-1/2}(x) = \cos x \geqslant \frac{\pi^2 - 4x^2}{\pi^2 + 4x^2} \text{ for all } x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]. \tag{8}$$

Observe that if we use (14) again, by changing p with $p - 1/2$ and taking $q = -1/2$ one has for all $p > -1/2$, $x \in \mathbb{R}$

$$\mathscr{J}_p(x) = \frac{2}{B\left(p+\frac{1}{2},\frac{1}{2}\right)} \int_0^{\pi/2} \mathscr{J}_{-1/2}(x\sin\theta)\cos^{2p}\theta\,\mathrm{d}\theta\,. \tag{9}$$

Following the proof of Theorem 13 it is easy to verify that from (8) and (9) we have

$$\mathscr{J}_p(x) = \cos x \geqslant \frac{\pi^2 - 4x^2}{\pi^2 + 4x^2} \text{ for all } p \geqslant -\frac{1}{2} \text{ and } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Thus, if we change p with $\kappa - 1$ and x with $x\sqrt{c}$, we get an other lower bound for the function λ_p. Namely, if $c \geqslant 0$ and $\kappa \geqslant 1/2$, then for all $x \in [-\pi/2, \pi/2]$ one has

$$\lambda_p(x) \geqslant \frac{\pi^2 - 4cx^2}{\pi^2 + 4cx^2}.$$

It is worth mentioning that for $c \in [0,1]$, $\kappa \geqslant 1/2$, the lower bound from (12) is better than the above lower bound, since direct application of (8) yields

$$\lambda_p(x) \geqslant \cos\left(\sqrt{\frac{c}{2\kappa}}x\right) \geqslant \frac{\pi^2 - 2cx^2/\kappa}{\pi^2 + 2cx^2/\kappa} \geqslant \frac{\pi^2 - 4cx^2}{\pi^2 + 4cx^2}.$$

4. Cusa's inequality and related inequalities

Nicolaus da Cusa (1401-1464) using geometrical constructions discovered the inequality

$$\frac{\sin x}{x} \leqslant \frac{2 + \cos x}{3} \Leftrightarrow \mathscr{J}_{1/2}(x) \leqslant \frac{2 + \mathscr{J}_{-1/2}(x)}{3} \text{ for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \tag{10}$$

where the comment about removable singularities applies just as in (10). Willebrod Snellius (1581-1626) in his book entitled "Cyclometicus" found a proof for (10), but his proof was quite obscure (for further details please see the book of J. Sándor [25]). The first scientist who found an acceptable (geometrical) proof for (10) was Christian Huygens (1629-1695). Huygens in his book "De circuli magnitudine inventa" used (10) in the approximation of π (for the history of this see [10], [13]). In 1999 F. Qi, L. H. Cui and S. L. Xu [21, p. 521] using Tchebysheff's integral inequality proved that

$$\frac{\sin x}{x} \geqslant \frac{1 + \cos x}{2} \text{ for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{11}$$

Recently, there has been a keen interest in Missouri Journal of Mathematical Sciences, regarding inequalities on $\mathscr{J}_{1/2}(x) = (\sin x)/x$ (see the paper [25] and the references therein). We note that in fact inequalities (10) and (11) may be used for a simple proof of the well known fact that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

On the other hand it is known that [18, p. 238]

$$\frac{2(1 + a\cos x)}{\pi} \leqslant \frac{\sin x}{x} \leqslant \frac{1 + a\cos x}{a + 1} \text{ for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and } a \in \left(0, \frac{1}{2}\right]. \tag{12}$$

This trigonometric inequality represent a partial answer to the problem E 1277 proposed by A. Oppenheim and was proved by W. B. Carver in American Mathematical

Monthly **65**, (1958), 206–209 . As a generalization of this inequality, using the same idea as in the proof of Theorem 13, we recently proved [8, Corollary 2.18] that if $a \in (0,1/2]$, $c \geqslant 0$ and $\kappa \geqslant 1/2$, then inequality

$$\frac{1+2a\kappa\lambda_p(x)}{a(2\kappa-1)+\pi/2} \leqslant \lambda_{p+1}(x) \leqslant \frac{1+2a\kappa\lambda_p(x)}{a(2\kappa-1)+(a+1)} \tag{13}$$

holds for all $|x\sqrt{c}| \leqslant \pi/2$. In what follows our aim is to improve (13) using (10) and (11). First note that Cusa's inequality (10) is better than the right hand side of (12). For this observe that

$$\frac{\mathrm{d}}{\mathrm{da}}\left(\frac{1+a\cos x}{1+a}\right) = \frac{\cos x - 1}{(1+a)^2} \leqslant 0 \text{ for all } x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \text{ and } a \in \left(0,\frac{1}{2}\right].$$

Thus the function

$$a \mapsto \frac{1+a\cos x}{1+a}$$

is decreasing on $(0,1/2]$, and hence the asserted result follows. Secondly observe that when $|x| \leqslant x_3 \simeq 0.7197987821\ldots$, inequality (11) is better than the left hand side of (12) for all $a \in (0,1/2]$. This is justified by the following inequality

$$\frac{1+\cos x}{2} - \frac{2(1+a\cos x)}{\pi} \geqslant \frac{1+\cos x}{2} - \frac{2}{\pi}\left(1+\frac{\cos x}{2}\right) \geqslant 0,$$

where $|x| \leqslant x_3$ and x_3 is the largest root of $4-\pi = (\pi-2)\cos x$. Thus a slightly improvement of (13) is the following result. Since the proofs of these inequalities go along the lines introduced in [8], they are not included in this paper.

THEOREM 14. *If $c \geqslant 0$ and $\kappa \geqslant 1/2$, then we have*

$$\frac{1+2\kappa\lambda_p(x)}{2\kappa+1} \leqslant \lambda_{p+1}(x) \leqslant \frac{1+\kappa\lambda_p(x)}{\kappa+1} \textit{ for all } x\sqrt{c} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]. \tag{15}$$

In what follows let us discuss the hyperbolic counterpart of inequality (11). As we can see the well known Tchebysheff integral inequality is also useful here. For the reader's convenience let us recall this inequality. If $f,g : [a,b] \to \mathbb{R}$ are integrable functions, both increasing or both decreasing and $p : [a,b] \to \mathbb{R}$ is a positive integrable function, then

$$\int_a^b p(t)f(t)\,\mathrm{d}t \int_a^b p(t)g(t)\,\mathrm{d}t \leqslant \int_a^b p(t)\,\mathrm{d}t \int_a^b p(t)f(t)g(t)\,\mathrm{d}t\,. \tag{16}$$

Note that if one of the functions f or g is decreasing and the other is increasing, then (16) is reversed. Using the same idea as in the proof of inequality (11), putting $p(t)=1$, $f(t)=\sinh t$, $g(t)=t$, $t \in [a,b]=[0,x]$, $x \in [0,\infty)$ in (16), we have

$$\int_0^x \sinh t\,\mathrm{d}t \int_0^x t\,\mathrm{d}t \leqslant \int_0^x \mathrm{d}t \int_0^x t\sinh t\,\mathrm{d}t\,.$$

A direct calculation yields

$$\frac{\sinh x}{x} \leqslant \frac{1+\cosh x}{2} \Leftrightarrow \mathscr{I}_{1/2}(x) \leqslant \frac{1+\mathscr{I}_{-1/2}(x)}{2} \text{ for all } x \in \mathbb{R}. \tag{17}$$

Looking for a generalization of (17) we obtain the following result.

THEOREM 18. *If* $c \leqslant 0$ *and* $\kappa \geqslant 1/2$, *then we have*

$$\lambda_{p+1}(x) \leqslant \frac{1+2\kappa\lambda_p(x)}{2\kappa+1} \text{ for all } x \in \mathbb{R}. \tag{19}$$

Proof. As a generalization of (16) bellow, recently Sz. András and Á. Baricz proved [4, Lemma 1] that if $c, x \in \mathbb{R}$ and $2p > 2q > -(b+1)$, then

$$\lambda_p(x) = \int_0^1 \lambda_q(tx) \frac{2t^{2q+b}(1-t^2)^{p-q-1}}{B\left(q+\frac{b+1}{2}, p-q\right)} \, \mathrm{d}t. \tag{20}$$

From this it follows that if $p > q > -1$, then

$$\mathscr{I}_p(x) = \frac{2}{B(q+1, p-q)} \int_0^1 \mathscr{I}_q(tx) t^{2q+1}(1-t^2)^{p-q-1} \, \mathrm{d}t, \tag{21}$$

and in particular taking $q = -1/2$ (changing p with $p+1$ and taking $q = 1/2$ respectively), we get that for all $p > -1/2$ and $x \in \mathbb{R}$

$$\mathscr{I}_p(x) = \frac{2}{B\left(p+\frac{1}{2}, \frac{1}{2}\right)} \int_0^{\pi/2} \mathscr{I}_{-1/2}(x \sin\theta) \cos^{2p}\theta \, \mathrm{d}\theta, \tag{22}$$

$$\mathscr{I}_{p+1}(x) = \frac{2}{B\left(p+\frac{1}{2}, \frac{3}{2}\right)} \int_0^{\pi/2} \mathscr{I}_{1/2}(x \sin\theta) \sin^2\theta \cos^{2p}\theta \, \mathrm{d}\theta. \tag{23}$$

Now changing in (17) x with $x\sin\theta$ and multiplying both sides of (17) with $\sin^2\theta\cos^{2p}\theta$, after integration we obtain

$$\mathscr{I}_{p+1}(x) \leqslant \frac{1+2(p+1)\mathscr{I}_p(x)}{2p+3}$$

for all $p \geqslant -1/2$ and $x \in \mathbb{R}$. Finally observe that $\mathscr{I}_{\kappa-1}(x\sqrt{-c}) = \lambda_p(x)$, thus the proof is complete. □

REMARK 4.15. Note that recently the author proved [8, Corollary 2.18] that if $c \in [0,1]$ and $\kappa \geqslant 1/2$, then $\lambda_{p+1}(x) \geqslant \lambda_p(x)$ for all $x \in [-\pi, \pi]$. The left hand side of inequality (15) when $c \in [0,1]$, $\kappa \geqslant 1/2$ and $x \in [-\pi/2, \pi/2]$ provides an improvement of the above mentioned inequality, since under the same assumptions

$$\lambda_{p+1}(x) \geqslant \frac{1+2\kappa\lambda_p(x)}{2\kappa+1} \geqslant \lambda_p(x).$$

Here we used the fact that $\lambda_p(x) \leqslant 1$ under hypothesis. For this let us recall the following integral representation formula obtained by Á. Baricz and E. Neuman [9, Lemma 2.1]

$$\lambda_p(x) = \begin{cases} \dfrac{2}{B\left(p+\frac{b}{2}, \frac{1}{2}\right)} \displaystyle\int_0^1 (1-t^2)^{p+\frac{b-2}{2}} \cos(tx\sqrt{c}) \, \mathrm{d}t, & c \geqslant 0 \\ \dfrac{2}{B\left(p+\frac{b}{2}, \frac{1}{2}\right)} \displaystyle\int_0^1 (1-t^2)^{p+\frac{b-2}{2}} \cosh(tx\sqrt{-c}) \, \mathrm{d}t, & c \leqslant 0, \end{cases} \tag{16}$$

which is valid for all $x \in \mathbb{R}$ and $\kappa > 1/2$. Since

$$2\int_0^1 (1-t^2)^{p+\frac{b-2}{2}}\,\mathrm{dt} = B\left(p+\frac{b}{2},\frac{1}{2}\right)$$

it follows that $\lambda_p(x) \leqslant 1$ when $c \geqslant 0$, $\kappa > 1/2$ and $x \in \mathbb{R}$. Because $\mathscr{J}_{-1/2}(x) = \cos x \leqslant 1$ for all $x \in \mathbb{R}$, from (9) it follows that $\mathscr{J}_p(x) \leqslant 1$ for all $p \geqslant -1/2$. Now using again the formula $\mathscr{J}_{\kappa-1}(x\sqrt{c}) = \lambda_p(x)$, we obtain $\lambda_p(x) \leqslant 1$ for all $c \geqslant 0$, $\kappa \geqslant 1/2$ and $x \in \mathbb{R}$.

Following the above argument it is clear that $\lambda_p(x) \geqslant 1$ for all $c \leqslant 0$, $\kappa \geqslant 1/2$ and $x \in \mathbb{R}$. This means that inequality (19) improves the inequality $\lambda_{p+1}(x) \leqslant \lambda_p(x)$, where $c \leqslant 0$, $\kappa > 0$ and $x \in \mathbb{R}$, obtained by the author [8, Corollary 2.18].

An other immediate application of the integral representation formula (16) is the following result.

THEOREM 17. *If $c \leqslant 0$ and $\kappa \geqslant 1$, then $\lambda_{p+1}(x) \geqslant \sqrt{\lambda_p(x)}$ for all $x \in \mathbb{R}$. Moreover if $c \in [0,1]$ and $\kappa \in [1/2,1]$, then $\lambda_{p+1}(x) \geqslant \sqrt{\lambda_p(x)}$ also holds for all $x \in [-\pi/2,\pi/2]$.*

Proof. Let us consider the function $\varphi_5(x) = \lambda_{p+1}^2(x) - \lambda_p(x)$. Since this function is even, it is enough to show the required inequality for $x \geqslant 0$ and $x \in [0,\pi/2]$, respectively. Applying (9) for p and $p+1$, we obtain

$$2\varphi_5'(x) = (-c)x\lambda_{p+1}(x)\left[\frac{2\lambda_{p+2}(x)}{\kappa+1} - \frac{1}{\kappa}\right].$$

Now if $c \leqslant 0$ and $\kappa \geqslant 1$, then in view of Remark 4.15 clearly $\lambda_{p+2}(x) \geqslant 1 \geqslant (\kappa+1)/(2\kappa)$, and thus φ_5 is increasing. Hence $\varphi_5(x) \geqslant \varphi_5(0) = 0$. Finally suppose that $c \in [0,1]$ and $\kappa \in [1/2,1]$. From (12) it follows that $\lambda_{p+1}(x) \geqslant 0$, provided $x \in [-\pi/2,\pi/2]$. Now using again Remark 4.15 one has $\lambda_{p+2}(x) \leqslant 1 \leqslant (\kappa+1)/(2\kappa)$, and thus φ_5 is increasing. This completes the proof. □

REMARK 4.17. Taking $b = 1$, $c = 1$ and $p = -1/2$ in Theorem 17 we obtain the inequality $\sin x \geqslant x\sqrt{\cos x}$, $x \in [-\pi/2,\pi/2]$, which was established by J. Sándor [25].

5. Extensions of Jordan's inequality

The following inequality is known as Jordan's inequality [18, p. 33]

$$1 \geqslant \frac{\sin x}{x} \geqslant \frac{2}{\pi} \text{ for all } x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]. \tag{18}$$

It has been studied by several mathematicians in order to sharpen this basic analytic inequality. In this section we present two recent results related to Jordan's inequality and we extend them to generalized Bessel functions, in order to obtain other lower and upper bounds for the function λ_p. Recall that Redheffer's inequality (10) and Jordan's inequality (18) do not imply each other. Recently, J. Sándor [24, 25, 26] proved that the function $\mathscr{J}_{1/2}(x) = (\sin x)/x$ is concave on $[0, \pi/2]$ and from this he deduced the following improvement of Jordan's inequality:

$$\frac{2}{\pi} + \frac{2}{\pi^2}(\pi - 2x) \geqslant \frac{\sin x}{x} \geqslant \frac{2}{\pi} + \frac{\pi - 2}{\pi^2}(\pi - 2x) \text{ for all } x \in \left[0, \frac{\pi}{2}\right]. \quad (19)$$

J. Sándor's idea was that, since $\mathscr{J}_{1/2}$ is concave, its graph lies above the line segment joining the points $(0,1)$ and $(\pi/2, 2/\pi)$ on the graph of $\mathscr{J}_{1/2}$ on $[0, \pi/2]$.

From this follows the right hand side of (19). Now from the left hand side just consider the tangent line to $\mathscr{J}_{1/2}$ at the point $(\pi/2, 2/\pi)$, which line lies above the graph of $\mathscr{J}_{1/2}$ on $[0, \pi/2]$. The inequality (19) was recently also proved by X. Zhang, G. Wang and Y. Chu [28] using the monotone form of l'Hospital's rule established by G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen [3] (see also their book [2]). There is an other improvement of Jordan's inequality proved by F. Qi and Q. D. Hao [22] (see also [21, p. 522]) using the intricate technique of calculus, namely

$$\frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2) \geqslant \frac{\sin x}{x} \geqslant \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \text{ for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \quad (20)$$

This new refinement of Jordan's inequality was rediscovered too by X. Zhang, G. Wang and Y. Chu [28] using also the monotone form of l'Hospital's rule. The right hand side of (20) was also proved by L. Debnath and C. J. Zhao [12] using a completely different method. Moreover L. Zhu [29] using also the monotone form of l'Hospital's rule extended (20) in the following way for $-\pi/2 \leqslant x \leqslant r \leqslant \pi/2$

$$\frac{\sin r}{r} + \frac{r - \sin r}{r^3}(r^2 - x^2) \geqslant \frac{\sin x}{x} \geqslant \frac{\sin r}{r} + \frac{\sin r - r\cos r}{2r^3}(r^2 - x^2). \quad (21)$$

For these, and related details, see the survey papers of J. Sándor [24] and F. Qi [20]. It is known that inequalities (19) and (20) cannot be compared on the whole interval $[0, \pi/2]$ (see the discussion bellow). Observe that (11) in particular case for $p = 1/2$ becomes the following:

$$\varphi_6(x) := \frac{2}{9}\left[2 + \frac{5}{2}\cos\left(\sqrt{\frac{3}{5}}x\right)\right] \geqslant \frac{\sin x}{x} \geqslant \cos\left(\frac{x}{\sqrt{3}}\right) \text{ for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \quad (22)$$

This result of E. Neuman, which is actually a refinement of Jordan's inequality (18), does not appear in any paper related to sharpenings of Jordan's inequality, even if direct computations and numerical experiments in Derive6 show the followings:

(1) The right hand side of (22) is better than the right hand side of (20) for all $x \in [-x_4, x_4]$, where $x_4 \simeq 1.204850991\ldots$ is the root of the equation $\cos(x/\sqrt{3}) = 3/\pi - 4x^2/\pi^3$ on $[0, \pi/2]$. The situation is reversed when $x \in [-\pi/2, -x_4]$ or $x \in [x_4, \pi/2]$.

(2) The right hand side of (22) is better than the right hand side of (19) for all $x \in [0, x_5]$, where $x_5 \simeq 1.475028163\ldots$ is the root of the equation $\cos(x/\sqrt{3}) = 1 - 2(\pi - 2)x/\pi^2$ on $[0, \pi/2]$. Further the situation is reversed when $x \in [x_5, \pi/2]$.

(3) The right hand side of (20) is better than the right hand side of (19) for all $x \in [\pi(\pi-3)/2, \pi/2]$. Moreover the situation is reversed for all $x \in [0, \pi(\pi-3)/2]$. Note that $\pi(\pi-3)/2 \simeq 0.2224132208\ldots$.

(4) If $x \in [-x_6, x_6]$, then $\cos(x/\sqrt{3}) \geqslant 2/\pi$, where $x_6 \simeq 1.525398501\ldots$ is the root of the equation $\cos(x/\sqrt{3}) = 2/\pi$. When $x \in [-\pi/2, -x_6]$ or $x \in [x_6, \pi/2]$ the above inequality is reversed. Note that $\pi/2 \simeq 1.570796327\ldots$.

(5) The left hand side of (22) is better than the left hand side of (20) for all $x \in [-x_7, x_7]$, where $x_7 \simeq 1.563220278\ldots$ is the root of the equation $\varphi_6(x) = 1-4(\pi-2)x^2/\pi^3$ on $[0, \pi/2]$. The situation is reversed when $x \in [-\pi/2, -x_7]$ or $x \in [x_7, \pi/2]$.

(6) The left hand side of (22) is better than the left hand side of (19) for all $x \in [0, x_8]$, where $x_8 \simeq 1.497945837\ldots$ is the root of the equation $\varphi_6(x) = 4(\pi-x)/\pi^2$ on $[0, \pi/2]$. The situation is reversed when $x \in [x_8, \pi/2]$.

(7) The left hand side of (20) is better than the left hand side of (19) for all $x \in [0, (\pi/2)(4-\pi)/(\pi-2)]$. Note that $(\pi/2)(4-\pi)/(\pi-2) \simeq 1.181142066\ldots$. Further the situation is reversed when $x \in [(\pi/2)(4-\pi)/(\pi-2), \pi/2]$.

(8) If $x \in [-\pi/2, \pi/2]$, then $\varphi_6(x) \leqslant 1$.

From the above discussion follows that it is worth extending inequalities (19) and (20) for the generalized and normalized Bessel functions. Before we state our first main result in this section let us recall the monotone form of the well-known l'Hospital rule [3]:

LEMMA 23. *For $\alpha, \beta \in \mathbb{R}$ let $f, g : [\alpha, \beta] \to \mathbb{R}$ be continuous on $[\alpha, \beta]$, and differentiable on (α, β). Further let $g'(x) \neq 0$ for all $x \in (\alpha, \beta)$. If f'/g' is (strictly) increasing (decreasing) on (α, β), then so are*

$$x \mapsto \frac{f(x) - f(\alpha)}{g(x) - g(\alpha)} \quad \text{and} \quad x \mapsto \frac{f(x) - f(\beta)}{g(x) - g(\beta)}.$$

Our first main result in this section reads as follows.

THEOREM 24. *The following assertions are true:*

(1) *If $\kappa \geqslant 1/2$ and $c \in [0,1]$, then for all $x \in [0, \pi/2]$ one has*

$$\begin{aligned} \lambda_p\left(\frac{\pi}{2}\right) + \left[1 - \lambda_p\left(\frac{\pi}{2}\right)\right]\frac{\pi - 2x}{\pi} &\leqslant \lambda_p(x) \\ &\leqslant \lambda_p\left(\frac{\pi}{2}\right) + \left[\left(\frac{c\pi}{2\kappa}\right)\lambda_{p+1}\left(\frac{\pi}{2}\right)\right]\frac{\pi - 2x}{\pi}. \end{aligned} \tag{5.8}$$

(2) *If $\kappa > 0$ and $c \in [0,1]$, then for all $x \in [-\pi/2, \pi/2]$ one has*

$$\begin{aligned} \lambda_p\left(\frac{\pi}{2}\right) + \left[\left(\frac{c}{4\kappa}\right)\lambda_{p+1}\left(\frac{\pi}{2}\right)\right]\frac{\pi^2 - 4x^2}{4} &\leqslant \lambda_p(x) \\ &\leqslant \lambda_p\left(\frac{\pi}{2}\right) + \left[1 - \lambda_p\left(\frac{\pi}{2}\right)\right]\frac{\pi^2 - 4x^2}{\pi^2}. \end{aligned} \tag{5.9}$$

Proof. (1) Since if $\kappa \geqslant 1/2$ and $c \in [0,1]$, then for all $x \in [0, \pi/2]$ the function λ_p is concave [8, Corollary 2.18], it follows that its graph lies above the line segment joining the points $(0,1)$ and $(\pi/2, 2/\pi)$ on the graph of λ_p on $[0,\pi/2]$. This property implies the left hand side of (5.8). For the right hand side it suffices to consider the tangent line to λ_p at the point $(\pi/2,2/\pi)$, which lies above the graph of λ_p on $[0,\pi/2]$.

(2) Since the function λ_p is even, clearly it is enough to show (5.9) for $x \in [0,\pi/2]$. Let us consider the functions $\varphi_7, \varphi_8 : [0,\pi/2] \to [0,\infty)$, defined by $\varphi_7(x) = \lambda_p(x) - \lambda_p(\pi/2)$ and $\varphi_8(x) = \pi^2/4 - x^2$. From (9) we obtain $\varphi_7'(x)/\varphi_8'(x) = [c\lambda_{p+1}(x)]/(4\kappa)$. Because if $c \in [0,1]$ and $\kappa \geqslant 1/2$ the function λ_p is decreasing on $[0,\pi/2]$ (see [8, Corollary 2.18]), it follows that φ_7'/φ_8' is decreasing on $[0,\pi/2]$. Clearly $\varphi_7(\pi/2) = \varphi_8(\pi/2) = 0$, thus from monotone form of l'Hospital's rule, i.e. Lemma 23 we get that φ_7/φ_8 is decreasing too on $[0,\pi/2]$. All that remains to show is that from l'Hospital's rule

$$\lim_{x\to\frac{\pi}{2}} \frac{\varphi_7(x)}{\varphi_8(x)} = \lim_{x\to\frac{\pi}{2}} \frac{\varphi_7'(x)}{\varphi_8'(x)} = \lim_{x\to\frac{\pi}{2}} \left(\frac{c}{4\kappa}\right)\lambda_{p+1}(x) = \left(\frac{c}{4\kappa}\right)\lambda_{p+1}\left(\frac{\pi}{2}\right).$$

Now the inequality (5.9) follows from the monotonicity and the limiting values of φ_7/φ_8. □

REMARK 5.10. Putting $c=1$, $b=1$ and $p=-1/2$ in (5.8) we get

$$1-\frac{2}{\pi}x \leqslant \cos x \leqslant 2\left(1-\frac{2}{\pi}x\right) \text{ for all } x \in \left[0,\frac{\pi}{2}\right]. \tag{11}$$

The left hand side of (11) is known as Kober's inequality [14]. Taking $c=1$, $b=1$ and $p=1/2$ in (5.8) (in (5.9) respectively) we reobtain (19) ((20) respectively). Observe that changing x with $\pi/2-x$ in (19) we obtain the following inequality

$$1-\frac{2}{\pi}x+\frac{\pi-2}{\pi^2}x(\pi-2x) \leqslant \cos x \leqslant 1-\frac{2}{\pi}x+\frac{2}{\pi^2}x(\pi-2x) \text{ for all } x \in \left[0,\frac{\pi}{2}\right], \tag{12}$$

which was proved by F. Qi and Q. D. Hao [22], X. Zhang, G. Wang and Y. Chu [28] using different methods. Now taking $c=1$, $b=1$ and $p=-1/2$ in (5.9) we obtain

$$\frac{\pi}{4}-\frac{1}{\pi}x^2 \leqslant \cos x \leqslant 1-\frac{4}{\pi^2}x^2 \text{ for all } x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]. \tag{13}$$

Straightforward simplifications and easy computations show that the right hand side of (13) is exactly the right hand side of (12), but the left hand side of (12) (which is a refinement of Kober's inequality) is better than the left hand side of (13) for all $x \in [0,\pi/2]$. Further the right hand side of (13) is better than the right hand side of (11) for all $x \in [0,\pi/2]$.

Finally note that (5.8) can be proved also using Lemma 23. Just consider the functions φ_7 and $\varphi_9 : [0,\pi/2] \to [0,\infty)$, defined by $\varphi_9(x) = \pi/2 - x$. We know that if $c \in [0,1]$ and $\kappa \geqslant 1/2$, then λ_p is concave on $[0,\pi/2]$, thus $x \mapsto \varphi_7'(x)/\varphi_9'(x) = -\lambda_p'(x)$ is increasing on $[0,\pi/2]$. Application of Lemma 23 gives that φ_7/φ_9 is increasing too on $[0,\pi/2]$, thus the required inequality (5.8) follows.

We end this section with the following extension of (21), which provides a generalization of (5.9). As we can see the result of L. Zhu [29] is in fact a typical result for

Bessel functions. Since the proofs of the next inequalities go along the lines introduced in the proof of (5.9), we state the following result without proof.

THEOREM 14. *If* $\kappa > 0$ *and* $c \in [0,1]$, *then for all* $-\pi/2 \leqslant x \leqslant r \leqslant \pi/2$ *we have*

$$\lambda_p(r) + \left[\left(\frac{c}{4\kappa}\right)\lambda_{p+1}(r)\right](r^2 - x^2) \leqslant \lambda_p(x) \leqslant \lambda_p(r) + \left[\frac{1-\lambda_p(r)}{r^2}\right](r^2 - x^2). \quad (15)$$

Moreover if $\kappa > 0$ *and* $c \leqslant 0$, *then* (15) *holds for all* $-\infty < x \leqslant r < \infty$.

REMARK 5.16. Choosing $c = 1$, $b = 1$ and $p = 1/2$ in (15) we obtain (21). Analogously taking $c = -1$, $b = 1$ and $p = 1/2$ in (15) we obtain the hyperbolic counterpart of (21)

$$\frac{\sinh r}{r} + \frac{r - \sinh r}{r^3}(r^2 - x^2) \geqslant \frac{\sinh x}{x} \geqslant \frac{\sinh r}{r} + \frac{\sinh r - r\cosh r}{2r^3}(r^2 - x^2),$$

whenever $-\infty < x \leqslant r < \infty$. Here we used the fact that

$$\mathscr{J}_{3/2}(x) = 3\left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right) \text{ and } \mathscr{I}_{3/2}(x) = -3\left(\frac{\sinh x}{x^3} - \frac{\cosh x}{x^2}\right).$$

6. The sine and hyperbolic sine integral

Let

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{dt} \text{ and } \mathrm{Shi}(x) = \int_0^x \frac{\sinh t}{t}\,\mathrm{dt}$$

be the sine and hyperbolic sine integral, which play an important role in various topics of Fourier analysis [30]. S. Koumandos [15] among other things recently proved that the sine integral is sub-additive on $[0,\infty)$. In this section we prove that hyperbolic sine integral is super-additive on $[0,\infty)$, moreover we show that these properties also holds for the following integral, which generalize the sine and hyperbolic sine integrals. For $c \in \mathbb{R}$ and $\kappa > 0$ let us consider

$$\varsigma_p(x) = \int_0^x \lambda_p(t)\,\mathrm{dt} = \sum_{n\geqslant 0} \frac{(-c/4)^n}{(\kappa)_n n!} \cdot \frac{x^{2n+1}}{2n+1}. \quad (17)$$

Clearly when $b = 1$, $c = 1$, then $\varsigma_{-1/2}(x) = \sin x$, $\varsigma_{1/2}(x) = \mathrm{Si}(x)$, and when $b = 1$, $c = -1$, then $\varsigma_{-1/2}(x) = \sinh x$, $\varsigma_{1/2}(x) = \mathrm{Shi}(x)$. Our main result in this section reads as follows.

THEOREM 18. *If* $c \in [0,1]$ *and* $\kappa \geqslant 1/2$, *then* ς_p *is sub-additive on* $[0,\pi]$. *Moreover if* $c \leqslant 0$ *and* $\kappa > 0$, *then* ς_p *is super-additive on* $[0,\infty)$.

Proof. It is well-known that if the function f, where $f(0) = 0$ has a continuous derivative on $[0,\infty)$ and the function $x \mapsto f(x)/x$ is decreasing (increasing), then f is sub-additive (super-additive). An easy calculation from (17) yields

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\varsigma_p(x)}{x}\right) = \frac{x\lambda_p(x) - \varsigma_p(x)}{x^2}. \quad (19)$$

Let us consider the function $\varphi_{10}(x) = x\lambda_p(x) - \varsigma_p(x)$. It follows that $\varphi_{10}'(x) = x\lambda_p'(x)$. If $c \in [0,1]$ and $\kappa \geqslant 1/2$, then it is known [8, Corollary 2.18] that λ_p is decreasing on $[0,\pi]$, thus φ_{10} is decreasing too on $[0,\pi]$. Hence $\varphi_{10}(x) \leqslant \varphi_{10}(0) = 0$ and therefore from (19) we obtain that $x \mapsto \varsigma_p(x)/x$ is decreasing on $[0,\pi]$. From this the sub-additivity property of ς_p follows.

If $c \leqslant 0$ and $\kappa > 0$, then from the series representation of λ_p, clearly λ_p has positive coefficients, and consequently is increasing on $[0,\infty)$. Thus the functions φ_{10} and $x \mapsto \varsigma_p(x)/x$ are also increasing on $[0,\infty)$, which completes the proof. □

REMARK 6.4. We note that using Tchebysheff integral inequality (16) we can deduce other inequalities involving ς_p and λ_p. For example choosing $p(t) = 1$, $f(t) = \lambda_{p+1}(t)$ ($f(t) = t\lambda_{p+1}(t)$ respectively) and $g(t) = t$, $t \in [a,b] = [0,x]$, $x \in [0,\infty)$ in (16), we have that if $\kappa > 0$ and $c < 0$, then

$$(-c)x\varsigma_{p+1}(x) \leqslant (4\kappa)\lambda_p(x) \text{ and } 2\varsigma_p(x) \leqslant x\lambda_p(x) \text{ for all } x \geqslant 0.$$

Here we used the differentiation formula (9) and the fact that when $\kappa > 0$ and $c < 0$ the functions $x \mapsto \lambda_p(x)$ and $x \mapsto x\lambda_p(x)$ are increasing on $[0,\infty)$.

In a recent paper H. Alzer and S. Koumandos [1] established sub- and super-additive properties of Fejér's sine polynomial $\sum_{m=1}^{n}(\sin mx)/m$. We note that if we consider the sum $\sum_{m=1}^{n} x\lambda_p(mx)$, then it is easy to verify that it is sub-additive (super-additive respectively) on $[0,\pi/m]$ (on $[0,\infty)$ respectively) if $c \in [0,1]$ and $\kappa \geqslant 1/2$ (if $c \leqslant 0$ and $\kappa > 0$ respectively).

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