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ON THE HYERS–ULAM–RASSIAS STABILITY PROBLEM FOR APPROXIMATELY k–ADDITIVE MAPPINGS AND FUNCTIONAL INEQUALITIES

KIL-WOUNG JUN AND HARK-MAHN KIM

(*communicated by Th. M. Rassias*)

Abstract. The purpose of this paper is to solve the generalized Hyers–Ulam stability problem for a k–additive functional equation

$$D_{m,i}f(x,y) = 0,$$

on the basis of direct method, where $k{=}4m{+}i$ is a positive integer for each $i = -1, 0, 1$ and 2.

1. Introduction

In 1940, S. M. Ulam [28] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

In 1941, D. H. Hyers [10] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon$$

for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leqslant \epsilon.$$

In 1978, Th. M. Rassias [23] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

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for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leqslant \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. Also, if the mapping from t to $f(tx)$ is continuous in t, where t belongs to $\mathbb{R}$, for each fixed x in E, then L is $\mathbb{R}$-linear.

In 1991, Z. Gajda [7] following the same approach as in Th. M. Rassias [23], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [7], as well as by Th. M. Rassias and P. Šemrl [24] that one cannot prove a Th. M. Rassias' type theorem when $p = 1$. The inequality (1.1) that was introduced for the first time by Th. M. Rassias [23] provided a lot of influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept of stability is known as *generalized Hyers–Ulam stability* or *Hyers–Ulam–Rassias stability* of functional equations (cf. the books of P. Czerwik[4], D. H. Hyers, G. Isac and Th. M. Rassias [12]).

P. Găvruta [8] provided a further generalization of Th. M. Rassias' Theorem. During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers–Ulam stability to a number of functional equations and mappings (see [9, 17, 18, 19, 25]).

Let both E_1 and E_2 be vector spaces. First, we introduce the following four functional equations of different types in sequence

$$2^{4m-1}\left[\sum_{k=0}^{2m}\binom{4m}{2k}f(x+(m-k)y)\right] = \sum_{k=0}^{2m-1}\binom{4m}{2k+1}f(2x+(2m-1-2k)y) \tag{1.3}$$

for all 2–dimensional vectors $(x, y) \in E_1 \times E_1$, where $m \geqslant 1$. If $m = 1$, this equation reduces to

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.4}$$

which is in fact a *cubic functional equation* and the authors [15] have proved that a mapping $f : E_1 \to E_2$ satisfies the equation (1.4) if and only if there exists a mapping $B : E_1 \times E_1 \times E_1 \to E_2$ such that $f(x) = B(x, x, x)$ for all $x \in E_1$, where B is symmetric for each fixed one variable and additive for each fixed two variables. Further they solved the generalized Hyers–Ulam stability problem for (1.4). In general we are going to investigate the generalized Hyers–Ulam stability problem for the equation (1.3).

Second, we consider the following functional equations

$$2^{4m}\left[\sum_{k=0}^{2m}\binom{4m+1}{2k}f(x+(m-k)y)\right] = \sum_{k=0}^{2m}\binom{4m+1}{2k+1}f(2x+(2m-1-2k)y) \tag{1.5}$$

for all 2–dimensional vectors $(x, y) \in E_1 \times E_1$, where $m \geqslant 1$. If $m = 1$, this equation reduces to

$$16[f(x+y) + 10f(x) + 5f(x-y)] = 5f(2x+y) + 10f(2x-y) + f(2x-3y),$$

which is equivalent to the following functional equation

$$f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y)+24f(y) \tag{1.6}$$

and has in fact as a solution $f(x)=cx^4$ with c an arbitrary constant when f is a real function. For this obvious reason, the above functional equation (1.6) is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping* [22]. In [3], Chung and Sahoo determined the general solution of the quartic equation (1.6) without assuming any regularity conditions on the unknown mapping f. On the other hand, it is easy to see that the solution f of (1.6) is even, thus the above equation can be written in the following way

$$f(2x+y)+f(2x-y)+6f(y)=4f(x+y)+4f(x-y)+24f(x),$$

of which the general solution is determined by a symmetric biquadratic mapping $B: E_1\times E_1\to E_2$ between real vector spaces E_1,E_2 such that $f(x)=V(x):=B(x,x)$ for all $x\in E_1$ [16, 20].

Next, we will try to find the general solution of the following functional equations

$$2^{4m+1}\left[\sum_{k=0}^{2m}\binom{4m+2}{2k+1}f(x+(m-k)y)\right]=\sum_{k=0}^{2m+1}\binom{4m+2}{2k}f(2x+(2m+1-2k)y) \tag{1.7}$$

for all 2–dimensional vectors $(x,y)\in E_1\times E_1$, where $m\geqslant 0$. If $m=0$, this equation yields

$$f(2x+y)+f(2x-y)=4f(x),$$

which is in fact a *Cauchy–Jensen additive functional equation*.

Finally we are going to investigate the generalized Hyers–Ulam stability problem for the equations

$$2^{4m+2}\left[\sum_{k=0}^{2m+1}\binom{4m+3}{2k+1}f(x+(m-k)y)\right]=\sum_{k=0}^{2m+1}\binom{4m+3}{2k}f(2x+(2m+1-2k)y) \tag{1.8}$$

for all 2–dimensional vectors $(x,y)\in E_1\times E_1$, where $m\geqslant 0$. If $m=0$, this equation reduces to

$$12f(x)+4f(x-y)=f(2x+y)+3f(2x-y),$$

which is equivalent to the original *quadratic functional equation*

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1.9}$$

and thus has general solution $f(x)=B(x,x)$ for a unique symmetric biadditive mapping $B: E_1\times E_1\to E_2$ ([1]). A stability problem for the quadratic functional equation (1.9) was solved by a lot of authors [12, 13, 26, 27]. Furthermore, Jun and Lee [14] have proved the Hyers–Ulam–Rassias stability of the Pexiderized quadratic equation (1.9).

In the present paper, we will solve the general solution of the equations (1.3), (1.5), (1.7) and (1.8) and we are going to investigate the generalized Hyers–Ulam stability problem for these equations collectively.

2. General solution

Let S be a commutative semigroup with a zero element and Y a Banach space. For functions $f : S \to Y$, we define the *linear difference operator* $\Delta_h f$ by $\Delta_h f(x) := f(x+h) - f(x)$, $x \in S$. Similarly, we define $\Delta^2_{h_1,h_2} f(x) := \Delta_{h_2}(\Delta_{h_1} f(x))$ and $\Delta^{n+1}_{h_1,\cdots,h_{n+1}} f(x) := \Delta_{h_{n+1}}\left(\Delta^n_{h_1,\cdots,h_n} f(x)\right)$, $n = 1, 2, \cdots$. Here we consider *k–additive mapping* $V_k : S^k \to Y$, that is, $V_k(x_1, x_2, \cdots, x_k)$ is additive in each of its variables when the others are fixed. We say that V_k is symmetric provided $V_k(x_1, x_2, \cdots, x_k) = V_k(y_1, y_2, \cdots, y_k)$ whenever $(y_1, y_2, \cdots, y_k)$ is any permutation of $(x_1, x_2, \cdots, x_k) \in S^k$. If $V_k : S^k \to Y$ is symmetric and k–additive, let $V_k{}^*$ denote the diagonalization of V_k so that $V_k{}^*(x) = V_k(x, x, \cdots, x)$ and note that $V_k{}^*(rx) = r^k V_k{}^*(x)$ whenever $x \in S$ and $r \in \mathbb{Q}$.

Now, the following functional equations

$$\Delta^n_h f(x) := \Delta^n_{h,\cdots,h} f(x) = 0, \quad \text{or} \quad \Delta^n_{h_1,\cdots,h_n} f(x) = 0$$

were studied and proved by M. Fréchet [6] and by S. Mazur and W. Orlicz [21], and then by D. Djoković [5]. The Hyers–Ulam stability of these equations controlled by a constant was established by D. H. Hyers [11] and by M. Albert and J. A. Baker [2].

LEMMA 2.1. *A mapping $f : E_1 \to Y$ satisfies the functional equation* (1.3) $((1.5), (1.7)$ *and* (1.8), *respectively*) *for all $x, y \in E_1$ if and only if there exists a symmetric k–additive mapping $V_k : E_1^k \to Y$ such that $f(x) = V_k(x, x, \cdots, x)$ for all $x \in E_1$, where $k = 4m - 1$* $(4m, 4m+1$ *and* $4m+2$, *respectively* $)$.

Proof. Let $f : E_1 \to Y$ satisfy the functional equation (1.3). By putting $x = 0$ or $y = 0$ in (1.3), it then follows easily that $f(0) = 0$ and $f(2x) = 2^k f(x)$ for all $x \in E_1$, where $k = 4m - 1$ $(4m, 4m+1$ and $4m+2$, respectively$)$. If we replace y by $2y$ in (1.3), one has

$$\sum_{k=0}^{2m} \binom{4m}{2k} f(x + 2(m-k)y) = \sum_{k=0}^{2m-1} \binom{4m}{2k+1} f(x + (2m-1-2k)y)$$

for all 2–dimensional vectors $(x, y) \in E_1 \times E_1$. Substitution of x with $x + 2my$ into the above equation yields

$$\sum_{k=0}^{2m} \binom{4m}{2k} f(x + (4m-2k)y) = \sum_{k=0}^{2m-1} \binom{4m}{2k+1} f(x + (4m-1-2k)y),$$

which is written in the form of bounded $4m$–th differences

$$\Delta^{4m}_y f(x) := \sum_{k=0}^{4m} (-1)^k \binom{4m}{4m-k} f(x + ky) = 0$$

for all 2–dimensional vectors $(x, y) \in E_1 \times E_1$. Thus f is a mapping vanishing identically its kth difference with equal increments, where $k = 4m$ $(4m+1, 4m+2$ and $4m+3$, respectively$)$. The other three cases for (1.5), (1.7), (1.8) are similar to the above argument. Since $f(2x) = 2^k f(x)$ for all $x \in E_1$, we conclude from [5, 21] that there exists a symmetric k–additive mapping $V_k : E_1^k \to Y$ such that $f(x) = V_k(x, x, \cdots, x)$ for all $x \in E_1$, where $k = 4m-1$ ($4m, 4m+1$ and $4m+2$, respectively).

The converse is obvious. □

Thus it is natural that a mapping $T : E_1 \to Y$ is called *k–additive* if the mapping T satisfies *k–additive functional equation*, so called,

$$D_{m,i}T(x,y) = 0,$$

for all 2–dimensional vectors $(x,y) \in E_1 \times E_1$, where $k = 4m+i \geqslant 1$ is a positive integer for each $i = -1, 0, 1$ and 2.

3. Hyers-Ulam-Rassias stability

For convenience, we denote the perturbing terms of the equations (1.3), (1.5), (1.7) and (1.8) by $D_{m,-1}f(x,y), D_{m,0}f(x,y), D_{m,1}f(x,y)$ and $D_{m,2}f(x,y)$, respectively, as follows:

$$D_{m,-1}f(x,y) :=$$
$$2^{4m-1}\left[\sum_{k=0}^{2m}\binom{4m}{2k}f(x+(m-k)y)\right] - \sum_{k=0}^{2m-1}\binom{4m}{2k+1}f(2x+(2m-1-2k)y),$$
$$D_{m,0}f(x,y) :=$$
$$2^{4m}\left[\sum_{k=0}^{2m}\binom{4m+1}{2k}f(x+(m-k)y)\right] - \sum_{k=0}^{2m}\binom{4m+1}{2k+1}f(2x+(2m-1-2k)y),$$
$$D_{m,1}f(x,y) :=$$
$$2^{4m+1}\left[\sum_{k=0}^{2m}\binom{4m+2}{2k+1}f(x+(m-k)y)\right] - \sum_{k=0}^{2m+1}\binom{4m+2}{2k}f(2x+(2m+1-2k)y),$$

and

$$D_{m,2}f(x,y) :=$$
$$2^{4m+2}\left[\sum_{k=0}^{2m+1}\binom{4m+3}{2k+1}f(x+(m-k)y)\right] - \sum_{k=0}^{2m+1}\binom{4m+3}{2k}f(2x+(2m+1-2k)y)$$

for all 2–dimensional vectors $(x,y) \in E_1 \times E_1$.

In the following theorem, we are going to investigate the generalized Hyers–Ulam stability problem for $(4m+i)$–additive functional equation

$$D_{m,i}f(x,y) = 0,$$

for each $i = -1, 0, 1$ and 2 subject to $(4m+i) \geqslant 1$. That is, we are looking for conditions that an approximate mapping f of which the perturbing term satisfies the inequality

$$\|D_{m,i}f(x,y)\| \leqslant \phi_i(x,y)$$

differing by an approximate remainder ϕ_i is asymptotically close to a true mapping T_i satisfying

$$D_{m,i}T_i(x,y) = 0$$

for each $i = -1, 0, 1$ and 2. We will also construct the true mapping near a given approximate mapping with small error controlled by ϕ_i.

Let X be a topological vector space and let Y be a Banach space unless we give any specific reference.

THEOREM 3.1. *Suppose that for a fixed $i = -1, 0, 1, 2$ and a fixed nonnegative integer m with $(4m+i) \geqslant 1$ and $m = \begin{cases} m \geqslant 1 & \text{if } i = -1 \text{ or } 0, \\ m \geqslant 0 & \text{if } i = 1 \text{ or } 2, \end{cases}$ a mapping $f : X \to Y$ satisfies*

$$\|D_{m,i}f(x,y)\| \leqslant \phi_i(x,y) \tag{3.1}$$

for all $x, y \in X$ and the approximate remainder $\phi_i : X^2 \to \mathbb{R}^+$ is a mapping such that the series

$$\sum_{k=0}^{\infty} \frac{\phi_i(2^k x, 2^k y)}{{r_{m,i}}^k} \quad \left(\sum_{k=1}^{\infty} {r_{m,i}}^k \phi_i(\frac{x}{2^k}, \frac{y}{2^k}), \text{ respectively} \right)$$

converges for all $x, y \in X$, where $r_{m,i} := 2^{4m+i}$. Then there is a unique mapping $T_{m,i} : X \to Y$ which satisfies the equation

$$D_{m,i}T_{m,i}(x,y) = 0,$$

that is, $T_{m,i}$ is $(4m+i)$–additive mapping, and the inequality

$$\begin{gathered} \|f(x) - T_{m,i}(x)\| \leqslant \frac{1}{{r_{m,i}}^2} \sum_{k=0}^{\infty} \frac{\phi_i(2^k x, 0)}{{r_{m,i}}^k} \\ \left(\|f(x) - T_{m,i}(x)\| \leqslant \frac{1}{{r_{m,i}}^2} \sum_{k=1}^{\infty} {r_{m,i}}^k \phi_i(\frac{x}{2^k}, 0) \right) \end{gathered} \tag{3.2}$$

for all $x \in X$. The mapping $T_{m,i}$ is given by

$$T_{m,i}(x) = \lim_{n\to\infty} \frac{f(2^n x)}{{r_{m,i}}^n} \quad \left(T_{m,i}(x) = \lim_{n\to\infty} {r_{m,i}}^n f(\frac{x}{2^n}) \right) \tag{3.3}$$

for all $x \in X$.

If, moreover, f is continuous, then $T_{m,i}(rx) = r^{4m+i} T_{m,i}(x)$ for all $r \in \mathbb{R}$ and all $x \in X$.

Proof. Step 1. We show that the following inequality

$$\left\| f(x) - \frac{f(2^n x)}{{r_{m,i}}^n} \right\| \leqslant \frac{1}{{r_{m,i}}^2} \sum_{k=0}^{n-1} \frac{\phi_i(2^k x, 0)}{{r_{m,i}}^k} \tag{3.4}$$

holds for all positive integer n and all $x \in X$, where $r_{m,i} := 2^{4m+i}$.

Substituting y with 0 in (3.1), we obtain

$$\left\|f(x)-\frac{f(2x)}{r_{m,i}}\right\| \leqslant \frac{\phi_i(x,0)}{{r_{m,i}}^2} \tag{3.5}$$

for all $x \in X$. To use induction argument we assume that the inequality (3.4) holds for all positive integer n and all $x \in X$. The inequality (3.5) and triangle inequality yield

$$\begin{aligned}\left\|f(x)-\frac{f(2^{n+1}x)}{{r_{m,i}}^{n+1}}\right\| &\leqslant \left\|f(x)-\frac{f(2^n x)}{{r_{m,i}}^n}\right\| + \frac{1}{{r_{m,i}}^n}\left\|f(2^n x)-\frac{f(2\cdot 2^n x)}{r_{m,i}}\right\| \\ &\leqslant \frac{1}{{r_{m,i}}^2}\sum_{k=0}^{n-1}\frac{\phi_i(2^k x,0)}{{r_{m,i}}^k} + \frac{\phi_i(2^n x,0)}{{r_{m,i}}^{n+2}} \\ &= \frac{1}{{r_{m,i}}^2}\sum_{k=0}^{n}\frac{\phi_i(2^k x,0)}{{r_{m,i}}^k}\end{aligned} \tag{3.6}$$

for all $x \in X$. Thus it follows by induction argument that the inequality (3.4) holds for all positive integer n and all $x \in X$.

Step 2. We claim that the sequence $\{\frac{f(2^n x)}{{r_{m,i}}^n}\}$ is Cauchy in the Banach space Y. In fact, we see that for $n > l > 0$

$$\begin{aligned}\left\|\frac{f(2^l x)}{{r_{m,i}}^l}-\frac{f(2^n x)}{{r_{m,i}}^n}\right\| &\leqslant \left\|\frac{f(2^l x)}{{r_{m,i}}^l}-\frac{f(2^{l+1}x)}{{r_{m,i}}^{l+1}}\right\| + \cdots + \left\|\frac{f(2^{n-1}x)}{{r_{m,i}}^{n-1}}-\frac{f(2^n x)}{{r_{m,i}}^n}\right\| \\ &\leqslant \frac{1}{{r_{m,i}}^2}\sum_{k=0}^{n-l-1}\frac{\phi_i(2^{l+k}x,0)}{{r_{m,i}}^{l+k}} \\ &\to 0 \qquad \text{as } l \to \infty.\end{aligned} \tag{3.7}$$

Therefore, we may define a mapping $T_{m,i} : X \to Y$ by

$$T_{m,i}(x) = \lim_{n\to\infty}\frac{f(2^n x)}{{r_{m,i}}^n}$$

for all $x \in X$. Then by letting $n \to \infty$ in (3.4), we arrive at the formula (3.2).

Step 3. We show that the mapping $T_{m,i}$ is a solution of the equation $D_{m,i}f(x,y) = 0$ for each $i = -1,0,1$ and 2.

If we substitute $2^n x, 2^n y$ for x, y in (3.1) respectively, and divide the resulting inequality by ${r_{m,i}}^n$, and then take the limit as $n \to \infty$, then we see that

$$\begin{aligned}\|D_{m,i}T_{m,i}(x,y)\| &= \lim_{n\to\infty}\frac{1}{{r_{m,i}}^n}\left\|D_{m,i}f(2^n x, 2^n y)\right\| \\ &\leqslant \lim_{n\to\infty}\frac{\phi_i(2^n x,2^n y)}{{r_{m,i}}^n} = 0\end{aligned}$$

for all $x, y \in X$. Hence we find that $T_{m,i}$ satisfies the equation $D_{m,i}f(x,y) = 0$ for each $i = -1,0,1$ and 2.

Step 4. Let's prove the uniqueness of the mapping $T_{m,i}$ satisfying the equation

$$D_{m,i}T_{m,i}(x,y) = 0$$

and the inequality (3.2) for each $i = -1, 0, 1$ and 2.

Assume that there is another mapping $S : X \to Y$ which satisfies the equation

$$D_{m,i}S(x,y) = 0$$

and the inequality (3.2) for each $i = -1, 0, 1$ and 2. Since $T_{m,i}$ and S are $(4m+i)$–additive, we obtain that for each $i = -1, 0, 1$ and 2,

$$S(2^n x) = {r_{m,i}}^n S(x) \quad \text{and} \quad T_{m,i}(2^n x) = {r_{m,i}}^n T_{m,i}(x)$$

for all $x \in X$ and $n \in \mathbb{N}$. Hence it follows from (3.2) that

$$\begin{aligned}
\|S(x) - T_{m,i}(x)\| &= \frac{1}{{r_{m,i}}^n}\|S(2^n x) - T_{m,i}(2^n x)\| \\
&\leqslant \frac{1}{{r_{m,i}}^n}(\|S(2^n x) - f(2^n x)\| + \|f(2^n x) - T_{m,i}(2^n x)\|) \\
&\leqslant \frac{2}{{r_{m,i}}^2}\sum_{k=0}^{\infty}\frac{\phi_i(2^{n+k}x, 0)}{{r_{m,i}}^{n+k}} \\
&\to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

for all $x \in X$. Hence we find immediately the uniqueness of $T_{m,i}$.

Step 5. The proof of assertion indicated by parentheses in the theorem is similarly verified by the following inequality due to (3.5) and (3.4)

$$\left\|f(x) - {r_{m,i}}^n f\left(\frac{x}{2^n}\right)\right\| \leqslant \frac{1}{{r_{m,i}}^2}\sum_{k=1}^{n}{r_{m,i}}^k \phi_i\left(\frac{x}{2^k}, 0\right)$$

for all $x \in X$. In this case, a mapping $T_{m,i} : X \to Y$ is well defined by

$$T_{m,i}(x) = \lim_{n\to\infty} {r_{m,i}}^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$ by the similar method to (3.7). In particular, it should be noted that $T_{m,i}(0) = 0$ since $\sum_{k=1}^{\infty} {r_{m,i}}^k \phi_i(0,0) < \infty$ and thus $f(0) = 0$ according to the fact of $\phi_i(0,0) = 0$.

Step 6. We prove that $T_{m,i}$ is homogeneous of degree $4m+i$ for all real numbers.

For each r in $\mathbb{R}$, we choose a sequence $\{q_k\}$ of rational numbers such that $\{q_k\} \to r$ as $k \to \infty$. Then $f(2^n q_k x) \to f(2^n r x)$ by continuity of f as $k \to \infty$. Since the mapping $T_{m,i}(x) = V_{4m+i}(x, x, \cdots, x)$ is $(4m+i)$-additive by Lemma 2.1, one gets $T_{m,i}(kx) = k^{4m+i}T_{m,i}(x)$ for all integers k. Thus we lead to the relation

$$T_{m,i}(x) = T_{m,i}\left(k \cdot \frac{x}{k}\right) = k^{4m+i}T_{m,i}\left(\frac{x}{k}\right)$$

for all integers $k \neq 0$. Consequently it follows easily that $T_{m,i}(qx) = q^{4m+i}T_{m,i}(x)$ for all rational number q. Hence we figure out

$$\begin{aligned} T_{m,i}(rx) &= \lim_{n\to\infty} \frac{f(2^n rx)}{{r_{m,i}}^n} = \lim_{n\to\infty}\lim_{k\to\infty} \frac{f(2^n q_k x)}{{r_{m,i}}^n} \\ &= \lim_{n\to\infty}\lim_{k\to\infty} \frac{{q_k}^{4m+i} f(2^n x)}{{r_{m,i}}^n} = \lim_{n\to\infty} \frac{r^{4m+i} f(2^n x)}{{r_{m,i}}^n} \\ &= r^{4m+i}T_{m,i}(x) \end{aligned}$$

for all $x \in X$. Therefore $T_{m,i}$ is homogeneous of degree $(4m+i)$ for all real numbers, as desired. This completes the proof of the theorem. □

From the main Theorem 3.1, we obtain the following corollary concerning the Hyers-Ulam-Rassias stability of the equation $D_{m,i}f(x,y) = 0$.

COROLLARY 3.2. *Let X and Y be a normed space and a Banach space respectively. Under the same notations given in Theorem* 3.1*, let $\varepsilon \geqslant 0, p \neq 4m+i$ be real numbers. Suppose that a mapping $f : X \to Y$ satisfies*

$$\|D_{m,i}f(x,y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p) \tag{3.8}$$

for all $x, y \in X$ (for all $x, y \in X \setminus \{0\}$ if $p \leqslant 0$). Then there is a unique mapping $T_{m,i} : X \to Y$ which satisfies the equation

$$D_{m,i}T_{m,i}(x,y) = 0$$

and the inequality

$$\|f(x) - T_{m,i}(x)\| \leqslant \frac{\varepsilon\|x\|^p}{r_{m,i}|2^p - r_{m,i}|} \tag{3.9}$$

for all $x \in X$ (for all $x \in X \setminus \{0\}$ if $p \leqslant 0$). The mapping $T_{m,i}$ is given by

$$T_{m,i}(x) = \lim_{n\to\infty} \frac{f(2^n x)}{{r_{m,i}}^n} \quad \text{if} \quad p < 4m+i$$

$$\left(T_{m,i}(x) = \lim_{n\to\infty} {r_{m,i}}^n f\left(\frac{x}{{r_{m,i}}^n}\right) \quad \text{if} \quad p > 4m+i \right)$$

for all $x \in X$.

Furthermore, if f is continuous, then for each $x \in X$, $T_{m,i}(rx) = r^{4m+i}T_{m,i}(x)$ for all $r \in \mathbb{R}$.

Proof. Taking $\phi_i(x,y) := \varepsilon(\|x\|^p + \|y\|^p)$ in Theorem 3.1 for each $i = -1, 0, 1$ and 2, we obtain the conclusions concerning the stability of the equation $D_{m,i}f_i(x,y) = 0$. If furthermore f is continuous, then the mapping $T_{m,i} : X \to Y$ satisfies $T_{m,i}(rx) = r^{4m+i}T_{m,i}(x)$ for all $x \in X$ and all $r \in \mathbb{R}$ by the same reasoning as the proof of Theorem 3.1. □

The following corollary implies the Hyers–Ulam stability of the equation $D_{m,i}f(x,y) = 0$, which is an immediate consequence of Theorem 3.1.

COROLLARY 3.3. *Let X and Y be a normed space and a Banach space, respectively, and let $\varepsilon \geqslant 0$ be a real number. Under the same notations given in Theorem* 3.1*, suppose that a mapping $f : X \to Y$ satisfies*

$$\|D_{m,i}f(x,y)\| \leqslant \varepsilon \tag{3.10}$$

for all $x, y \in X$. Then there is a unique mapping $T_{m,i} : X \to Y$ defined by $T_{m,i}(x) = \lim_{n\to\infty} \frac{f(2^n x)}{{r_{m,i}}^n}$ which satisfies the equation

$$D_{m,i}T_{m,i}(x,y) = 0$$

and the inequality

$$\|f(x) - T_{m,i}(x)\| \leqslant \frac{\varepsilon}{r_{m,i}(r_{m,i}-1)} \tag{3.11}$$

for all $x \in X$.

Further, if f is continuous, then $T_{m,i}(rx) = r^{4m+i}T_{m,i}(x)$ holds for all $x \in X$ and all $r \in \mathbb{R}$.

Thus as an application of Corollary 3.3, we answer the Hyers–Ulam question as follows: Given $\epsilon > 0$, there exists a $\delta(\epsilon) := r_{m,i}(r_{m,i}-1)\epsilon > 0$ such that if a mapping $f : X \to Y$ satisfies

$$\|D_{m,i}f(x,y)\| \leqslant \delta$$

for all $x, y \in X$, then there is a $(4m+i)$–additive mapping $T_{m,i} : X \to Y$ which satisfies the equation

$$D_{m,i}T_{m,i}(x,y) = 0$$

and the inequality

$$\|f(x) - T_{m,i}(x)\| \leqslant \epsilon$$

for all $x \in X$.

The following corollary states that a continuous approximate real function of the equation $D_{m,i}f(x,y) = 0$ can be approximated by a polynomial of degree $(4m+i)$, exactly.

COROLLARY 3.4. *Let ϕ_i be a such mapping given in Theorem* 3.1*. Suppose that a mapping $f : \mathbb{R} \to \mathbb{R}$ satisfies*

$$\|D_{m,i}f(x,y)\| \leqslant \phi_i(x,y)$$

for all $x, y \in \mathbb{R}$. If f is continuous, then there is a unique polynomial $T_{m,i} : \mathbb{R} \to \mathbb{R}$ defined by $T_{m,i}(x) = T_{m,i}(1)x^{4m+i}$ which satisfies the equation

$$D_{m,i}T_{m,i}(x,y) = 0$$

and the inequality

$$\|f(x) - T_{m,i}(1)x^{4m+i}\| \leqslant \frac{1}{{r_{m,i}}^2}\sum_{k=0}^{\infty}\frac{\phi_i(2^k x, 0)}{{r_{m,i}}^k}$$

$$\left(\|f(x) - T_{m,i}(1)x^{4m+i}\| \leqslant \frac{1}{{r_{m,i}}^2}\sum_{k=1}^{\infty}{r_{m,i}}^k\phi_i\left(\frac{x}{2^k}, 0\right)\right)$$

for all $x \in X$.

Proof. From Theorem 3.1, we obtain the conclusions concerning the stability of the equation $D_{m,i}f(x,y) = 0$ for each $i = -1, 0, 1$ and 2. Under the assumption that f is continuous, the mapping $T_{m,i} : \mathbb{R} \to \mathbb{R}$ satisfies $T_{m,i}(xr) = x^{4m+i}T_{m,i}(r)$ for all $x \in X$ and all $r \in \mathbb{R}$ by the same reasoning as the proof of Theorem 3.1. Thus $T_{m,i}(x) = x^{4m+i}T_{m,i}(1)$ is a polynomial satisfying the results stated in the corollary. □

4. Stability of Eq. (1.3) in Banach modules

In the last part of this paper, let B be a unital Banach algebra with norm $|\cdot|$, and let ${}_B\mathbb{M}_1$ and ${}_B\mathbb{M}_2$ be left Banach B–modules with norms $||\cdot||$ and $\|\cdot\|$ respectively.

As an application of the main Theorem 3.1 we are going to prove the generalized Hyers-Ulam stability problem of the functional equation (1.3) in Banach modules over a unital Banach algebra.

THEOREM 4.1. *Let $\phi_{-1} : {}_B\mathbb{M}_1 \times {}_B\mathbb{M}_1 \to \mathbb{R}^+$ be the mapping satisfying the conditions in Theorem 3.1. Suppose that for some positive integer m, a mapping $f : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ satisfies*

$$\begin{aligned} \|D^u_{m,-1}f(x,y)\| :&= \Big\|2^{4m-1}\Big[\sum_{k=0}^{2m}\binom{4m}{2k}f(ux+(m-k)uy)\Big] \\ &\qquad - u^{4m-1}\sum_{k=0}^{2m-1}\binom{4m}{2k+1}f(2x+(2m-1-2k)y)\Big\| \\ &\leqslant \phi_{-1}(ux,uy) \end{aligned} \tag{4.1}$$

for all $u \in B(|u| = 1)$ and for all $x, y \in {}_B\mathbb{M}_1$. If f is continuous, then there is a unique $(4m-1)$–additive mapping $T_{-1} : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ defined by (3.3) with $i = -1$ which satisfies the equation

$$D_{m,-1}T_{-1}(x,y) = 0, \quad T_{-1}(bx) = b^{4m-1}T_{-1}(x) \tag{4.2}$$

and the inequality (3.2) with $i = -1$ for all $b \in B$ and for all $x, y \in {}_B\mathbb{M}_1$.

Proof. By Theorem 3.1 it follows from the inequality of the statement for $u = 1$ that there is a unique $(4m-1)$–additive mapping $T_{-1} : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ defined by (3.3) with $i = -1$ which satisfies the equation

$$D_{m,-1}T_{-1}(x,y) = 0$$

and the inequality (3.2) with $i = -1$ for all $x, y \in {}_B\mathbb{M}_1$. Under the assumption that f is continuous, the mapping T_{-1} satisfies

$$T_{-1}(rx) = r^{4m-1}T_{-1}(x), \quad \forall x \in {}_B\mathbb{M}_1, \forall r \in \mathbb{R}.$$

Replacing x, y by $2^n x, 2^n y$ in (4.1), respectively, and dividing it by ${r_{m,-1}}^n$, we figure out

$$\|D^u_{m,-1}T_{-1}(x,y)\| = \lim_{n\to\infty}\frac{\|D^u_{m,-1}f(2^n x, 2^n y)\|}{{r_{m,-1}}^n} \leqslant \lim_{n\to\infty}\frac{\|\phi_{-1}(2^n ux, 2^n uy)\|}{{r_{m,-1}}^n} = 0$$

for all $u \in B(|u|=1)$ and for all $x, y \in {}_B\mathbb{M}_1$. Thus one concludes that T_{-1} satisfies

$$\begin{aligned} 0 &= D^u_{m,-1} T_{-1}(x,0) \\ &= 2^{4m-1} \sum_{k=0}^{2m} \binom{4m}{2k} T_{-1}(ux) - u^{4m-1} \sum_{k=0}^{2m-1} \binom{4m}{2k+1} T_{-1}(2x), \end{aligned}$$

which yields

$$T_{-1}(ux) = u^{4m-1} T_{-1}(x) \tag{4.3}$$

for all $u \in B(|u|=1)$ and for all $x \in {}_B\mathbb{M}_1$.

The last equality is also true for $u=0$ vacuously. Now for each element $b \in B$ $(b \neq 0)$ we figure out by (4.3)

$$\begin{aligned} T_{-1}(bx) &= T_{-1}(|b| \cdot \frac{b}{|b|} x) = |b|^{4m-1} \cdot T_{-1}(\frac{b}{|b|} x) \\ &= |b|^{4m-1} \cdot \frac{b^{4m-1}}{|b|^{4m-1}} \cdot T_{-1}(x) = b^{4m-1} T_{-1}(x) \end{aligned}$$

for all $b \in B(b \neq 0)$ and all $x \in {}_B\mathbb{M}_1$. So the mapping T_{-1} satisfies

$$T_{-1}(bx) = b^{4m-1} T_{-1}(x)$$

for all $b \in B$ and for all $x \in {}_B\mathbb{M}_1$, as desired. This completes the proof of the theorem. □

Since $\mathbb{C}$ is a Banach algebra, the Banach spaces M_1 and M_2 are considered as Banach modules over $\mathbb{C}$. Thus we have the following corollary.

COROLLARY 4.2. *Let ϕ_{-1} be the mapping defined in Theorem 4.1. Let M_1 and M_2 be Banach spaces over the complex field $\mathbb{C}$. Suppose that a mapping $f : M_1 \to M_2$ satisfies*

$$\|D^u_{m,-1} f(x,y)\| \leqslant \phi_{-1}(ux, uy)$$

for all $u \in \mathbb{C}(|u|=1)$ and for all $x, y \in M_1$. If f is continuous, then there is a unique $(4m-1)$–additive mapping $T_{-1} : M_1 \to M_2$ which satisfies the equation (4.2) and the inequality (3.2) with $m=-1$ for all $b \in \mathbb{C}$ and for all $x, y \in M_1$.

THEOREM 4.3. *Let ϕ_{-1} be the mapping defined in Theorem 4.1. Suppose that a mapping $f : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ satisfies*

$$\begin{aligned} \Big\| 2^{4m-1} u^{4m-1} \Big[\sum_{k=0}^{2m} \binom{4m}{2k} f(x+(m-k)y) \Big] - \sum_{k=0}^{2m-1} \binom{4m}{2k+1} f(2ux+(2m-1-2k)uy) \Big\| \\ \leqslant \phi_{-1}(ux, uy) \end{aligned}$$

for all $u \in B(|u|=1)$ and for all $x, y \in {}_B\mathbb{M}_1$. If f is continuous, then there is a unique $(4m-1)$–additive mapping $T_{-1} : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ defined by (3.3) with $m=-1$ which satisfies the equation (4.2) and the inequality (3.2) with $m=-1$ for all $b \in B$ and for all $x, y \in {}_B\mathbb{M}_1$.

Proof. The proof is similar to that of Theorem 4.1. □

The generalized Hyers–Ulam stability problem in Banach modules over a unital Banach algebra for other cases,

$$\|D^u_{m,0}f(x,y)\| \leqslant \phi_0(ux,uy),$$
$$\|D^u_{m,1}f(x,y)\| \leqslant \phi_1(ux,uy),$$
$$\text{and} \qquad \|D^u_{m,2}f(x,y)\| \leqslant \phi_2(ux,uy)$$

is proved by the same manner as that of Theorem 4.1.

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Kil-Woung Jun
Department of Mathematics
Chungnam National University
Daejeon, 305-764
Korea
e-mail: kwjun@math.cnu.ac.kr

Corresponding author: Hark-Mahn Kim
Department of Mathematics
Chungnam National University
Daejeon, 305-764
Korea
e-mail: hmkim@math.cnu.ac.kr