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INTERNAL CUBIC SYMMETRIC FORMS IN A SMALL NUMBER OF VARIABLES

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Abstract. We consider means of the form $\mu = \sqrt[3]{f}$, where f is a cubic symmetric form in n variables, and we show that if $n \leqslant 4$ and μ is internal on the points $(x_1, \cdots, x_n)$ where $x_i = 0$ or 1, then μ is internal for all points $(x_1, \cdots, x_n)$ with $x_i \geqslant 0$ for all i. We highlight the similarity between this internality problem and the parallel problem pertaining to copositive symmetric cubic forms.

1. Introduction

An n-*dimensional mean*, or simply a *mean*, is usually defined to be a function $\mu : [0,\infty)^n \longrightarrow \mathbb{R}$ that is *internal* in the sense that it satisfies the internality property

$$\min\{x_1, \cdots, x_n\} \leqslant \mu(x_1, \cdots, x_n) \leqslant \max\{x_1, \cdots, x_n\} \tag{1}$$

for all $x_1, \cdots, x_n \geqslant 0$; see [4] and [3, Chapter 8.7, page 266]. All means considered in this note are also *symmetric* and *homogeneous* in the sense that

$$\begin{aligned} \mu(x_{\sigma(1)}, \cdots, x_{\sigma(n)}) &= \mu(x_1, \cdots, x_n) \text{ for all permutations } \sigma \text{ of } x_1, \cdots, x_n, \\ \mu(\lambda x_1, \cdots, \lambda x_n) &= \lambda \mu(x_1, \cdots, x_n) \text{ for all } \lambda > 0. \end{aligned}$$

Means of the form Q/S and $\sqrt{Q}$, where $S = x_1 + \cdots + x_n$ and Q is a (real) symmetric quadratic form in $x_1, \cdots, x_n$, are characterized in [6, Theorems 3 and 5], and means of the more general form $\alpha S \pm \sqrt{Q}$ are characterized in [1, Theorem 3]. In all of these cases, the internality property (1) is reduced to a set of simple conditions on α and on the coefficients of Q. Using [6, Theorem 5] and [1, Theorem 3], one can easily check that for the functions Q/S and $\alpha S \pm \sqrt{Q}$ to be internal, it is sufficient that they satisfy the internality property (1) at the two test n-tuples $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(n)}$ only, where $\mathbf{v}^{(j)}$ is defined by

$$\mathbf{v}^{(j)} = (\overbrace{1, \cdots, 1}^{j}, \overbrace{0, \cdots, 0}^{n-j}). \tag{2}$$

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On the other hand, 2-dimensional means of the form $C(x,y)/Q(x,y)$, where C and Q are symmetric forms of degree 3 and 2, respectively, are characterized in [2, Theorem 1 (iii)] and it is seen there that internality at the test tuples $(1,1)$ and $(1,0)$ does not guarantee internality for all $x,y \geqslant 0$; see [2, Section 6].

In this note, we consider functions of the form $\mu = \sqrt[3]{f(x_1,\cdots,x_n)}$, where f is a symmetric cubic form, and we show that if $n \leqslant 4$, then such a μ is internal if it satisfies the internality property (1) for the test tuples $\mathbf{v}^{(1)},\cdots,\mathbf{v}^{(n)}$ given in (2). The proof uses a very weak theorem of calculus and does not generalize to $n \geqslant 5$. However, the authors believe that the same result holds for all n, and discuss a similar situation that pertains to co-positive forms and that supports this belief.

2. A weak lemma from calculus

Let $\mathbb{R}[x]$ be the ring of polynomials in one variable x and with real coefficients, and let $f(x) \in \mathbb{R}[x]$. If all the coefficients of f are non-negative, then $f(x) \geqslant 0$ for all $x \geqslant 0$. Equivalently, if the derivatives $f^{(j)}(a)$ are non-negative for all $j \geqslant 0$, then $f(x) \geqslant 0$ for all $x \geqslant a$. Repeated application of this simple fact yields Theorem 2 which is the main result of this paper. This theorem states that if the symmetric cubic form $f(x_1,\cdots,x_n)$ satisfies the internality property (1) for the test tuples $\mathbf{v}^{(1)},\cdots,\mathbf{v}^{(n)}$, and if $n \leqslant 4$, then f is internal. To prove this theorem, we assume that the symmetric cubic form $f(x_1,\cdots,x_n)$ satisfies (1) for $\mathbf{v}^{(1)},\cdots,\mathbf{v}^{(n)}$, and fixing $x_1 \leqslant \cdots \leqslant x_{n-1}$, we let

$$F(x_n) = x_n^3 - f(x_1,\cdots,x_n), \quad G(x_n) = f(x_1,\cdots,x_n) - x_1^3.$$

We are to show that $F(x_n) \geqslant 0$ and $G(x_n \geqslant 0$ for all $x_n \geqslant x_{n-1}$. Interestingly, it turns out that if $n \leqslant 4$, then the derivatives $F^{(j)}$ and $G^{(j)}$ ($0 \leqslant j \leqslant 3$) are non-negative at $x_n = x_{n-1}$. This trivially implies that $F(x_n) \geqslant 0$ and $G(x_n) \geqslant 0$ for all $x_n \geqslant x_{n-1}$. It also turns out that if $n \geqslant 5$, then the internality of f at $\mathbf{v}^{(1)},\cdots,\mathbf{v}^{(n)}$ does not imply that $F^{(j)}$ and $G^{(j)}$ ($0 \leqslant j \leqslant 3$) are non-negative at $x_n = x_{n-1}$. However, the authors believe that no matter what n is, the internality of f at $\mathbf{v}^{(1)},\cdots,\mathbf{v}^{(n)}$ does imply that $F(x_n) \geqslant 0$ for all $x_n \geqslant x_{n-1}$. This belief is supported by examining the similar situation for co-positive symmetric cubic forms, where it is well-known that a symmetric cubic form $f(x_1,\cdots,x_n)$ is non-negative for all $x_j \geqslant 0$ if it is non-negative for $\mathbf{v}^{(1)},\cdots,\mathbf{v}^{(n)}$; see [5, Theorem 3.7] and [7]. However, fixing $x_1 \leqslant \cdots \leqslant x_{n-1}$ as before and letting $H(x_n) = f(x_1,\cdots,x_n)$, we see that internality at these test n-tuples implies that the derivatives $H^{(j)}$ ($0 \leqslant j \leqslant 3$) are non-negative at $x_n = x_{n-1}$ only if $n \leqslant 4$. In view of this, and in view of the extremely weak nature of our main tool, Lemma 1, the failure of our proof for $n \geqslant 5$ should not be discouraging. We thus emphasize that the real value of Theorem 2 is in its statement, and not in its proof, and we hope that this first step would motivate a better approach that works for all n.

To facilitate calculations and to prepare for a computer-assisted verification, we set

$$\begin{aligned} \Omega^{(n)} &= \{(x_1,\cdots,x_n) \in \mathbb{R}^n : 0 < x_1 \leqslant \cdots \leqslant x_n\}, \quad (3)\\ S^{(n)} &= \{(r_1,\cdots,r_t) \in \mathbb{Z}^t : 0 < r_1 \leqslant \cdots \leqslant r_t \leqslant n-1\}, \end{aligned}$$

and we define the operator $\Delta_s^{(n)} : \mathbb{R}[x_1,\dots,x_n] \to \mathbb{R}[x_1,\dots,x_n]$ for $s \in S^{(n)}$ inductively by

$$\Delta_s^{(n)} = \begin{cases} \text{the identity} & \text{if } s \text{ is the empty sequence} \\ \displaystyle\sum_{j=n-r+1}^{n} \partial/\partial x_j & \text{if } s \text{ is the singleton } (r), \\ \Delta_u^{(n)}\Delta_v^{(n)} & \text{if } s=(r_1,\cdots,r_{t+1}),\ v=(r_1,\cdots,r_t),\ u=(r_{t+1}). \end{cases}$$

It is clear that $\Delta_s^{(n)}(f)$ vanishes if $s=(r_1,\cdots,r_t)$ and $t > \deg f$.

LEMMA 1. *Let $f=f(x_1,\cdots,x_n)$ be a (not necessarily symmetric) form in the variables $x_1,\cdots,x_n$. If $\Delta_s^{(n)}(f)(1,\cdots,1) \geqslant 0$ for all $s \in S^{(n)}$, then $f(x_1,\cdots,x_n) \geqslant 0$ for all $(x_1,\cdots,x_n) \in \Omega^{(n)}$.*

Proof. If $n=1$, then $S^{(n)}$ consists of the empty sequence, and the assumption $f(1)=\Delta_\Phi^{(1)}(f)(1) \geqslant 0$ implies, by homogeneity, that $f(x_1) \geqslant 0$ for all $x_1 \geqslant 0$.

Suppose the lemma is true for $n-1$, and let $f=f(x_1,\cdots,x_n)$ be a form for which $\Delta_s^{(n)}(f)(1,\cdots,1) \geqslant 0$ for all $s \in S^{(n)}$. To prove that $f(x_1,\cdots,x_n) \geqslant 0$ for all $(x_1,\cdots,x_n) \in \Omega^{(n)}$, let

$$f_j(x_1,\cdots,x_{n-1}) = \frac{\partial^j f}{\partial x_n^j}(x_1,\cdots,x_{n-1},x_{n-1}).$$

Then it is sufficient to prove that $f_j(x_1,\cdots,x_{n-1}) \geqslant 0$ for all $j \geqslant 0$ and for all $(x_1,\cdots,x_{n-1}) \in \Omega^{(n-1)}$. By the inductive assumption, it is sufficient to prove that $\Delta_s^{(n-1)} f_j(1,\cdots,1) \geqslant 0$ for all $j \geqslant 0$ and for all $s \in S^{(n)}$ This follows immediately from the fact that if $s=(r_1,\cdots,r_t) \in S^{(n-1)}$, then

$$u = (\overbrace{1,\cdots,1}^{j}, r_1,\cdots,r_t) \in S^{(n)}$$

and

$$\Delta_s^{(n-1)}(f_j)(1,\cdots,1) = \Delta_u^{(n)}(f)(1,\cdots,1).$$

This completes the proof. □

3. The proof of the main theorem

We now use Lemma 1 to prove our main result.

THEOREM 2. *Let $f=f(x_1,\cdots,x_n)$ be a symmetric cubic form and suppose that $\sqrt[3]{f}$ has the internality property (1) at the test n-tuples $\mathbf{v}^{(1)},\cdots,\mathbf{v}^{(n)}$ given in (2). If $n \leqslant 4$, then $\sqrt[3]{f}$ is internal.*

Proof. It is easy to see that the vector space V_n of symmetric cubic forms in n variables is generated by the three forms

$$A = A_n = \sum_{j=1}^{n} x_j^3, \quad B = B_n = \left(\sum_{j=1}^{n} x_j^2\right)\left(\sum_{j=1}^{n} x_j\right), \quad C = C_n = \left(\sum_{j=1}^{n} x_j\right)^3. \qquad (4)$$

In fact, A, B, C form a basis of V_n when $n \geqslant 3$, and A, B form a basis of V_2. Thus a symmetric cubic form f is of the form

$$f = aA + bB + cC, \qquad (5)$$

where c may be assumed 0 if $n = 2$. Let

$$J_k = f(\mathbf{v}^{(k)}) = ak + bk^2 + ck^3.$$

Then the assumption that $\sqrt[3]{f}$ satisfies (1) for $\mathbf{v}^{(1)}, \cdots, \mathbf{v}^{(n)}$ means that

$$J_n = 1 \text{ and } 0 \leqslant J_k \leqslant 1 \text{ for } k = 1, \cdots, n-1.$$

We shall prove that these inequalities imply internality of $\sqrt[3]{f}$. In fact, letting $\mathbf{x}$ stand for $(x_1, \cdots, x_n)$, we shall prove the following statements:

1. The inequalities $J_k \geqslant 0$ for $k = 1, \cdots, n$ imply that $f(\mathbf{x}) \geqslant 0 \ \forall\, \mathbf{x} \in \Omega^{(n)}$.
2. The inequalities $J_n \geqslant 1$ and $J_k \geqslant 0$ for $k = 1, \cdots, n-1$ imply that $f(\mathbf{x}) \geqslant x_1^3 \ \forall\, \mathbf{x} \in \Omega^{(n)}$.
3. The inequalities $J_k \leqslant 1$ for $k = 1, \cdots, n$ imply that $f(\mathbf{x}) \leqslant x_n^3 \ \forall\, \mathbf{x} \in \Omega^{(n)}$.

Let $\Delta_s(h)(1, \cdots, 1)$ be denoted by $\delta_s(h)$. To prove the first claim, we show that $\delta_s(f) \geqslant 0$ for every $s \in S^{(n)}$ by expressing it as a positive linear combination of $J_1, \cdots, J_n$. This is done in the column $\delta_s(f)$ of the accompanying tables and is computer-verified. Note that f is as given in (5) and that c is taken to be 0 when $n = 2$. Thus we have proved that if f is non-negative for the test tuples (2), then it is non-negative for all non-negative tuples, a statement that is true for all n [5, Theorem 3.7].

The second claim follows immediately from observing that $\delta_s(f) \geqslant \delta_s(x_1^3)$ for every $s \in \Omega^{(n)}$. This is done by comparing the columns $\delta_s(f)$ and $\delta_s(x_1^3)$ and using the inequality $J_n \geqslant 1$.

Similarly, the third claim follows from observing that $\delta_s(f) \leqslant \delta_s(x_n^3)$ for every $s \in S^{(n)}$. Again, this is done by comparing the columns $\delta_s(f)$ and $\delta_s(x_n^3)$ and using the inequality $J_k \leqslant 1$ for every k.

$n = 2$ $J_2 = 2a + 4b,\ J_1 = a + b$			
s	$\delta_s(f)$	$\delta_s(x_n^3)$	$\delta_s(x_1^3)$
Φ	$2a + 4b = J_2$	1	1
1	$3a + 6b = (3/2)J_2$	3	0
11	$6a + 8b = 4J_1 + J_2$	6	0
111	$6a + 6b = 6J_1$	6	0

Table 1. $n = 2$

$n=3$ $J_3 = 3a+9b+27c,\ J_2 = 2a+4b+8c,\ J_1 = a+b+c$			
s	$\delta_s(f)$	$\delta_s(x_n^3)$	$\delta_s(x_1^3)$
Φ	$3a+9b+27c = J_3$	1	1
1	$3a+9b+27c = J_3$	3	0
2	$6a+18b+54c = 2J_3$	3	0
11	$6a+10b+18c = 2J_2+2J_1$	6	0
12	$6a+14b+36c = J_3+J_2+J_1$	6	0
22	$12a+28b+72c = 2(J_3+J_2+J_1)$	6	0
111	$6a+6b+6c = 6J_1$	6	0
112	$6a+8b+12c = J_2+4J_1$	6	0
122	$6a+12b+24c = 3J_2$	6	0
222	$12a+24b+48c = 6J_2$	6	0

Table 2. $n=3$

$n=4$ $J_4 = 4a+16b+64c,\ J_3 = 3a+9b+27c,\ J_2 = 2a+4b+8c,\ J_1 = a+b+c$			
s	$\delta_s(f)$	$\delta_s(x_n^3)$	$\delta_s(x_1^3)$
Φ	$4a+16b+64c = J_4$	1	1
1	$3a+12b+48c = (3/4)J_4$	3	0
2	$6a+24b+96c = (3/2)J_4$	3	0
3	$9a+36b+144c = (9/4)J_4$	3	0
11	$6a+12b+24c = 3J_2$	6	0
12	$6a+16b+48c = (1/2)J_4+2J_2$	6	0
13	$6a+20b+72c = J_4+J_2$	6	0
22	$12a+32b+96c = J_4+4J_2$	6	0
23	$12a+40b+144c = 2(J_4+J_2)$	6	0
33	$18a+60b+216c = 3(J_4+J_2)$	6	0
111	$6a+6b+6c = 6J_1$	6	0
112	$6a+8b+12c = J_2+4J_1$	6	0
113	$6a+10b+18c = 2J_2+2J_1$	6	0
122	$6a+12b+24c = 3J_2$	6	0
123	$6a+14b+36c = J_3+J_2+J_1$	6	0
133	$6a+18b+54c = 2J_3$	6	0
222	$12a+24b+48c = 6J_2$	6	0
223	$12a+28b+72c = 2(J_3+J_2+J_1)$	6	0
233	$12a+36b+108c = 4J_3$	6	0
333	$18a+54b+162c = 6J_3$	6	0

Table 3. $n=4$

□

4. Summary and concluding remarks

Let f be a symmetric cubic form in n variables $\mathbf{x} = (x_1, \cdots, x_n)$ and let $\mu = \sqrt[3]{f}$. Let $\mathbf{v}^{(1)}, \cdots, \mathbf{v}^{(n)}$ be the test tuples defined in (2).

We have shown that if μ satisfies the internality condition (1) for $\mathbf{v}^{(1)}, \cdots, \mathbf{v}^{(n)}$, and if $n \leqslant 4$, then μ satisfies (1) for all $\mathbf{x} \geqslant 0$. The proof we gave for $n \leqslant 4$ breaks down when $n \geqslant 5$.

Our proof for $n \leqslant 4$ was also used to answer the similar question whether

$$f(\mathbf{v}^{(i)}) \geqslant 0 \text{ for } 1 \leqslant i \leqslant n$$

implies that

$$f(\mathbf{x}) \geqslant 0 \text{ for all } \mathbf{x} \geqslant 0$$

and was shown to fail for $n \geqslant 5$. However, this question does have an affirmative answer for all n as shown in [5, Theorem 3.7] and [7]. We remark that the case $n \leqslant 3$ of this problem had already been established in [9], and that the case when f is quartic is investigated in [8].

The authors expect [10] to be useful for settling the problem described above.

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