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INEQUALITY OF O'NEIL–TYPE FOR CONVOLUTIONS ASSOCIATED WITH THE LAPLACE–BESSEL DIFFERENTIAL OPERATOR AND APPLICATIONS

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Abstract. In this paper we prove an O'Neil-type inequality for the convolution operator (B–convolution) associated with the Laplace-Bessel differential operator. By using an O'Neil-type inequality for rearrangements we obtain a pointwise rearrangement estimate of the B–convolution. As an application, we obtain necessary and sufficient conditions on the parameters for the boundedness of the fractional B-maximal operator and B–fractional integral operator with rough kernels from the spaces $L_{p,\gamma}$ to $L_{q,\gamma}$ and from the spaces $L_{1,\gamma}$ to the weak spaces $WL_{q,\gamma}$.

1. Introduction

The potential type integral operators associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}$$

(see [1, 4, 6, 7, 11]), are playing an important role in harmonic analysis, theory of functions and partial differential equations. Here we study the convolution (B-convolution), the fractional maximal function (fractional B-maximal function) and fractional integral (B-fractional integral) with rough kernel, associated with the Laplace-Bessel differential operator.

Let $\mathbb{R}^n_+ = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : \ x_n > 0\}$, and define

$$L_{p,\gamma} \equiv L_{p,\gamma}(\mathbb{R}^n_+) = \left\{ f \ : \|f\|_{L_{p,\gamma}} \equiv \left(\int_{\mathbb{R}^n_+} |f(x)|^p x_n^\gamma dx \right)^{1/p} < \infty \right\},$$

where $\gamma > 0$ is a fixed parameter and $1 \leqslant p < \infty$.

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Denote by T^y the shift operator (B–shift) acting according to the law

$$T^y f(x) = C_\gamma \int_0^\pi f\left(x' - y', \sqrt{x_n^2 - 2x_n y_n \cos\alpha + y_n^2}\right) \sin^{\gamma-1} \alpha d\alpha,$$

where $C_\gamma \equiv \left(\int_0^\pi \sin^{\gamma-1}\alpha d\alpha\right)^{-1} = \pi^{-\frac{1}{2}} \Gamma\left(\frac{\gamma+1}{2}\right) [\Gamma\left(\frac{\gamma}{2}\right)]^{-1}$ and $x = (x', x_n)$, $x' \in R^{n-1}$.

We remark that the B-shift is closely related to the Laplace-Bessel differential operator Δ_B. The shift operator T^y generates the corresponding convolution (B-convolution)

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_+} f(y) T^y g(x) y_n^\gamma dy.$$

The paper is organized as follows. In Section 2, we give some lemmas needed to facilitate the proofs of our theorems. In Section 3, we show that an O'Neil-type inequality for rearrangements of the B-convolution holds. In Section 4, we prove an O'Neil-type inequality for B-convolutions. In Section 5, we obtain rearrangement estimates for the fractional B-maximal function and B-fractional integral. We prove the boundedness of the fractional B-maximal operator and B–fractional integral operator with rough kernels from the spaces $L_{p,\gamma}$ to $L_{q,\gamma}$ and from the spaces $L_{1,\gamma}$ to the weak spaces $WL_{q,\gamma}$. We show that the conditions on the parameters ensuring the boundedness cannot be weakened.

2. Some auxiliary lemmas

In this section we formulate some lemmas that will be needed later.

For the B-shift operator the following two lemmas hold.

LEMMA 1. *1. Let* $1 \leqslant p \leqslant \infty$, $f \in L_{p,\gamma}(\mathbb{R}^n_+)$, *then for all* $y \in \mathbb{R}^n_+$

$$\|T^y f(\cdot)\|_{L_{p,\gamma}} \leqslant \|f\|_{L_{p,\gamma}}. \tag{1}$$

(see [10]).

2. Let $1 \leqslant p, r \leqslant q \leqslant \infty$, $1/p' + 1/q = 1/r$, $pp' = p + p'$, $f \in L_{p,\gamma}(\mathbb{R}^n_+)$, $g \in L_{r,\gamma}(\mathbb{R}^n_+)$. *Then* $f \otimes g \in L_{q,\gamma}(\mathbb{R}^n_+)$ *and*

$$\|f \otimes g\|_{L_{q,\gamma}} \leqslant \|f\|_{L_{p,\gamma}} \|g\|_{L_{r,\gamma}}. \tag{2}$$

LEMMA 2. *For any measurable set* $\mathscr{A} = (\mathscr{A}', \mathscr{A}_n) \subset \mathbb{R}^n_+$, $\mathscr{A}' = \mathscr{A}_1 \times ... \times \mathscr{A}_{n-1} \subset \mathbb{R}^{n-1}$, $\mathscr{A}_n \subset (0,\infty)$ *and for any* $y \in \mathbb{R}^n_+$ *the following equality holds*

$$\int_{\mathscr{A}} T^y g(x)\, x_n^\gamma dx = C_\gamma \int_{(y,0)+\tilde{\mathscr{A}}} g\left(z', \sqrt{z_n^2 + z_{n+1}^2}\right) d\mu(z, z_{n+1}), \tag{3}$$

where $\tilde{\mathscr{A}} = \mathscr{A}' \times (-m, m) \times [0, m)$, $m = \sup \mathscr{A}_n$, $d\mu(z, z_{n+1}) = z_{n+1}^{\gamma-1} dz dz_{n+1}$.

The proof of Lemma 2 is straightforward after applying the following substitutions

$$z' = x', \quad z_n = x_n \cos\alpha, \quad z_{n+1} = x_n \sin\alpha. \tag{4}$$

The following two Hardy inequalities (see [12]) have an important role in proving our main results:

LEMMA 3. *Let $1 < p \leqslant q < \infty$. There exists a constant C independent of the function φ such that*

$$\left(\int_0^\infty \left(\int_0^t \varphi(s)ds\right)^q w(t)dt\right)^{1/q} \leqslant C\left(\int_0^\infty \varphi(t)^p v(t)dt\right)^{1/p} \tag{5}$$

if and only if

$$K = \sup_{r>0}\left(\int_r^\infty w(t)dt\right)^{1/q}\left(\int_0^r v(t)^{1-p'}dt\right)^{1/p'} < \infty, \tag{6}$$

where $p + p' = pp'$. Moreover, if C is the best constant in (5), then

$$K \leqslant C \leqslant k(p,q)K. \tag{7}$$

Here the constant $k(p,q)$ in (7) can be written in various forms. For example (see [14])

$$k(p,q) = p^{1/q}(p')^{1/p'} \text{ or } k(p,q) = q^{1/q}(q')^{1/p'} \text{ or } k(p,q) = \left(1+\frac{q}{p'}\right)^{1/q}\left(1+\frac{p'}{q}\right)^{1/p'}.$$

LEMMA 4. *Let $1 < p \leqslant q < \infty$ and let v and w be two functions measurable and positive a.e. on $(0,\infty)$. Then there exists a constant C independent of the function φ such that*

$$\left(\int_0^\infty \left(\int_t^\infty \varphi(s)ds\right)^q w(t)dt\right)^{1/q} \leqslant C\left(\int_0^\infty \varphi(t)^p v(t)dt\right)^{1/p} \tag{8}$$

if and only if

$$K_1 = \sup_{r>0}\left(\int_0^r w(t)dt\right)^{1/q}\left(\int_r^\infty v(t)^{1-p'}dt\right)^{1/p'} < \infty.$$

Moreover, the best constant C in (8) satisfies the inequalities $K_1 \leqslant C \leqslant k(p,q)K_1$.

3. O'Neil-type inequality for rearrangements of B-convolutions

In this section, we establish a relation between shift operator $T^y f$ and γ-rearrangement of f. We show that for the B-convolution an O'Neil-type inequality for rearrangements holds.

Let $f:\mathbb{R}_+^n\to\mathbb{R}$ be a measurable function and for any measurable set E, $|E|_\gamma=\int_E x_n^\gamma dx$. We define γ-rearrangement of f in decreasing order by

$$f^*(t)=\inf\{s>0\,:\,f_*(s)\leqslant t\},\quad \forall t\in[0,\infty),$$

where $f_*(s)$ denotes the γ-distribution function of f given by

$$f_*(s)=\left|\{x\in\mathbb{R}_+^n:|f(x)|>s\}\right|_\gamma .$$

We note the following properties of γ-rearrangement of functions (see [3, 13]):

1) if $0<p<\infty$, then

$$\int_{\mathbb{R}_+^n}|f(x)|^p x_n^\gamma dx=\int_0^\infty (f^*(t))^p\,dt; \tag{9}$$

2) for any $t>0$,

$$\sup_{|E|_\gamma=t}\int_E |f(x)|x_n^\gamma dx=\int_0^t f^*(s)ds; \tag{10}$$

3)

$$\int_{\mathbb{R}_+^n}|f(x)g(x)|x_n^\gamma dx\leqslant\int_0^\infty f^*(t)g^*(t)dt. \tag{11}$$

The function f^{**} on $(0,\infty)$ is defined by $f^{**}(t)=\frac{1}{t}\int\limits_0^t f^*(s)ds,\quad t>0.$

We denote by $WL_{p,\gamma}(\mathbb{R}_+^n)$ the weak $L_{p,\gamma}$ space of all measurable functions f with finite norm

$$\|f\|_{WL_{p,\gamma}}=\sup_{t>0}t^{1/p}f^*(t),\qquad 1\leqslant p<\infty.$$

LEMMA 5. *For any measurable set $\mathscr{A}\subset\mathbb{R}_+^n$ and for any $y\in\mathbb{R}_+^n$*

$$\sup_{|\mathscr{A}|_\gamma=t}\int_{\mathscr{A}} T^y|f(x)|x_n^\gamma dx=C_\gamma\int_0^t f^*(s)ds. \tag{12}$$

Proof. By Lemma 2 we have

$$\int_{\mathscr{A}} T^y|f(x)|x_n^\gamma dx=C_\gamma\int_{(y,0)+\tilde{\mathscr{A}}}\left|\overline{f}(z,z_{n+1})\right|d\mu(z,z_{n+1}), \tag{13}$$

where $\overline{f}(z,z_{n+1})=f\left(z',\sqrt{z_n^2+z_{n+1}^2}\right)$, $z_{n+1}>0$, $d\mu(z,z_{n+1})=z_{n+1}^{\gamma-1}dzdz_{n+1}$. For the function $\overline{f}(z,z_{n+1})$ the analogous equality (10) is also valid (see, for example [3])

$$\sup_{\mu(\tilde{\mathscr{A}})=t}\int_{\tilde{\mathscr{A}}}|\overline{f}(z,z_{n+1})|d\mu(z,z_{n+1})=C_\gamma\int_0^t\left(\overline{f}\right)_\mu^*(s)ds, \tag{14}$$

where $\left(\overline{f}\right)_{\mu}^{*}(s) = \inf\left\{t > 0 \;:\; \mu\left(\left\{(z, z_{n+1}) \;:\; \left|\overline{f}(z, z_{n+1})\right| > t\right\}\right) \leqslant s\right\}$.

Note that $\mu\left((y,0) + \tilde{\mathscr{A}})\right) = |\mathscr{A}|_{\gamma}$ and $\left(\overline{f}\right)_{\mu}^{*}(s) = f^{*}(s)$. Indeed, taking into account (4) we have

$$\mu\left(\left\{(z, z_{n+1}) \in \mathbb{R}_{+}^{n+1} : \left|\overline{f}(z, z_{n+1})\right| > t\right\}\right) = \int\limits_{\{x \in \mathbb{R}_{+}^{n} : |f(x)| > t\}} x_n^{\gamma} dx = f_{*}(t).$$

Consequently,

$$\left(\overline{f}\right)_{\mu}^{*}(s) = \inf\left\{t > 0 \;:\; f_{*}(t) \leqslant s\right\} = f^{*}(s).$$

By (13) and (14) we have

$$\begin{aligned}\sup_{|\mathscr{A}|_{\gamma}=t} \int_{\mathscr{A}} T^{y}|f(x)| x_n^{\gamma} dx &= C_{\gamma} \sup_{\mu(\overline{\mathscr{A}})=t} \int_{(y,0)+\overline{\mathscr{A}}} |\overline{f}(z, z_{n+1})| d\mu(z, z_{n+1}) \\ &= C_{\gamma} \int_0^t \left(\overline{f}\right)_{\mu}^{*}(s) ds = C_{\gamma} \int_0^t f^{*}(s) ds.\end{aligned}$$

Thus Lemma 5 is proved. □

The following theorem is one of our main results which shows that an O'Neil-type inequality for rearrangements of the B-convolution holds. The methods of the proof used here are close to those in [8].

THEOREM 1. *Let f, g be positive measurable functions on $\mathbb{R}_{+}^{n}$. Then for all* $0 < t < \infty$

$$(f \otimes g)^{**}(t) \leqslant C_{\gamma}\left(f^{**}(t) \int_0^t g^{**}(u) du + \int_t^{\infty} f^{*}(u) g^{**}(u) du\right). \tag{15}$$

Proof. For $t > 0$ we choose a measurable set E_t such that

$$\left\{x \in \mathbb{R}_{+}^{n} : |f(x)| > f^{*}(t)\right\} \subset E_t \subset \left\{x \in \mathbb{R}_{+}^{n} : |f(x)| \geqslant f^{*}(t)\right\}.$$

Let

$$f_1(x) = \left(f(x) - f^{*}(t)\right) \chi_{E_t}(x), \qquad f_2(x) = f(x) - f_1(x).$$

For any measurable set $\mathscr{A} \subset \mathbb{R}_{+}^{n}$ with measure $|\mathscr{A}|_{\gamma} = t$, we have

$$\int_{\mathscr{A}} (g \otimes f_1)(x) x_n^{\gamma} dx = \int_{\mathbb{R}_{+}^{n}} f_1(y) y_n^{\gamma} dy \int_{\mathscr{A}} T^{y} g(x) x_n^{\gamma} dx.$$

Hence by Lemma 5, we obtain

$$\begin{aligned}\int_{\mathscr{A}} (g \otimes f_1)(x) x_n^{\gamma} dx &\leqslant C_{\gamma} \int_0^t g^{*}(u) du \int_{\mathbb{R}_{+}^{n}} f_1(y)\, y_n^{\gamma}\, dy \\ &\leqslant C_{\gamma} \int_0^t g^{**}(u) du \int_{\mathbb{R}_{+}^{n}} f_1(y) y_n^{\gamma} dy \\ &= C_{\gamma}\left(\int_{E_t} f(y) y_n^{\gamma} dy - t f^{*}(t)\right) \int_0^t g^{**}(u) du.\end{aligned}$$

Thus by (10) we have

$$(g\otimes f_1)^{**}(t)=\frac{1}{t}\sup_{|\mathscr{A}|_\gamma=t}\int_{\mathscr{A}}(g\otimes f_1)(x)x_n^\gamma dx$$
$$\leqslant C_\gamma\left(f^{**}(t)-f^*(t)\right)\int_0^t g^{**}(u)du.$$

Next, estimate $(g\otimes f_2)^{**}(t)$. Taking into account Lemma 5 and equality (10) we have

$$\left(T^{\cdot}g(x)\right)^*(s)\leqslant\left(T^{\cdot}g(x)\right)^{**}(s)=\frac{1}{s}\sup_{|\mathscr{A}|_\gamma=s}\int_{\mathscr{A}}T^y g(x)y_n^\gamma dy=C_\gamma g^{**}(s), \qquad (16)$$

hence by (11) we get

$$(g\otimes f_2)(x)\leqslant\int_0^\infty(f_2)^*(u)\left(T^{\cdot}g(x)\right)^*(u)du$$
$$\leqslant C_\gamma\int_0^\infty(f_2)^*(u)g^{**}(u)du$$
$$=C_\gamma\left(f^*(t)\int_0^t g^{**}(u)du+\int_t^\infty f^*(u)g^{**}(u)du\right).$$

Consequently by (10) we have

$$(g\otimes f_2)^{**}(t)\leqslant C_\gamma\left(f^*(t)\int_0^t g^{**}(u)du+\int_t^\infty f^*(u)g^{**}(u)du\right).$$

Therefore we obtain (15). □

THEOREM 2. *If* $g\in WL_{r,\gamma}(\mathbb{R}^n_+)$, $1<r<\infty$, *then*

$$(f\otimes g)^*(t)\leqslant(f\otimes g)^{**}(t)\leqslant$$
$$C_1\|g\|_{WL_{r,\gamma}}\left(t^{-\frac{1}{r}}\int_0^t f^*(s)ds+\int_t^\infty s^{-\frac{1}{r}}f^*(s)ds\right), \qquad (17)$$

where $C_1=C_\gamma r'(1+r')$.

Proof. Since $f\in WL_{r,\gamma}(\mathbb{R}^n_+)$, we have

$$g^*(t)\leqslant\|g\|_{WL_{r,\gamma}}t^{-\frac{1}{r}},\quad g^{**}(t)\leqslant r'\|g\|_{WL_{r,\gamma}}t^{-\frac{1}{r}}.$$

Taking into account inequality (15) we get the inequality (17). □

4. O'Neil-type inequality for the B-convolutions

In this section we prove an O'Neil-type inequality for the B-convolutions.

THEOREM 3. *1. Let* $1 < p < q < \infty$, $1/p' + 1/q = 1/r$, $f \in L_{p,\gamma}(\mathbb{R}_+^n)$, $g \in WL_{r,\gamma}(\mathbb{R}_+^n)$. *Then* $f \otimes g \in L_{q,\gamma}(\mathbb{R}_+^n)$ *and*

$$\|f \otimes g\|_{L_{q,\gamma}} \leqslant A_1 \|f\|_{L_{p,\gamma}} \|g\|_{WL_{r,\gamma}}, \tag{18}$$

where $A_1 = C_1\left(p^{1/q}q^{1/p'} + (p')^{1/q}(q')^{1/p'}\right)$.

2. Let $p = 1$, $1 < q < \infty$, $f \in L_{1,\gamma}(\mathbb{R}_+^n)$, $g \in WL_{q,\gamma}(\mathbb{R}_+^n)$. *Then* $f \otimes g \in WL_{q,\gamma}(\mathbb{R}_+^n)$ *and*

$$\|f \otimes g\|_{WL_{q,\gamma}} \leqslant A_2 \|f\|_{L_{1,\gamma}} \|g\|_{WL_{q,\gamma}}, \tag{19}$$

where $A_2 = 2C_1$.

Proof. 1. Let $f \in L_{p,\gamma}(\mathbb{R}_+^n)$, $g \in WL_{r,\gamma}(\mathbb{R}_+^n)$, $1 < p < q < \infty$ and $1/r = 1/p' + 1/q$. By using equality (9) and inequality (17) we get

$$\begin{aligned}\|f \otimes g\|_{L_{q,\gamma}} &= \left\|(f \otimes g)^*\right\|_{L_q(0,\infty)} \\ &\leqslant C_1\left(\int_0^\infty t^{-1/r}\left(\int_0^t f^*(s)ds\right)^q dt\right)^{1/q} \\ &\quad + C_1\left(\int_0^\infty\left(\int_t^\infty s^{-1/r}f^*(s)ds\right)^q dt\right)^{1/q}.\end{aligned}$$

By Lemma 3, for the validity of the inequality

$$\left(\int_0^\infty t^{-1/r}\left(\int_0^t f^*(s)ds\right)^q dt\right)^{1/q} \leqslant C_2\left(\int_0^\infty f^*(t)^p dt\right)^{1/p}$$

it is necessary and sufficient condition that

$$\sup_{t>0}\left(\int_t^\infty s^{-1/r}ds\right)^{1/q}\left(\int_0^t ds\right)^{1/p'} = \left(q/r-1\right)^{-1/q}\sup_{t>0} t^{1/r'-1/p+1/q} < \infty.$$

Note that $C_2 \leqslant \left(q/r-1\right)^{-1/q} q^{1/q}(q')^{1/p'} = (p')^{1/q}(q')^{1/p'}$ and $1/p - 1/q = 1/r'$.

Furthermore, by Lemma 4, for the validity of the inequality

$$\left(\int_0^\infty\left(\int_t^\infty s^{-\frac{1}{r}}f^*(s)ds\right)^q dt\right)^{1/q} \leqslant C_3\left(\int_0^\infty f^*(t)^p dt\right)^{1/p}$$

it is necessary and sufficient condition that

$$\sup_{t>0}\left(\int_0^t ds\right)^{1/q}\left(\int_t^\infty s^{-p'/r}ds\right)^{1/p'} = \left(p'/r-1\right)^{-1/p'}\sup_{t>0} t^{1/r'-1/p+1/q} < \infty.$$

Note that $C_3 \leqslant \left(p'/r-1\right)^{-1/p'} p^{1/q}(p')^{1/p'} = p^{1/q}q^{1/p'}$ and $1/p-1/q=1/r'$.

By using these inequalities and applying equality (9) we obtain

$$\|f \otimes g\|_{L_{q,\gamma}} \leqslant C_1(C_2+C_3)\, \|f\|_{L_{p,\gamma}}\, \|g\|_{WL_{r,\gamma}} .$$

2. Let $p=1$, $1<q<\infty$, $f \in L_{1,\gamma}(\mathbb{R}^n_+)$ and $g \in WL_{q,\gamma}(\mathbb{R}^n_+)$.
By inequality (17) and equality (9) we obtain

$$\begin{aligned}
\|f \otimes g\|_{WL_{q,\gamma}} &= \sup_{t>0} t^{1/q}(f \otimes g)^*(t) \\
&\leqslant C_1\|g\|_{WL_{q,\gamma}} \sup_{t>0} t^{1/q}\left(t^{-\frac{1}{r}}\int_0^t f^*(s)ds + \int_t^\infty s^{-\frac{1}{r}} f*(s)ds\right) \\
&= C_1\|g\|_{WL_{q,\gamma}}\left(\sup_{t>0}\int_0^t f*(s)ds + \sup_{t>0} t^{1/q}\int_t^\infty s^{-1/q}f^*(s)ds\right) \\
&\leqslant 2C_1\|g\|_{WL_{q,\gamma}}\|f^*\|_{L_1(0,\infty)} = 2C_1\|f\|_{L_{1,\gamma}}\|g\|_{WL_{q,\gamma}}.
\end{aligned}$$

Thus the proof is completed. □

5. Boundedness of the B-fractional integral operator with rough kernels in $L_{p,\gamma}$ spaces

We define the fractional B-maximal function with a rough kernel by

$$M_{\Omega,\alpha,\gamma}f(x) = \sup_{r>0}\frac{1}{r^{n+\gamma-\alpha}}\int_{B(0,r)}|\Omega(y)|\; T^y|f(x)|\; y_n^\gamma\, dy \tag{20}$$

and the B–fractional integral with a rough kernel by

$$I_{\Omega,\alpha,\gamma}f(x) = \int_{\mathbb{R}^n_+}\frac{\Omega(y)}{|y|^{n+\gamma-\alpha}}\, T^y f(x)\, y_n^\gamma\, dy, \tag{21}$$

where $\Omega \in L_{s,\gamma}(S^{n-1}_+)$, $s \geqslant 1$, $S^{n-1}_+ = \{x \in \mathbb{R}^n_+ : |x|=1\}$, and Ω is homogeneous of degree zero on $\mathbb{R}^n_+$, i.e., $\Omega(tx)=\Omega(x)$ for all $t>0$, $x \in \mathbb{R}^n_+$.

It is easy to see that, when $\Omega \equiv 1$, $M_{\Omega,\alpha,\gamma}$ and $I_{\Omega,\alpha,\gamma}$ are the usual fractional B–maximal operator $M_{\alpha,\gamma}$ ([6]) and the B–Riesz potential $I_{\alpha,\gamma}$ ([1, 5, 11]), respectively.

Note that, it can be easily verified that

$$g(x) = |x|^{\alpha-n-\gamma} \in WL_{(n+\gamma)/(n+\gamma-\alpha),\gamma}(\mathbb{R}^n_+), \quad 0<\alpha<n,$$

moreover in this case

$$g_*(t) = \omega(n,\gamma)\, t^{-(n+\gamma)/(n+\gamma-\alpha)}, \;\; g^*(t) = \left(\omega(n,\gamma)\, t^{-1}\right)^{1-\alpha/(n+\gamma)},$$

$$\|g\|_{WL_{(n+\gamma)/(n+\gamma-\alpha),\gamma}} = \omega(n,\gamma)^{1-\alpha/(n+\gamma)},$$

where $\omega(n,\gamma) = |B(0,1)|_\gamma$, and $B(0,1) = \{x \in \mathbb{R}^n_+ : |x|<1\}$.

If we take

$$g(x)=\frac{\Omega(x)}{|x|^{n+\gamma-\alpha}},\quad 0<\alpha<n,$$

where Ω is homogeneous of degree zero on $\mathbb{R}^n_+$ and $\Omega\in L_{(n+\gamma)/(n+\gamma-\alpha),\gamma}(S^{n-1}_+)$, then

$$\begin{aligned} g_*(t) &= \frac{A}{n+\gamma}\,t^{-(n+\gamma)/(n+\gamma-\alpha)},\quad g^*(t)=\left(\frac{A}{(n+\gamma)t}\right)^{1-\alpha/(n+\gamma)},\\ g^{**}(t) &= \frac{n+\gamma}{\alpha}\,g^*(t), \end{aligned}$$

where

$$A=\|\Omega\|^{(n+\gamma)/(n+\gamma-\alpha)}_{L_{(n+\gamma)/(n+\gamma-\alpha),\gamma}(S^{n-1}_+)}$$

and therefore $g\in WL_{(n+\gamma)/(n+\gamma-\alpha),\gamma}(\mathbb{R}^n_+)$ and $\|g\|_{WL_{(n+\gamma)/(n+\gamma-\alpha),\gamma}}=(\frac{A}{n+\gamma})^{1-\alpha/(n+\gamma)}$.

COROLLARY 1. *Let $0<\alpha<n+\gamma$, Ω be homogeneous of degree zero on $\mathbb{R}^n_+$ and $\Omega\in L_{(n+\gamma)/(n+\gamma-\alpha),\gamma}(S^{n-1}_+)$. Then the following inequalities hold*

$$\begin{aligned} &\left(I_{\Omega,\alpha,\gamma}f\right)^*(t)\leqslant\left(I_{\Omega,\alpha,\gamma}f\right)^{**}(t)\\ &\qquad\leqslant C_4\,\|\Omega\|_{L_{(n+\gamma)/(n+\gamma-\alpha),\gamma}(S^{n-1}_+)}\left(t^{\frac{\alpha}{n+\gamma}-1}\int_0^t f^*(s)\,ds+\int_t^\infty s^{\frac{\alpha}{n+\gamma}-1}f^*(s)\,ds\right), \end{aligned}$$

where $C_4=\alpha^{-2}(n+\gamma)^{\frac{\alpha}{n+\gamma}+1}\,C_\gamma$.

LEMMA 6. *Suppose that $\Omega\in L_{s,\gamma}(S^{n-1}_+)$, $s\geqslant 1$, $0<\alpha<n+\gamma$, then*

$$M_{\Omega,\alpha,\gamma}f(x)\leqslant\frac{2^{n+\gamma-\alpha}}{1-2^{\alpha-n-\gamma}}\,I_{|\Omega|,\alpha,\gamma}(|f|)(x).$$

Proof. Denote

$$I_{|\Omega|,\alpha,\gamma,j}(|f|)(x)=\int_{B(0,2^j)\setminus B(0,2^{j-1})}\frac{|\Omega(y)|}{|y|^{n+\gamma-\alpha}}\,T^y|f(x)|\,y_n^\gamma\,dy,$$

then

$$I_{|\Omega|,\alpha,\gamma}(|f|)(x)=\sum_{j\in\mathbb{Z}}I_{|\Omega|,\alpha,\gamma,j}(|f|)(x). \tag{22}$$

Since

$$\begin{aligned} &I_{|\Omega|,\alpha,\gamma,j}(|f|)(x)\\ &\quad\geqslant 2^{j(\alpha-n-\gamma)}\int_{B(0,2^j)\setminus B(0,2^{j-1})}|\Omega(y)|\,T^y|f(x)|\,y_n^\gamma\,dy\\ &\quad=2^{j(\alpha-n-\gamma)}\left[\int_{B(0,2^j)}|\Omega(y)|\,T^y|f(x)|\,y_n^\gamma\,dy-\int_{B(0,2^{j-1})}|\Omega(y)|\,T^y|f(x)|\,y_n^\gamma\,dy\right]\\ &\quad=\frac{1}{2^{j(n+\gamma-\alpha)}}\int_{B(0,2^j)}|\Omega(y)|\,T^y|f(x)|\,y_n^\gamma\,dy\\ &\qquad-\frac{2^{\alpha-n-\gamma}}{2^{(j-1)(n+\gamma-\alpha)}}\int_{B(0,2^{j-1})}|\Omega(y)|\,T^y|f(x)|\,y_n^\gamma\,dy, \end{aligned}$$

we have

$$I_{|\Omega|,\alpha,\gamma,j}(|f|)(x)+\frac{2^{\alpha-n-\gamma}}{2^{(j-1)(n+\gamma-\alpha)}}\int_{B(0,2^{j-1})}|\Omega(y)|\,T^y|f(x)|\,y_n^\gamma\,dy$$
$$\geqslant\frac{1}{2^{j(n+\gamma-\alpha)}}\int_{B(0,2^j)}|\Omega(y)|\,T^y|f(x)|\,y_n^\gamma\,dy.$$

If we take the supremum with respect to $j\in\mathbb{Z}$ in the both sides of the above inequality, then we get

$$\sup_{j\in\mathbb{Z}}I_{|\Omega|,\alpha,\gamma,j}(|f|)(x)\geqslant(1-2^{\alpha-n-\gamma})\sup_{j\in\mathbb{Z}}\frac{1}{2^{j(n+\gamma-\alpha)}}\int_{B(0,2^j)}|\Omega(y)|\,T^y|f(x)|\,y_n^\gamma\,dy. \quad (23)$$

On the other hand, it is easy to see that

$$M_{|\Omega|,\alpha,\gamma}(f)(x)\leqslant 2^{n+\gamma-\alpha}\sup_{j\in\mathbb{Z}}\frac{1}{2^{j(n+\gamma-\alpha)}}\int_{B(0,2^j)}|\Omega(y)|\,T^y|f(x)|\,y_n^\gamma\,dy. \quad (24)$$

Thus, the proof of Lemma 6 follows from (22), (23) and (24). □

From Corollary 1 and Lemma 6 we get the following

COROLLARY 2. *Suppose that* Ω *is homogeneous of degree zero on* $\mathbb{R}^n_+$ *and* $\Omega\in L_{(n+\gamma)/(n+\gamma-\alpha),\gamma}(S^{n-1}_+)$, $0<\alpha<n+\gamma$, *then for all* $0<t<\infty$

$$\begin{aligned}\left(M_{\Omega,\alpha,\gamma}f\right)^*(t) &\leqslant \left(M_{\Omega,\alpha,\gamma}f\right)^{**}(t)\\ &\leqslant C_4'\left(t^{\frac{\alpha}{n+\gamma}-1}\int_0^t f^*(s)\,ds+\int_t^\infty s^{\frac{\alpha}{n+\gamma}-1}f^*(s)\,ds\right),\end{aligned}$$

where $C_4'=\frac{2^{n+\gamma-\alpha}}{1-2^{\alpha-n-\gamma}}\,C_4$.

COROLLARY 3. *For the* B*-Riesz potential*

$$I_{\alpha,\gamma}f(x)=\int_{\mathbb{R}^n_+}T^y|x|^{\alpha-n-\gamma}f(y)y_n^\gamma dy, \quad 0<\alpha<n+\gamma,$$

for all $0<t<\infty$

$$\begin{aligned}\left(I_{\alpha,\gamma}f\right)^*(t) &\leqslant \left(I_{\alpha,\gamma}f\right)^{**}(t)\\ &\leqslant C_5\left(t^{\frac{\alpha}{n+\gamma}-1}\int_0^t f^*(s)ds+\int_t^\infty s^{\frac{\alpha}{n+\gamma}-1}f^*(s)ds\right),\end{aligned}$$

where $C_5=C_\gamma\left(\frac{n+\gamma}{\alpha}\right)\left(1+\frac{n+\gamma}{\alpha}\right)\omega(n,\gamma)^{1-\frac{\alpha}{n+\alpha}}$.

COROLLARY 4. *Let* Ω *be homogeneous of degree zero on* $\mathbb{R}^n_+$ *and* $\Omega\in L_{(n+\gamma)/(n+\gamma-\alpha),\gamma}(S^{n-1}_+)$, $0<\alpha<n$. *Then*

1) If $1<p<\frac{n+\gamma}{\alpha}$, $f\in L_{p,\gamma}(\mathbb{R}^n_+)$ *and* $1/p-1/q=\alpha/(n+\gamma)$, *then* $I_{\Omega,\alpha,\gamma}f\in L_{q,\gamma}(\mathbb{R}^n_+)$ *and*

$$\|I_{\Omega,\alpha,\gamma}f\|_{L_{q,\gamma}}\leqslant A_3\,\|\Omega\|_{L_{(n+\gamma)/(n+\gamma-\alpha),\gamma}(S^{n-1}_+)}\,\|f\|_{L_{p,\gamma}},$$

where $A_3 = C_\gamma \alpha^{-2}(n+\gamma)^{\alpha/(n+\gamma)+1}\left(p^{1/q}q^{1/p'} + (p')^{1/q}(q')^{1/p'}\right)$.

2) If $p = 1$, $f \in L_{1,\gamma}(\mathbb{R}^n_+)$ *and* $1 - 1/q = \alpha/(n+\gamma)$, *then* $I_{\Omega,\alpha,\gamma}f \in WL_{q,\gamma}(\mathbb{R}^n_+)$ *and*

$$\|I_{\Omega,\alpha,\gamma}f\|_{WL_{q,\gamma}} \leqslant A_4 \|\Omega\|_{L_{(n+\gamma)/(n+\gamma-\alpha),\gamma}(S^{n-1}_+)} \|f\|_{L_{1,\gamma}},$$

where $A_4 = C_\gamma \alpha^{-2}(n+\gamma)^{\alpha/(n+\gamma)+1}$.

COROLLARY 5. *Let* $0 < \alpha < n$. *Then*

1) If $1 < p < \frac{n+\gamma}{\alpha}$, $f \in L_{p,\gamma}(\mathbb{R}^n_+)$ *and* $1/p - 1/q = \alpha/(n+\gamma)$, *then* $I_{\alpha,\gamma}f \in L_{q,\gamma}(\mathbb{R}^n_+)$ *and*

$$\|I_{\alpha,\gamma}f\|_{L_{q,\gamma}} \leqslant A_5 \|f\|_{L_{p,\gamma}},$$

where $A_5 = C_\gamma\,(pq+q-p)\left(pq/(q-p)^2\right)\omega(n,\gamma)^{1/p'+1/q}\left(p^{1/q}q^{1/p'} + (p')^{1/q}(q')^{1/p'}\right)$.

2) If $p = 1$, $f \in L_{1,\gamma}(\mathbb{R}^n_+)$ *and* $1 - 1/q = \alpha/(n+\gamma)$, *then* $I_{\alpha,\gamma}f \in WL_{q,\gamma}(\mathbb{R}^n_+)$ *and*

$$\|I_{\alpha,\gamma}f\|_{WL_{q,\gamma}} \leqslant A_6 \|f\|_{L_{1,\gamma}},$$

where $A_6 = C_\gamma\,(pq+q-p)\left(pq/(q-p)^2\right)\omega(n,\gamma)^{1/p'+1/q}$.

Note that, Corollary 5 was proved in [1], [5] and [11] by using other methods but in those studies the constants were not calculated explicitly.

Next we obtain necessary and sufficient conditions on the parameters for the fractional B-maximal operator and B-fractional integral operator with rough kernels to be bounded from the spaces $L_{p,\gamma}$ to $L_{q,\gamma}$ and from the spaces $L_{1,\gamma}$ to the weak spaces $WL_{q,\gamma}$.

THEOREM 4. *Let* $0 < \alpha < n+\gamma$, Ω *be homogeneous of degree zero on* $\mathbb{R}^n_+$ *and* $\Omega \in L_{(n+\gamma)/(n+\gamma-\alpha),\gamma}(S^{n-1}_+)$. *Then*

1) If $1 < p < (n+\gamma)/\alpha$, *then the condition* $1/p - 1/q = \alpha/(n+\gamma)$ *is necessary and sufficient for the boundedness of* $I_{\Omega,\alpha,\gamma}$ *from* $L_{p,\gamma}(\mathbb{R}^n_+)$ *to* $L_{q,\gamma}(\mathbb{R}^n_+)$.

2) If $p = 1$, *then the condition* $1 - 1/q = \alpha/(n+\gamma)$ *is necessary and sufficient for the boundedness of* $I_{\Omega,\alpha,\gamma}$ *from* $L_{1,\gamma}(\mathbb{R}^n_+)$ *to* $WL_{q,\gamma}(\mathbb{R}^n_+)$.

Proof. Sufficiency of Theorem 4 follows from Theorem 3.

Necessity. 1) Suppose that the operator $I_{\Omega,\alpha,\gamma}$ is bounded from $L_{p,\gamma}(\mathbb{R}^n_+)$ to $L_{q,\gamma}(\mathbb{R}^n_+)$ and $1 < p < (n+\gamma)/\alpha$.

Define $f_t(x) =: f(tx)$ for $t > 0$. Then it can be easily shown that

$$\|f_t\|_{L_{p,\gamma}} = t^{-\frac{n+\gamma}{p}} \|f\|_{L_{p,\gamma}}, \quad (I_{\Omega,\alpha,\gamma}f_t)(x) = t^{-\alpha} I_{\Omega,\alpha,\gamma}f(tx),$$

and

$$\left\|I_{\Omega,\alpha,\gamma}f_t\right\|_{L_{q,\gamma}} = t^{-\alpha-\frac{n+\gamma}{q}} \left\|I_{\Omega,\alpha,\gamma}f\right\|_{L_{q,\gamma}}.$$

Since the operator $I_{\Omega,\alpha,\gamma}$ is bounded from $L_{p,\gamma}(\mathbb{R}^n_+)$ to $L_{q,\gamma}(\mathbb{R}^n_+)$, we have

$$\left\|I_{\Omega,\alpha,\gamma}f\right\|_{L_{q,\gamma}} \leqslant C\|f\|_{L_{p,\gamma}},$$

where C is independent of f. Then we get

$$\left\|I_{\Omega,\alpha,\gamma}f\right\|_{L_{q,\gamma}}=t^{\alpha+\frac{n+\gamma}{q}}\left\|I_{\Omega,\alpha,\gamma}f_t\right\|_{L_{q,\gamma}}\leqslant Ct^{\alpha+\frac{n+\gamma}{q}}\|f_t\|_{L_{p,\gamma}}=Ct^{\alpha+\frac{n+\gamma}{q}-\frac{n+\gamma}{p}}\|f\|_{L_{p,\gamma}}.$$

If $\frac{1}{p}<\frac{1}{q}+\frac{\alpha}{n+\gamma}$, then for all $f\in L_{p,\gamma}(\mathbb{R}^n_+)$ we have $\left\|I_{\Omega,\alpha,\gamma}f\right\|_{L_{q,\gamma}}=0$ as $t\to 0$.

If $\frac{1}{p}>\frac{1}{q}+\frac{\alpha}{n+\gamma}$, then for all $f\in L_{p,\gamma}(\mathbb{R}^n_+)$ we have $\left\|I_{\Omega,\alpha,\gamma}f\right\|_{L_{q,\gamma}}=0$ as $t\to\infty$.

Therefore we get the equality $\frac{1}{p}=\frac{1}{q}+\frac{\alpha}{n+\gamma}$.

2) Suppose that the operator $I_{\Omega,\alpha,\gamma}$ is bounded from $L_{1,\gamma}(\mathbb{R}^n_+)$ to $WL_{q,\gamma}(\mathbb{R}^n_+)$. It is easy to show that

$$\|f_t\|_{L_{1,\gamma}}=t^{-n-\gamma}\|f\|_{L_{1,\gamma}},\quad (I_{\Omega,\alpha,\gamma}f_t)(x)=t^{-\alpha}(I_{\Omega,\alpha,\gamma}f)(tx),$$

and

$$\left\|I_{\Omega,\alpha,\gamma}f_t\right\|_{WL_{q,\gamma}}=t^{-\alpha-\frac{n+\gamma}{q}}\left\|I_{\Omega,\alpha,\gamma}f\right\|_{WL_{q,\gamma}}.$$

By the boundedness of $I_{\Omega,\alpha,\gamma}$ from $L_{1,\gamma}(\mathbb{R}^n_+)$ to $WL_{q,\gamma}(\mathbb{R}^n_+)$, we have

$$\left\|I_{\Omega,\alpha,\gamma}f\right\|_{WL_{q,\gamma}}\leqslant C\|f\|_{L_{1,\gamma}},$$

where C is independent of f. Then we have

$$(I_{\Omega,\alpha,\gamma}f_t)_*(\tau)=t^{-n-\gamma}(I_{\Omega,\alpha,\gamma}f)_*(t^{\alpha}\tau),$$

$$\left\|I_{\Omega,\alpha,\gamma}f_t\right\|_{WL_{q,\gamma}}=t^{-\alpha-\frac{n+\gamma}{q}}\left\|I_{\Omega,\alpha,\gamma}f\right\|_{WL_{q,\gamma}},$$

and

$$\left\|I_{\Omega,\alpha,\gamma}f\right\|_{WL_{q,\gamma}}=t^{\alpha+\frac{n+\gamma}{q}}\|I_{\Omega,\alpha,\gamma}f_t\|_{WL_{q,\gamma}}\leqslant Ct^{\alpha+\frac{n+\gamma}{q}}\|f_t\|_{L_{1,\gamma}}=Ct^{\alpha+\frac{n+\gamma}{q}-n-\gamma}\|f\|_{L_{1,\gamma}}.$$

If $1<\frac{1}{q}+\frac{\alpha}{n+\gamma}$, then for all $f\in L_{1,\gamma}(\mathbb{R}^n_+)$ we have $\left\|I_{\Omega,\alpha,\gamma}f\right\|_{WL_{q,\gamma}}=0$ as $t\to 0$.

If $1>\frac{1}{q}+\frac{\alpha}{n+\gamma}$, then for all $f\in L_{1,\gamma}(\mathbb{R}^n_+)$ we have $\left\|I_{\Omega,\alpha,\gamma}f\right\|_{WL_{q,\gamma}}=0$ as $t\to\infty$.

Therefore we get the equality $1=\frac{1}{q}+\frac{\alpha}{n+\gamma}$. □

COROLLARY 6. *Let* $0<\alpha<n+\gamma$.

1) If $1<p<(n+\gamma)/\alpha$, *then the condition* $1/p-1/q=\alpha/(n+\gamma)$ *is necessary and sufficient for the boundedness of* $I_{\alpha,\gamma}$ *from* $L_{p,\gamma}(\mathbb{R}^n_+)$ *to* $L_{q,\gamma}(\mathbb{R}^n_+)$.

2) If $p=1$, *then the condition* $1-1/q=\alpha/(n+\gamma)$ *is necessary and sufficient for the boundedness of* $I_{\alpha,\gamma}$ *from* $L_{1,\gamma}(\mathbb{R}^n_+)$ *to* $WL_{q,\gamma}(\mathbb{R}^n_+)$.

COROLLARY 7. *Let* $0<\alpha<n+\gamma$, Ω *be homogeneous of degree zero on* $\mathbb{R}^n_+$ *and* $\Omega\in L_{(n+\gamma)/(n+\gamma-\alpha),\gamma}(S^{n-1}_+)$.

1) If $1<p<r'$, *then the condition* $1/p-1/q=\alpha/(n+\gamma)$ *is necessary and sufficient for the boundedness of* $M_{\Omega,\alpha,\gamma}$ *from* $L_{p,\gamma}(\mathbb{R}^n_+)$ *to* $L_{q,\gamma}(\mathbb{R}^n_+)$.

2) If $p=1$, *then the condition* $1-1/q=\alpha/(n+\gamma)$ *is necessary and sufficient for the boundedness of* $M_{\Omega,\alpha,\gamma}$ *from* $L_{1,\gamma}(\mathbb{R}^n_+)$ *to* $WL_{q,\gamma}(\mathbb{R}^n_+)$.

Proof. Sufficiency of Corollary 7 follows from Theorem 4 and Lemma 6.

Necessity. 1) Let $M_{\Omega,\alpha,\gamma}$ be bounded from $L_{p,\gamma}(\mathbb{R}^n_+)$ to $L_{q,\gamma}(\mathbb{R}^n_+)$ for $1 < p < \frac{n+\gamma}{\alpha}$.

Then we have

$$M_{\Omega,\alpha,\gamma} f_t(x) = t^{-\alpha} M^{\alpha}_{\gamma} f(tx),$$

and

$$\left\|M_{\Omega,\alpha,\gamma} f_t\right\|_{L_{q,\gamma}} = t^{-\alpha-\frac{n+\gamma}{q}} \left\|M_{\Omega,\alpha,\gamma} f\right\|_{L_{q,\gamma}}.$$

By the same argument in Theorem 4 we obtain $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+\gamma}$.

2) Let $M_{\Omega,\alpha,\gamma}$ be bounded from $L_{1,\gamma}(\mathbb{R}^n_+)$ to $WL_{q,\gamma}(\mathbb{R}^n_+)$.

Then we have

$$M_{\Omega,\alpha,\gamma} f_t(x) = t^{-\alpha} M^{\alpha}_{\Omega,\alpha,\gamma} f(tx),$$

and

$$\left\|M_{\Omega,\alpha,\gamma} f_t\right\|_{WL_{q,\gamma}} = t^{-\alpha-\frac{n+\gamma}{q}} \left\|M_{\Omega,\alpha,\gamma} f\right\|_{WL_{q,\gamma}}.$$

Hence we obtain the equality $1 - \frac{1}{q} = \frac{\alpha}{n+\gamma}$. □

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