A NOTE ON GRÜSS TYPE INEQUALITIES
VIA CAUCHY'S MEAN VALUE THEOREM

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Abstract. The main aim of the present note is to establish two new weighted Grüss type integral inequalities by using a fairly elementary analysis.

1. Introduction

The well known Grüss integral inequality [4] can be stated as follows (see [5, p. 296]):

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \left(\frac{1}{b-a} \int_a^b f(x) \, dx \right) \left(\frac{1}{b-a} \int_a^b g(x) \, dx \right) \right|$$

$$\leqslant \frac{1}{4}(P-p)(Q-q), \tag{1.1}$$

provided that f and g are two integrable functions on $[a,b]$ such that

$$p \leqslant f(x) \leqslant P, \quad q \leqslant g(x) \leqslant Q,$$

for all $x \in [a,b]$, where p, P, q, Q are real constants.

Over the years the inequality (1.1) is studied extensively by many investigators and numerous generalizations, variants and extensions have appeared in the literature, see [1, 3, 5–9] and the references cited therein. The main aim of this note is to establish two new weighted integral inequalities similar to the Grüss inequality given in (1.1). The analysis used in the proofs is based on the application of Cauchy's mean value theorem and our results provide new estimates on this type of inequality.

2. Statement of Results

In what follows, R and $'$ denotes the set of real numbers and the derivative of a function. For continuous functions $p, q : [a, b] \to R$ which are differentiable on (a, b) and $w : [a, b] \to [0, \infty)$ an integrable function such that $\int_a^b w(x)\, dx > 0$, we use the notation

$$G[w, p, q] = \int_a^b w(x)\, p(x)\, q(x)\, dx - \frac{\left(\int_a^b w(x)\, p(x)\, dx\right)\left(\int_a^b w(x)\, q(x)\, dx\right)}{\int_a^b w(x)\, dx},$$

to simplify the details of presentation and if $q'(t) \neq 0$ for each $t \in (a, b)$ we define

$$\left\|\frac{p'}{q'}\right\|_\infty = \sup_{t \in (a, b)} \left|\frac{p'(t)}{q'(t)}\right| < \infty.$$

Our main result on weighted Grüss type inequality is given in the following theorem.

THEOREM 1. *Let $f, g, h : [a, b] \to R$ be continuous on $[a, b]$ and differentiable on (a, b) and $w : [a, b] \to [0, \infty)$ be an integrable function such that $\int_a^b w(x)\, dx > 0$. If $h'(t) \neq 0$ for each $t \in (a, b)$, then*

$$|G[w, f, g]| \leqslant \frac{1}{2}\left\{|G[w, g, h]|\left\|\frac{f'}{h'}\right\|_\infty + |G[w, f, h]|\left\|\frac{g'}{h'}\right\|_\infty\right\}. \tag{2.1}$$

A slight variant of Theorem 1 is embodied in the following theorem.

THEOREM 2. *Let f, g, h be as in Theorem 1. Then*

$$|G[w, f, g]| \leqslant \left|\int_a^b w(x)\, h^2(x)\, dx - \frac{\left(\int_a^b w(x)\, h(x)\, dx\right)^2}{\int_a^b w(x)\, dx}\right|\left\|\frac{f'}{h'}\right\|_\infty\left\|\frac{g'}{h'}\right\|_\infty. \tag{2.2}$$

3. Proofs of Theorems 1 and 2

Let $x, y \in [a, b]$ with $y \neq x$. Applying Cauchy's mean value theorem, there exist points c and d between y and x such that (see [2,10])

$$f(x) - f(y) = \frac{f'(c)}{h'(c)}\{h(x) - h(y)\}, \tag{3.1}$$

$$g(x) - g(y) = \frac{g'(d)}{h'(d)}\{h(x) - h(y)\}. \tag{3.2}$$

Multiplying both sides of (3.1) and (3.2) by $g(x)$ and $f(x)$ respectively and adding the resulting identities we get

$$2f(x)g(x) - g(x)f(y) - f(x)g(y)$$
$$= \frac{f'(c)}{h'(c)}\{g(x)h(x) - g(x)h(y)\} + \frac{g'(d)}{h'(d)}\{f(x)h(x) - f(x)h(y)\}. \tag{3.3}$$

Multiplying both sides of (3.3) by $w(y)$ and integrating the resulting identity with respect to y over $[a,b]$ we have

$$2\left(\int_a^b w(y)\,dy\right)f(x)g(x) - g(x)\int_a^b w(y)f(y)\,dy - f(x)\int_a^b w(y)g(y)\,dy$$
$$= \frac{f'(c)}{h'(c)}\left\{\left(\int_a^b w(y)\,dy\right)g(x)h(x) - g(x)\int_a^b w(y)h(y)\,dy\right\}$$
$$+ \frac{g'(d)}{h'(d)}\left\{\left(\int_a^b w(y)\,dy\right)f(x)h(x) - f(x)\int_a^b w(y)h(y)\,dy\right\}. \tag{3.4}$$

Next, multiplying both sides of (3.4) by $w(x)$ and integrating the resulting identity with respect to x over $[a,b]$ we have

$$2\left(\int_a^b w(y)\,dy\right)\int_a^b w(x)f(x)g(x)\,dx - \left(\int_a^b w(x)g(x)\,dx\right)\left(\int_a^b w(y)f(y)\,dy\right)$$
$$- \left(\int_a^b w(x)f(x)\,dx\right)\left(\int_a^b w(y)g(y)\,dy\right)$$
$$= \frac{f'(c)}{h'(c)}\left\{\left(\int_a^b w(y)\,dy\right)\int_a^b w(x)g(x)h(x)\,dx\right.$$
$$\left. - \left(\int_a^b w(x)g(x)\,dx\right)\left(\int_a^b w(y)h(y)\,dy\right)\right\}$$
$$+ \frac{g'(d)}{h'(d)}\left\{\left(\int_a^b w(y)\,dy\right)\int_a^b w(x)f(x)h(x)\,dx\right.$$
$$\left. - \left(\int_a^b w(x)f(x)\,dx\right)\left(\int_a^b w(y)h(y)\,dy\right)\right\}. \tag{3.5}$$

From (3.5) it is easy to observe that

$$G\left[w,f,g\right] = \frac{1}{2}\left\{\frac{f'(c)}{h'(c)}G\left[w,g,h\right] + \frac{g'(d)}{h'(d)}G\left[w,f,h\right]\right\}. \tag{3.6}$$

Using the properties of modulus, from (3.6) we have

$$\left|G\left[w,f,g\right]\right| \leqslant \frac{1}{2}\left\{\left|G\left[w,g,h\right]\right|\left\|\frac{f'}{h'}\right\|_{\infty} + \left|G\left[w,f,h\right]\right|\left\|\frac{g'}{h'}\right\|_{\infty}\right\}.$$

This is the required inequality in (2.1) and the proof of Theorem 1 is complete. To prove Theorem 2, first we multiply both sides of (3.1) and (3.2) by $w(y)$ and integrate the resulting identities with respect to y over $[a,b]$ to get

$$\left(\int_a^b w(y)\,dy\right)f(x) - \int_a^b w(y)f(y)\,dy$$

$$= \frac{f'(c)}{h'(c)}\left\{\left(\int_a^b w(y)\,dy\right)h(x) - \int_a^b w(y)h(y)\,dy\right\}, \tag{3.7}$$

and

$$\left(\int_a^b w(y)\,dy\right)g(x) - \int_a^b w(y)g(y)\,dy$$

$$= \frac{g'(d)}{h'(d)}\left\{\left(\int_a^b w(y)\,dy\right)h(x) - \int_a^b w(y)h(y)\,dy\right\}. \tag{3.8}$$

Multiplying the left sides and right sides of (3.7) and (3.8) we get

$$\left(\int_a^b w(y)\,dy\right)^2 f(x)g(x) - \left(\int_a^b w(y)\,dy\right)f(x)\left(\int_a^b w(y)g(y)\,dy\right)$$

$$- \left(\int_a^b w(y)\,dy\right)g(x)\left(\int_a^b w(y)f(y)\,dy\right)$$

$$+ \left(\int_a^b w(y)f(y)\,dy\right)\left(\int_a^b w(y)g(y)\,dy\right)$$

$$= \left\{\left(\int_a^b w(y)\,dy\right)^2 h^2(x) + \left(\int_a^b w(y)h(y)\,dy\right)^2\right.$$

$$\left.- 2\left(\int_a^b w(y)\,dy\right)h(x)\left(\int_a^b w(y)h(y)\,dy\right)\right\}\left\{\frac{f'(c)}{h'(c)}\right\}\left\{\frac{g'(d)}{h'(d)}\right\}. \tag{3.9}$$

Multiplying both sides of (3.9) by $w(x)$ and integrating the resulting identity with respect to x over $[a,b]$ we get

$$\left(\int_a^b w(y)\,dy \right)^2 \int_a^b w(x) f(x) g(x)\,dx$$

$$- \left(\int_a^b w(y)\,dy \right) \left(\int_a^b w(x) f(x)\,dx \right) \left(\int_a^b w(y) g(y)\,dy \right)$$

$$- \left(\int_a^b w(y)\,dy \right) \left(\int_a^b w(x) g(x)\,dx \right) \left(\int_a^b w(y) f(y)\,dy \right)$$

$$+ \left(\int_a^b w(x)\,dx \right) \left(\int_a^b w(y) f(y)\,dy \right) \left(\int_a^b w(y) g(y)\,dy \right)$$

$$= \left\{ \left(\int_a^b w(y)\,dy \right)^2 \left(\int_a^b w(x) h^2(x)\,dx \right) + \left(\int_a^b w(x)\,dx \right) \left(\int_a^b w(y) h(y)\,dy \right)^2 \right.$$

$$\left. -2 \left(\int_a^b w(y)\,dy \right) \left(\int_a^b w(x) h(x)\,dx \right) \left(\int_a^b w(y) h(y)\,dy \right) \right\}$$

$$\times \left\{ \frac{f'(c)}{h'(c)} \right\} \left\{ \frac{g'(d)}{h'(d)} \right\}.$$

$$(3.10)$$

From (3.10) it is easy to observe that

$$G[w, f, g] = \left\{ \int_a^b w(x) h^2(x)\,dx - \frac{\left(\int_a^b w(x) h(x)\,dx \right)^2}{\int_a^b w(x)\,dx} \right\} \times \left\{ \frac{f'(c)}{h'(c)} \right\} \left\{ \frac{g'(d)}{h'(d)} \right\}.$$

$$(3.11)$$

Using the properties of modulus, from (3.11) we get the desired inequality in (2.2). The proof of Theorem 2 is complete.

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