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ON BEHAVIOUR OF THE RIESZ AND GENERALIZED RIESZ POTENTIALS AS ORDER TENDS TO ZERO

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Abstract. In this paper, we present the Riesz potentials I^{α} and the generalized Riesz potentials I_{ν}^{α} as the families of positive linear operators, depending on parameter $\alpha > 0$. We investigate their pointwise convergence and convergence in the norm as $\alpha \to 0$. We investigate also the order of approximation of these families and show in particular that the order of approximation at the Lipschitz points is independent from Lipschitz degree.

1. Introduction

The Riesz potentials I^{α} and the generalized Riesz potentials I_{ν}^{α} are defined in terms of Fourier and Fourier-Bessel transforms, F and F_{ν}, by the formulas

$$F\left(I^{\alpha}f\right)(x) = |x|^{-\alpha} F(f)(x)\,, \quad x \in \mathbb{R}^n\,,\ \alpha > 0\,, \tag{1.1}$$

$$F_{\nu}\left(I_{\nu}^{\alpha}f\right)(x) = |x|^{-\alpha} F_{\nu}(f)(x)\,, \quad x \in \mathbb{R}_{+}^{n}\,,\ \alpha > 0\,, \tag{1.2}$$

where the equalities are understood in the sense of distribution theory. These potentials are interpreted as a negative fractional powers of the Laplace operator $(-\Delta) = -\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$, and the singular Laplace-Bessel differential operator $(-\Delta_{\nu}) = -\left(\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + \frac{2\nu}{x_n}\frac{\partial}{\partial x_n}\right)$, respectively. The boundedness properties of these potentials and their inverses on the relevant $L_p\left(\mathbb{R}^n, dm\right)$ spaces were studied by many authors (see [13], [12], [11], [2], [4], [5] and references therein).

In this paper, we consider the potentials I^{α} and I_{ν}^{α} as families of linear positive operators depending on a parameter $\alpha > 0$, then we investigate the approximation properties of these families as α tends to zero. Note that the classical Riesz and Bessel kernels as approximations of the identity has been studied by T. Kurokawa [8]. (See Remark 3.5 at the end of Section 3).

The paper is organized as follows. In Section 2 we give background with the basic notations, definitions and auxiliary lemmas. The main results of the paper are given in

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Sections 3 and 4. In particular, in Section 3, it has been shown that $\lim_{\alpha\to 0}\left(I_\nu^\alpha f\right)(x) = f(x)$, if x is the "Lebesgue point" of $f \in L_{p,\nu}\left(\mathbb{R}_+^n, x_n^{2\nu}dx\right)$. The same statement is true also for the classical Riesz potentials. In Section 4 we investigate the order of approximation of a given function f by means of the Riesz potentials, using the modulus of continuity of f. We show also that the order of approximation at the Lipschitz points is independent from the Lipschitz degree of function f.

2. Preliminaries

Let $x = (x_1, \dots, x_{n-1}, x_n) \equiv (x', x_n) \in \mathbb{R}^n$; $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\}$. We denote by $L_{p,\nu} \equiv L_{p,\nu}\left(\mathbb{R}_+^n\right)$ the class of measurable functions f on $\mathbb{R}_+^n$ with the norm

$$\|f\|_{p,\nu} = \left(\int_{\mathbb{R}_+^n} |f(x)|^p x_n^{2\nu} dx\right)^{1/p} < \infty,$$

where $\nu > 0$ is a fixed parameter, $1 \leqslant p < \infty$ and $dx = dx_1...dx_n$. The notation $L_p \equiv L_p(\mathbb{R}^n)$ will be used for the Lebesgue space of functions with the norm $\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p}$. We set $S_+^{n-1} = \{x \in \mathbb{R}_+^n : |x| = 1\}$ and $\left|S_+^{n-1}\right| = \int_{S_+^{n-1}} \theta_n^{2\nu} d\theta$.

The Fourier-Bessel transform of the function $f : \mathbb{R}_+^n \to \mathbb{C}$ is defined by

$$\left(F_\nu f\right)(y) = \int_{\mathbb{R}_+^n} f(x)\, e^{-i x'\cdot y'} J_{\nu-\frac{1}{2}}(x_n y_n)\, x_n^{2\nu} dx, \quad y \in \mathbb{R}_+^n.$$

Here $x' \cdot y' = x_1 y_1 + \cdots + x_{n-1}y_{n-1}$, $J_\lambda(\tau) = 2^\lambda \Gamma(\lambda+1)\,\tau^{-\lambda}\,\mathscr{J}_\lambda(\tau)$, $\mathscr{J}_\lambda(\tau)$ is the Bessel function of the first kind. The Fourier-Bessel harmonic analysis is adopted to the generalized convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}_+^n} f(y)\left(T^y g(x)\right) y_n^{2\nu} dy, \; x \in \mathbb{R}_+^n,$$

where T^y is the generalized translation operator, acting according to the law

$$T^y g(x) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu)\,\Gamma\left(\frac{1}{2}\right)} \int_0^\pi g\left(x'-y', \sqrt{x_n^2 - 2x_n y_n \cos\alpha + y_n^2}\right) \sin^{2\nu-1}\alpha\, d\alpha \tag{2.1}$$

(see [9], [7], [16],[4], [5] and [1]). The translation operator (2.1) represents the ordinary (Euclidean) translation in $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and the generalized (Bessel) translation in x_n- variable. It is known that (see, e.g. [10])

$$\begin{aligned} &a) \quad \|T^y f\|_{p,\nu} \leqslant \|f\|_{p,\nu}, \;\; \forall y \in \mathbb{R}_+^n, \; 1 \leqslant p < \infty; \\ &b) \quad \|T^y f - f\|_{p,\nu} \to 0 \;\text{ as }\; |y| \to 0. \end{aligned} \tag{2.2}$$

By using (2.2)-a) and the Riesz-Thorin interpolation theorem, it is not difficult to prove the corresponding Young inequality for the generalized convolution:

$$\|f \otimes g\|_{r,\nu} \leqslant \|f\|_{p,\nu} \|g\|_{q,\nu}, \quad 1 \leqslant p,\ q,\ r \leqslant \infty,\ \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1. \tag{2.3}$$

Given a function $g : \mathbb{R}^n_+ \to \mathbb{C}$, we introduce the generalized maximal function

$$M_\nu g(x) = \sup_{r>0} \frac{1}{r^{n+2\nu}\omega(n,\ \nu)} \int_{B_r^+} |T^x g(y)|\, y_n^{2\nu} dy,$$

where

$$B_r^+ = \{y : y \in \mathbb{R}^n_+,\ |y| < r\} \text{ and } \omega(n,\ \nu) = \int_{B_1^+} y_n^{2\nu} dy.$$

LEMMA 2.1. *(See [14], [6]). Let $f \in L_{p,\nu}$. Then*

$$\|M_\nu f\|_{p,\nu} \leqslant c\, \|f\|_{p,\nu}, \quad 1 < p \leqslant \infty;$$

$$m\{x : |M_\nu f(x)| > \lambda\} \leqslant c\, \|f\|_{1,\nu} / \lambda, \quad \forall \lambda > 0,$$

where $m(E) = \int_E x_n^{2\nu} dx,\ \ E \subset \mathbb{R}^n_+.$

We now give the notion of "$d_\nu-$point" of $f : \mathbb{R}^n_+ \to \mathbb{C}$. Remind that a point $x \in \mathbb{R}^n$ is called $d-$point of function $f : \mathbb{R}^n \to \mathbb{C}$, if

$$\lim_{h\to 0} \frac{1}{h^n \Omega_n} \int_{|y-x|\leqslant h} f(y)\, dy = f(x), \ \ \textit{where } \ \Omega_n = \int_{|x|<1} dx.$$

Analogously we give the following

DEFINITION 2.2. A point $x \in \mathbb{R}^n_+$ is called $d_\nu-$point of function $f : \mathbb{R}^n_+ \to \mathbb{C}$, if

$$\lim_{r\to 0} \frac{1}{r^{n+2\nu}\omega(n,\ \nu)} \int_{B_r^+} \Big(T^y f(x) - f(x)\Big) y_n^{2\nu} dy = 0. \tag{2.4}$$

By making use of the Lemma 2.1 and relevant maximal function technique (see [13], p. 5–9) one can be prove that almost all points of $\mathbb{R}^n_+$ are the $d_\nu-$points of $f \in L_{p,\nu}$. Note that the definition of $d-$point associated with nonizotropic distance is given in [3].

The Riesz potentials $I^\alpha f$ and the generalized Riesz potentials $I^\alpha_\nu f$, initially defined in terms of Fourier and Fourier-Bessel transforms by (1.1) and (1.2) respectively, can be represented as the integral operators of convolution type. Namely,

$$(I^\alpha f)(x) = c_n(\alpha) \int_{\mathbb{R}^n} |y|^{\alpha-n} f(y-x)\, dy,$$

where

$$c_n(\alpha) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^{\alpha}\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}, \quad 0 < \alpha < n, \tag{2.5}$$

and

$$\left(I_\nu^\alpha f\right)(x) = c_{n,\nu}(\alpha) \int_{\mathbb{R}_+^n} |y|^{\alpha-n-2\nu} \left(T^y f(x)\right) y_n^{2\nu} dy,$$

where

$$c_{n,\nu}(\alpha) = \frac{\Gamma\left(\frac{n+2\nu-\alpha}{2}\right)}{2^{\alpha-1}\pi^{\frac{n-1}{2}}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}, \quad 0 < \alpha < n+2\nu. \tag{2.6}$$

The operator $I^\alpha f$ is well defined for $f \in L_p(\mathbb{R}^n)$, $1 \leqslant p < \infty$, if $0 < \alpha < \frac{n}{p}$, and the operator $I_\nu^\alpha f$ is well defined for $f \in L_{p,\nu}$, $1 \leqslant p < \infty$, if $0 < \alpha < \frac{n+2\nu}{p}$ (see, [13], [12], [4] and [5]).

LEMMA 2.3. *For any* $r > 0$

$$\lim_{\alpha\to 0} c_{n,\nu}(\alpha) \int_{B_r^+} |y|^{\alpha-n-2\nu} y_n^{2\nu} dy = 1. \tag{2.7}$$

Proof. Setting $S_+^{n-1} = \{y : y \in \mathbb{R}_+^n,\ |y| = 1\}$ and by passing to polar coordinates $y = \rho\theta,\ 0 < \rho < r,\ \theta = (\theta_1, \ldots, \theta_{n-1}, \theta_n) \in S_+^{n-1}$, we have

$$\int_{B_r^+} |y|^{\alpha-n-2\nu} y_n^{2\nu} dy = \frac{r^\alpha}{\alpha} \int_{S_+^{n-1}} \theta_n^{2\nu} d\theta = \frac{r^\alpha}{\alpha} \left|S_+^{n-1}\right|.$$

Since (see [7], p. 15)

$$\left|S_+^{n-1}\right| \overset{\text{def}}{=} \int_{S_+^{n-1}} \theta_n^{2\nu} d\theta = \pi^{\frac{(n-1)}{2}} \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma\left(\frac{n+2\nu}{2}\right)},$$

we have

$$\int_{B_r^+} |y|^{\alpha-n-2\nu} y_n^{2\nu} dy = \frac{1}{\alpha} r^\alpha \pi^{\frac{(n-1)}{2}} \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma\left(\frac{n+2\nu}{2}\right)}, \tag{2.8}$$

and therefore, by (2.6) and (2.8),

$$c_{n,\nu}(\alpha) \int_{B_r^+} |y|^{\alpha-n-2\nu} y_n^{2\nu} dy = \frac{\Gamma\left(\frac{n+2\nu-\alpha}{2}\right) r^\alpha}{\Gamma\left(\frac{n+2\nu}{2}\right) 2^\alpha \Gamma\left(1+\frac{\alpha}{2}\right)}. \tag{2.9}$$

Now the relation (2.7) is a simple consequence of (2.9). □

REMARK 2.4. It follows from (2.6) that

$$\lim_{\alpha\to 0} \frac{c_{n,\nu}(\alpha)}{\alpha} = \frac{\Gamma\left(\frac{n+2\nu}{2}\right)}{\pi^{\frac{(n-1)}{2}}\Gamma\left(\nu+\frac{1}{2}\right)},$$

and therefore,

$$c_{n,\nu}(\alpha) \sim c\alpha \quad \text{as} \quad \alpha \to 0 \;\; ; \;\; c = \Gamma\left(\frac{n+2\nu}{2}\right) \Big/ \pi^{\frac{(n-1)}{2}} \Gamma\left(\nu + \frac{1}{2}\right). \tag{2.10}$$

The similar arguments show that

$$\lim_{\alpha \to 0} c_n(\alpha) \int\limits_{|y|<r} |y|^{\alpha-n} dy = 1, \; \forall r > 0, \;\text{ and }\; c_n(\alpha) \sim c\alpha \quad \text{as} \quad \alpha \to 0.$$

(here $c = \Gamma\left(\frac{n}{2}\right) / 2\pi^{n/2}$).

This Remark will be used below.

Here and below the letter c is used for a constant that can be different at each occurrence. The notation C designates the class of all continuous functions on $\mathbb{R}^n$, and $C(K)$ stands for the space of continuous functions on compact K with the sup-norm. We will write "$\varphi(\alpha) = o(1)$ as $\alpha \to 0$" if $\lim\limits_{\alpha \to 0} \varphi(\alpha) = 0$, and "$\varphi(\alpha) = \mathscr{O}(\psi(\alpha))$ as $\alpha \to 0$" if $|\varphi(\alpha)| \leqslant c\psi(\alpha)$ as $\alpha \to 0$.

3. The approximation properties of the families $I_\nu^\alpha f$ and $I^\alpha f$

In this section we investigate a convergence of the families $I_\nu^\alpha f$ and $I^\alpha f$ as $\alpha \to 0$.

THEOREM 3.1. *Let $f \in L_{p,\nu}$, $1 \leqslant p < \infty$.*

(i) If $\lim\limits_{x \to x_0} f(x) = l$, $l \in \mathbb{C}$, then $\lim\limits_{\alpha \to 0} (I_\nu^\alpha f)(x_0) = l$. (3.1)

In particular, if f is continuous at $x_0 \in \mathbb{R}_+^n$, then $\lim\limits_{\alpha \to 0} (I_\nu^\alpha f)(x_0) = f(x_0)$.

(ii) If f is real function and $\lim\limits_{x \to x_0} f(x) = \pm\infty$, then $\lim\limits_{\alpha \to 0} (I_\nu^\alpha f)(x_0) = \pm\infty$.

(iii) If $f \in L_{p,\nu} \cap C$ for some $1 \leqslant p < \infty$, then for any compact $K \subset \mathbb{R}_+^n$

$$\lim_{\alpha \to 0} \|I_\nu^\alpha f - f\|_{C(K)} = 0.$$

Proof. (i) Since I_ν^α is a linear operator, it can be assumed that f is a real valued function and $-\infty < l < \infty$. Now the condition $\lim\limits_{x \to x_0} f(x) = l$ yields that for any $\varepsilon > 0$ one can find $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|T^y f(x_0) - l| < \varepsilon \;\; , \;\; \forall y \in B_\delta^+. \tag{3.2}$$

We have

$$(I_\nu^\alpha f)(x_0) - l \;=\; c_{n,\nu}(\alpha) \int\limits_{\mathbb{R}_+^n} |y|^{\alpha-n-2\nu} \left(T^y f(x_0)\right) y_n^{2\nu} dy - l$$

$$
\begin{aligned}
&= c_{n,\nu}(\alpha) \int\limits_{B_\delta^+} |y|^{\alpha-n-2\nu} \Big(T^y f(x_0) - l\Big) y_n^{2\nu} dy \\
&\quad + l\left(c_{n,\nu}(\alpha) \int\limits_{B_\delta^+} |y|^{\alpha-n-2\nu} y_n^{2\nu} dy - 1 \right) \\
&\quad + c_{n,\nu}(\alpha) \int\limits_{\mathbb{R}_+^n \setminus B_\delta^+} |y|^{\alpha-n-2\nu} \Big(T^y f(x_0)\Big) y_n^{2\nu} dy \\
&\equiv i_1(\alpha) + i_2(\alpha) + i_3(\alpha).
\end{aligned}
$$

By the Lemma 2.3, $\lim\limits_{\alpha\to 0} i_2(\alpha) = 0.$

Using (3.2) and Lemma 2.3 we have

$$
\begin{aligned}
|i_1(\alpha)| &\leqslant c_{n,\nu}(\alpha) \int\limits_{B_\delta^+} |y|^{\alpha-n-2\nu} |T^y f(x_0) - l|\, y_n^{2\nu} dy \\
&\leqslant \varepsilon c_{n,\nu}(\alpha) \int\limits_{B_\delta^+} |y|^{\alpha-n-2\nu} y_n^{2\nu} dy = \mathscr{O}(1)\,\varepsilon \quad \text{as} \quad \alpha \to 0.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary we have that $\lim\limits_{\alpha\to 0} i_1(\alpha) = 0.$

Let us estimate $i_3(\alpha)$. By making use of the Hölder's inequality (cf. (2.3) with $r = \infty$), and (2.2) we have

$$
\begin{aligned}
|i_3(\alpha)| &\leqslant c_{n,\nu}(\alpha) \|T^y f(x_0)\|_{p,\nu} \left(\int\limits_{B_r^+} |y|^{p'(\alpha-n-2\nu)} y_n^{2\nu} dy \right)^{1/p'} \\
&\leqslant A_p(\alpha)\, c_{n,\nu}(\alpha)\, \|f\|_{p,\nu}\, \delta^{\alpha-(n+2\nu)/p},
\end{aligned}
$$

where $\frac{1}{p} + \frac{1}{p'} = 1,$, and

$$
A_p(\alpha) = \begin{cases} 1 & , \quad \text{if } p = 1 \\ \left|S_+^{n-1}\right|^{1/p'} \Big((n+2\nu)(p'-1) - \alpha p' \Big)^{-1/p'} & , \quad \text{if } p > 1. \end{cases} \tag{3.3}
$$

Since $A_p(\alpha) = \mathscr{O}(1)$ as $\alpha \to 0$ and $\lim\limits_{\alpha\to 0} c_{n,\nu}(\alpha) = 0$, it follows that $\lim\limits_{\alpha\to 0} i_3(\alpha) = 0$. Thus, (3.1) is proved for a finite l.

(ii) The following calculations show that (3.1) is true also for $l = \pm\infty$. Let $\lim\limits_{x\to x_0} f(x) = +\infty$ (the case of $l = -\infty$ is proved analogously). For a given $M > 0$

there exists $\delta > 0$ such that $T^y f(x_0) > M$ for any $y \in B_\delta^+$. Using this observation we get

$$\begin{aligned}(I_\nu^\alpha f)(x_0) &= c_{n,\nu}(\alpha) \int_{B_\delta^+} |y|^{\alpha-n-2\nu} \Big(T^y f(x_0)\Big) y_n^{2\nu} dy \\ &\quad + c_{n,\nu}(\alpha) \int_{\mathbb{R}_+^n \setminus B_\delta^+} |y|^{\alpha-n-2\nu} \Big(T^y f(x_0)\Big) y_n^{2\nu} dy \\ &\geqslant Mc_{n,\nu}(\alpha) \int_{B_\delta^+} |y|^{\alpha-n-2\nu} y_n^{2\nu} dy \\ &\quad - c_{n,\nu}(\alpha) \int_{\mathbb{R}_+^n \setminus B_\delta^+} |y|^{\alpha-n-2\nu} |T^y f(x_0)| y_n^{2\nu} dy.\end{aligned}$$

By the Hölder's inequality,

$$\int_{\mathbb{R}_+^n \setminus B_\delta^+} |y|^{\alpha-n-2\nu} |T^y f(x_0)| y_n^{2\nu} dy \leqslant A_p(\alpha) \|f\|_{p,\nu} \delta^{\alpha-(n+2\nu)/p},$$

where $A_p(\alpha)$ is defined as in (3.3). Now it follows that

$$(I_\nu^\alpha f)(x_0) \geqslant Mc_{n,\nu}(\alpha) \int_{B_\delta^+} |y|^{\alpha-n-2\nu} y_n^{2\nu} dy - A_p(\alpha) c_{n,\nu}(\alpha) \|f\|_{p,\nu} \delta^{\alpha-(n+2\nu)/p}.$$

Owing to Lemma 2.3, the last expression yields

$$\liminf_{\alpha\to 0} (I_\nu^\alpha f)(x_0) \geqslant M, \ \forall M > 0,$$

and therefore

$$\lim_{\alpha\to 0} (I_\nu^\alpha f)(x_0) = +\infty.$$

(iii) Let $f \in L_{p,\nu} \cap C$, $1 \leqslant p < \infty$ and $K \subset \mathbb{R}_+^n$ be any compact set. Since f is continuous on $\mathbb{R}_+^n$ and uniformly continuous on K, given $\varepsilon > 0$ there exists $\delta > 0$ such that $\sup_{x\in K} |T^y f(x) - f(x)| < \varepsilon$ for any $y \in B_\delta^+$. For given $x \in K$ we have

$$\begin{aligned}(I_\nu^\alpha f)(x) - f(x) &= c_{n,\nu}(\alpha) \int_{B_\delta^+} |y|^{\alpha-n-2\nu} \Big(T^y f(x) - f(x)\Big) y_n^{2\nu} dy \\ &\quad + f(x) \left(c_{n,\nu}(\alpha) \int_{B_\delta^+} |y|^{\alpha-n-2\nu} y_n^{2\nu} dy - 1 \right) \\ &\quad + c_{n,\nu}(\alpha) \int_{\mathbb{R}_+^n \setminus B_\delta^+} |y|^{\alpha-n-2\nu} \Big(T^y f(x)\Big) y_n^{2\nu} dy. \qquad (3.4)\end{aligned}$$

Further,

$$\|I_\nu^\alpha f - f\|_{C(K)} \leqslant \varepsilon c_{n,\nu}(\alpha) \int_{B_\delta^+} |y|^{\alpha-n-2\nu} y_n^{2\nu} dy + \|f\|_{C(K)} \left| c_{n,\nu}(\alpha) \int_{B_\delta^+} |y|^{\alpha-n-2\nu} y_n^{2\nu} dy - 1 \right| + A_p(\alpha) \|f\|_{p,\nu} c_{n,\nu}(\alpha),$$

where $A_p(\alpha)$ is defined by (3.3). Now by making use of Lemma 2.3 and the fact that $\lim_{\alpha\to 0} c_{n,\nu}(\alpha) = 0$ we have

$$\lim_{\alpha\to 0} \|I_\nu^\alpha f - f\|_{C(K)} = 0.$$

The proof of the theorem is completed. □

The following analog of the Theorem 3.1 for the potentials I^α can be proved by the same way.

THEOREM 3.2. *Let $f \in L_p$, $1 \leqslant p < \infty$.*
(i) If $\lim_{x\to x_0} f(x) = l$, then $\lim_{\alpha\to 0} (I^\alpha f)(x_0) = l$.
In particular, if f is continuous at $x_0 \in \mathbb{R}^n$, then $\lim_{\alpha\to 0} (I^\alpha f)(x_0) = f(x_0)$.
(ii) If f is real function and $\lim_{x\to x_0} f(x) = \pm\infty$, then $\lim_{\alpha\to 0} (I^\alpha f)(x_0) = \pm\infty$.
(iii) If $f \in L_p \cap C$ for some $1 \leqslant p < \infty$, then for any compact $K \subset \mathbb{R}^n$

$$\lim_{\alpha\to 0} \|I^\alpha f - f\|_{C(K)} = 0.$$

The following statement strengthens the part (i) of Theorem 3.1, and shows that the family $I_\nu^\alpha f$, $\alpha > 0$ converges to f as $\alpha \to 0$ at the $d_\nu-$ points of $f \in L_{p,\nu}$ (i.e. almost everywhere).

THEOREM 3.3. *Let $f \in L_{p,\nu}$, $1 \leqslant p < \infty$. Then*

$$\lim_{\alpha\to 0} (I_\nu^\alpha f)(x) = f(x)$$

for any $d_\nu-$point x of f (i.e., almost everywhere).

Proof. Let x be a $d_\nu-$point of f. Then owing to (2.4), given $\varepsilon > 0$ there exists $\delta > 0$ such that for $0 < \rho \leqslant \delta$

$$\left| \int_{B_\rho^+} \Big(T^y f(x) - f(x) \Big) y_n^{2\nu} dy \right| \leqslant \omega(n,\nu) \rho^{n+2\nu} \varepsilon. \tag{3.5}$$

The last two terms of the right hand side of the equality (3.4) tend to zero as $\alpha \to 0$ by the same argument as in proof of the Theorem 3.1 Let us show that the first term in (3.4) tends to zero at any $d_\nu-$ point of $f \in L_{p,\nu}$. Setting

$$A(\alpha) = c_{n,\nu}(\alpha) \int_{B_\delta^+} |y|^{\alpha-n-2\nu} \Big(T^y f(x) - f(x) \Big) y_n^{2\nu} dy,$$

and passing to the polar coordinates, $y = t\theta,\ \theta \in S_+^{n-1},\ 0 < t \leqslant \delta,$ we get

$$A(\alpha) = c_{n,\nu}(\alpha) \int_0^\delta \int_{S_+^{n-1}} t^{\alpha-n-2\nu} \Big(T^{t\theta} f(x) - f(x) \Big) \theta_n^{2\nu} t^{n+2\nu-1} dt d\theta. \tag{3.6}$$

Set

$$F(\rho) = \int_0^\rho \to \int_{S_+^{n-1}} \Big(T^{t\theta} f(x) - f(x) \Big) \theta_n^{2\nu} t^{n+2\nu-1} dt d\theta,\ \ 0 < \rho \leqslant \delta.$$

By virtue of (3.5) we have

$$|F(\rho)| \leqslant \omega(n,\ \nu)\, \rho^{n+2\nu} \varepsilon. \tag{3.7}$$

Taking in mind that

$$dF(\rho) = \left(\int_{S_+^{n-1}} \Big(T^{\rho\theta} f(x) - f(x) \Big) \theta_n^{2\nu} d\theta \right) \rho^{n+2\nu-1} d\rho,$$

we have from (3.6)

$$\begin{aligned} A(\alpha) &= c_{n,\nu}(\alpha) \int_0^\delta \rho^{\alpha-n-2\nu} dF(\rho) \\ &= c_{n,\nu}(\alpha) \Big(\rho^{\alpha-n-2\nu} F(\rho)\Big|_0^\delta - (\alpha - n - 2\nu) \Big) \int_0^\delta F(\rho)\, \rho^{\alpha-n-2\nu-1} d\rho. \end{aligned}$$

By (3.7),

$$\lim_{\rho \to 0} \rho^{\alpha-n-2\nu} F(\rho) = 0$$

for any $\alpha > 0,$ and therefore,

$$A(\alpha) = c_{n,\nu}(\alpha)\, F(\delta)\, \delta^{\alpha-n-2\nu} + c_{n,\nu}(\alpha)\,(n + 2\nu - \alpha) \int_0^\delta F(\rho)\, \rho^{\alpha-n-2\nu-1} d\rho.$$

We have by (3.7)

$$\left| \int_0^\delta F(\rho)\, \rho^{\alpha-n-2\nu-1} d\rho \right| \leqslant \omega(n,\ \nu)\, \varepsilon \int_0^\delta \rho^{\alpha-1} d\rho = \omega(n,\ \nu)\, \frac{\delta^\alpha}{\alpha} \varepsilon.$$

Since $c_{n,\nu}(\alpha) = o(1)$ as $\alpha \to 0$ and $\frac{1}{\alpha} c_{n,\nu}(\alpha) = \mathcal{O}(1)$ as $\alpha \to 0$ (cf. (2.10)), it follows that

$$A(\alpha) = o(1) + \mathcal{O}(1)\, \varepsilon = \mathcal{O}(1)\, \varepsilon \ \text{ as } \alpha \to 0.$$

The last estimate shows that $\lim\limits_{\alpha\to 0} A(\alpha) = 0$ and the proof of the theorem is completed. □

By the same way it may be proved the following theorem on convergence of $(I^{\alpha} f)(x)$ as $\alpha \to 0$ in every $d-$point of $f \in L_p$, $1 \leqslant p < \infty$.

THEOREM 3.4. *Let* $f \in L_p(\mathbb{R}^n)$, $1 \leqslant p < \infty$. *Then*

$$\lim_{\alpha\to 0} (I^{\alpha} f)(x) = f(x)$$

for any $d-$*point* x *of* f *(i.e., almost everywhere).*

REMARK 3.5. The statement of Theorem 3.4 has been proved by T. Kurokawa [8], using another method.

4. An estimate of approximation

In this section we estimate the order of approximation of a function f by means of the families $I^{\alpha} f$ and $I^{\alpha}_{\nu} f$ as $\alpha \to 0$.

Let $K \subset \mathbb{R}^n$ be a compact and $\omega_f(\delta)$ be the modulus of continuity of f on K, that is (see [15])

$$\omega_f(\delta) = \sup_{|y|<\delta} \| f(x+y) - f(x) \|_{C(K)} .$$

THEOREM 4.1. *Let* $f \in L_p \cap C$, $1 \leqslant p < \infty$. *Then there exists* $c = const$ *such that*

$$\| I^{\alpha} f - f \|_{C(K)} \leqslant c\,\omega_f(\alpha) \quad \text{as } \alpha \to 0. \tag{4.1}$$

Proof. For $x \in K$ we can write (cf. (3.4))

$$\begin{aligned} |(I^{\alpha} f)(x) - f(x)| \leqslant\ & c_n(\alpha) \int\limits_{|y|\leqslant 1} |y|^{\alpha-n} |f(x+y) - f(x)|\, dy \\ & + |f(x)| \left| c_n(\alpha) \int\limits_{|y|\leqslant 1} |y|^{\alpha-n}\, dy - 1 \right| \\ & + c_n(\alpha) \int\limits_{|y|>1} |y|^{\alpha-n} |f(x+y)|\, dy. \end{aligned} \tag{4.2}$$

We estimate each integral separately. To estimate the first expression on the right of (4.2), we will use the following well known property of the modulus of continuity:

$$|f(x+y) - f(x)| \leqslant \omega_f(|y|) \leqslant \omega_f(\alpha) \left(1 + \frac{|y|}{\alpha}\right), \quad \alpha > 0.$$

By making use of this estimate and Remark 2.4 we have

$$c_n(\alpha)\int\limits_{|y|\leqslant 1}|y|^{\alpha-n}|f(x+y)-f(x)|\,dy\leqslant$$

$$c_n(\alpha)\omega_f(\alpha)\left(\frac{1}{\alpha}\int\limits_{|y|\leqslant 1}|y|^{\alpha-n+1}\,dy+\int\limits_{|y|\leqslant 1}|y|^{\alpha-n}\,dy\right)\leqslant c\omega_f(\alpha)\ \text{ as } \alpha\to 0. \quad (4.3)$$

Further,

$$c_n(\alpha)\int\limits_{|y|\leqslant 1}|y|^{\alpha-n}\,dy-1=\frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{2^{\alpha}\Gamma\left(\frac{\alpha}{2}+1\right)}-1=\frac{\Gamma\left(\frac{n-\alpha}{2}\right)-\Gamma\left(\frac{n}{2}\right)2^{\alpha}\Gamma\left(\frac{\alpha}{2}+1\right)}{\Gamma\left(\frac{n}{2}\right)2^{\alpha}\Gamma\left(\frac{\alpha}{2}+1\right)}.$$

Since by L'Hospital law the limit

$$\lim_{\alpha\to 0}\frac{\Gamma\left(\frac{n-\alpha}{2}\right)-\Gamma\left(\frac{n}{2}\right)2^{\alpha}\Gamma\left(\frac{\alpha}{2}+1\right)}{\alpha}$$

exists and is finite, we have

$$c_n(\alpha)\int\limits_{|y|\leqslant 1}|y|^{\alpha-n}\,dy-1=\mathscr{O}(1)\alpha\ \text{ as } \alpha\to 0. \quad (4.4)$$

Finally, the Hölder's inequality with $\frac{1}{p}+\frac{1}{p'}=1$ yields

$$\begin{aligned} c_n(\alpha)\int\limits_{|y|>1}|y|^{\alpha-n}|f(x+y)|\,dy &\leqslant \|f\|_p\left(p'\left(\frac{n}{p}-\alpha\right)\right)^{-1/p'}c_n(\alpha)\\ &\leqslant c\,\|f\|_p\,c_n(\alpha)\leqslant c\,\|f\|_p\,\alpha. \end{aligned} \quad (4.5)$$

(More simple calculations show that the same estimate is true when $p=1$).

Now the estimation (4.1) follows from (4.3), (4.4) and (4.5), by taking into account the following inequality:

$$\alpha\leqslant\frac{2}{\omega_f(1)}\omega_f(\alpha),\ \ (\alpha<1). \qquad \square$$

Our next goal is to estimate the order of approximation of function f at the Lipschitz points. It follows from estimation (4.1) that if $f\in Lip\beta$, then

$$\|I^{\alpha}f-f\|_{C(K)}\leqslant c\alpha^{\beta}\ \text{ as } \alpha\to 0.$$

However the next theorem shows that the order of approximation at Lipschitz point is independent from the Lipschitz degree β. We state this interesting result for generalized Riesz potentials I_{ν}^{α}. Note that the same statement is also true for classical Riesz potentials I^{α}.

DEFINITION 4.2. A point $x \in \mathbb{R}_+^n$ is called a Lipschitz point of degree $\beta > 0$ of function $f : \mathbb{R}_+^n \to \mathbb{C}$, if there exists $\delta > 0$ and $c > 0$ such that the inequality

$$\sup_{|y| \leqslant h} |T^y f(x) - f(x)| \leqslant ch^\beta.$$

holds for any $h,\ 0 < h \leqslant \delta$.

THEOREM 4.3. *Let $f \in L_{p,\nu} \cap C$, $1 \leqslant p < \infty$, and $x \in \mathbb{R}_+^n$ be a Lipschitz point (of degree $\beta > 0$) of function f. Then*

$$(I_\nu^\alpha f)(x) - f(x) = \mathscr{O}(1)\,\alpha \ \text{ as } \ \alpha \to 0.$$

Proof. For the simplicity of the calculations, suppose $\delta = 1$ in the Definition 4.2. As in (3.4) we have

$$\begin{aligned}(I_\nu^\alpha f)(x) - f(x) &= c_{n,\nu}(\alpha) \int_{B_1^+} |y|^{\alpha-n-2\nu} \Big(T^y f(x) - f(x)\Big) y_n^{2\nu} dy \\ &\quad + f(x) \left(c_{n,\nu}(\alpha) \int_{B_1^+} |y|^{\alpha-n-2\nu} y_n^{2\nu} dy - 1 \right) \\ &\quad + c_{n,\nu}(\alpha) \int_{\mathbb{R}_+^n \setminus B_1^+} |y|^{\alpha-n-2\nu} \Big(T^y f(x)\Big) y_n^{2\nu} dy \\ &\equiv j_1(\alpha) + j_2(\alpha) + j_3(\alpha). \end{aligned} \tag{4.6}$$

Since x is a Lipschitz point (of degree β) we get

$$|T^y f(x) - f(x)| \leqslant c\,|y|^\beta,\ \ y \in B_1^+,$$

and therefore

$$\begin{aligned}|j_1(\alpha)| &\leqslant c c_{n,\nu}(\alpha) \int_{B_1^+} |y|^{\alpha-n-2\nu+\beta} |y_n|^{2\nu} dy \\ &= c\left|S_+^{n-1}\right| c_{n,\nu}(\alpha) \frac{1}{\alpha+\beta} \overset{(2.10)}{=} \mathscr{O}(1)\,\alpha \ \text{ as } \alpha \to 0. \end{aligned} \tag{4.7}$$

Let us estimate $j_3(\alpha)$. Using the Hölder inequality we have

$$\begin{aligned}|j_3(\alpha)| &\leqslant c_{n,\nu}(\alpha) \|f\|_{p,\nu} \left(\int_{\mathbb{R}_+^n \setminus B_1^+} |y|^{(\alpha-n-2\nu)p'} y_n^{2\nu} dy \right)^{1/p'} \\ &= A_p(\alpha) \|f\|_p\, c_{n,\nu}(\alpha), \end{aligned}$$

where $A_p(\alpha)$ is defined by (3.3). Owing to (2.10) and (3.3) we get

$$j_3(\alpha) = \mathscr{O}(1)\,\alpha \ \text{ as } \alpha \to 0. \tag{4.8}$$

Let us estimate now the term $j_2(\alpha)$. By passing to the polar coordinates we have

$$\int_{B_1^+} |y|^{\alpha-n-2\nu} |y_n|^{2\nu} dy = \frac{|S_+^{n-1}|}{\alpha},$$

where

$$|S_+^{n-1}| = \int_{S_+^{n-1}} \theta_n^{2\nu} d\theta = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma\left(\frac{n+2\nu}{2}\right)}.$$

The simple calculations show that

$$\begin{aligned} j_2(\alpha) &\equiv c_{n,\nu}(\alpha) \int_{B_1^+} |y|^{\alpha-n-2\nu} |y_n|^{2\nu} dy - 1 \\ &= \frac{\Gamma\left(\frac{n+2\nu-\alpha}{2}\right) - 2^\alpha \Gamma\left(1+\frac{\alpha}{2}\right) \Gamma\left(\frac{n+2\nu}{2}\right)}{2^\alpha \Gamma\left(1+\frac{\alpha}{2}\right) \Gamma\left(\frac{n+2\nu}{2}\right)} \end{aligned}$$

By making use of the L'Hospital law we have

$$\begin{aligned} \lim_{\alpha\to 0} \frac{j_2(\alpha)}{\alpha} &= \frac{1}{\Gamma\left(\frac{n+2\nu}{2}\right)} \lim_{\alpha\to 0} \frac{\Gamma\left(\frac{n+2\nu-\alpha}{2}\right) - 2^\alpha \Gamma\left(1+\frac{\alpha}{2}\right) \Gamma\left(\frac{n+2\nu}{2}\right)}{\alpha} \\ &= -\frac{1}{2}\left(\frac{\Gamma'\left(\frac{n+2\nu}{2}\right)}{\Gamma\left(\frac{n+2\nu}{2}\right)} + 2\ln 2 + \Gamma'(1)\right). \end{aligned}$$

Therefore,

$$j_2(\alpha) = \mathscr{O}(1)\,\alpha \quad \text{as} \quad \alpha \to 0. \tag{4.9}$$

Finally, using (4.6), (4.7), (4.8) and (4.9) we get

$$(I_\nu^\alpha f)(x) - f(x) = \mathscr{O}(1)\,\alpha \quad \text{as} \quad \alpha \to 0.$$

The proof is completed. □

REFERENCES

[1] I. A. ALIEV AND B. RUBIN, *Parabolic potentials and wavelet transforms with the generalized translation*, Studia Math. 145 (1) (2001), 1–16.

[2] I. A. ALIEV AND B. RUBIN, *Wavelet-like transforms for admissible semi-groups; inversion formulas for potentials and Radon transforms*, The Journal of Fourier Analysis and Applications, 11 (3) (2005), 334–352.

[3] A. ARAL, *On generalized potential-type integral operators with non-isotropic kernel*, Intern. Math. Journal, Vol 3, (2003), No:12, 1367–1378.

[4] A. D. GADJIEV AND I. A. ALIEV, *On classes of operators of potential type, generated by a generalized shift*, Reports of enlarged session of the I. N. Vekua Inst. of Applied Mathematics, 3, N2, Tbilisi, 1988 (in Russian).

[5] A. D. GADJIEV AND I. A. ALIEV, *Riesz and Bessel potentials generated by the generalized translation and their inverses*, Proc. IV All-Union Winter Conf. "Theory of functions and approximation", Saratov (Russia)-1988; In the book "Theory of functions and approximation", Printed in Saratov University (1990), 47–53 (in Russian).

[6] V. S. GULIEV, *Sobolev theorem for the B-Riesz potentials*, Dokl. Rus. Akad. Nauk (RAN), 358 (4) (1998), 450–451 (in Russian).
[7] I. A. KIPRIYANOV, *Singular Elliptic Boundary Problems*, Nauka, Moscow, Fizmatlit, 1997 (in Russian).
[8] T. KUROKAWA, *On the Riesz and Bessel kernels as approximations of the identity*, Sci. Rep. Kagoshima Univ., No:30(1981), 31–45.
[9] B. M. LEVITAN, *Expansion in Fourier series and integrals in Bessel functions*, Uspekhi. Matem. Nauk, 6 (2) (1951), 102–143 (in Russian).
[10] J. LÖFSTRÖM AND J. PEETRE, *Approximation theorems connected with generalized translation*, Math. Ann., 181 (1969), 255–268.
[11] B. RUBIN, *Fractional Integrals and Potentials, Pitman Monograps and Surveys in Pure and Appl. Math.*, 82, Longman, Harlow, 1996.
[12] S. G. SAMKO, A. A. KILBAS AND O. I. MARICHEV, *Fractional Integrals and Derivatives*, Theory and Applications. Gordon and Breach Science Publishers, 1993.
[13] E. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N. J., 1970.
[14] K. STEMPAK, *La theorie de Littlewood-Paley pour la transformation de Fourier-Bessel*, C. R. Acad. Sci. Paris, ser. I (303) (1986), 15–18.
[15] A. F. TIMAN, *Theory of Approximation of Functions of a Real Variable*, New York, 1963.
[16] K. TRIMÉCHE, *Generalized Wavelets and Hypergroups*, Gordon and Breach Science Publishers, (1997).

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