Mathematical
Inequalities
& Applications
Volume 11, Number 1 (2007), 113–126

REFINEMENTS OF JENSEN'S INEQUALITY OF MERCER'S TYPE FOR OPERATOR CONVEX FUNCTIONS

A. MATKOVIĆ, J. PEČARIĆ AND I. PERIĆ

(*communicated by F. Hansen*)

Abstract. Refinements of Jensen's inequality for operator convex functions, which are generalizations of Mercer's result, are proved. Obtained results are used to refine monotonicity properties for power means of Mercer's type, and a comparision theorem for quasi-arithmetic means of Mercer's type for operators.

1. Introduction

We assume that H and K are Hilbert spaces, $\mathscr{B}(H)$ and $\mathscr{B}(K)$ are C^*-algebras of all bounded operators on the appropriate Hilbert space and $\mathbf{P}[\mathscr{B}(H),\mathscr{B}(K)]$ is the set of all positive linear maps from $\mathscr{B}(H)$ to $\mathscr{B}(K)$. We denote by $C([m,M])$ the set of all real valued continuous functions on an interval $[m,M]$. Let $A_1,\dots,A_k \in \mathscr{B}(H)$ be selfadjoint operators with spectra in $[m,M]$ for some scalars $m < M$ and let $\Phi_1,\dots,\Phi_k \in \mathbf{P}[\mathscr{B}(H),\mathscr{B}(K)]$ be positive linear maps with $\sum_{j=1}^{k}\Phi_j(1_H) = 1_K$.

In [5] the following theorem is proved:

THEOREM A. *If $f \in C([m,M])$ is a convex function on $[m,M]$, then*

$$\begin{aligned} f\left(m1_K + M1_K - \sum_{j=1}^{k}\Phi_j(A_j)\right) &\leqslant \frac{M1_K - \sum_{j=1}^{k}\Phi_j(A_j)}{M-m}\cdot f(M) + \frac{\sum_{j=1}^{k}\Phi_j(A_j) - m1_K}{M-m}\cdot f(m) \\ &\leqslant f(m)1_K + f(M)1_K - \sum_{j=1}^{k}\Phi_j(f(A_j)). \end{aligned}$$

In this paper we refine this result for operator convex function. For related results in the real case see [3], [6] and [1]. We shall start with the following definition (see [2, p. 6])

Mathematics subject classification (2000): 47A63, 47A64.
Key words and phrases: Jensen's inequality, operator convex functions, monotonicity, power means, quasi-arithmetic means.

DEFINITION. A real valued continuous function f defined on an interval I is said to be *operator convex* if

$$f\left((1-\lambda)A+\lambda B\right) \leqslant (1-\lambda)f(A)+\lambda f(B) \tag{1.1}$$

for all $\lambda \in [0,1]$ and for all selfadjiont operators A and B on a Hilbert space H whose spectra are contained in I. A real valued continuous function f is said to be *operator concave* if the reversed inequality (1.1) holds.

In [4] the following generalization of discrete Jensen's operator inequality is proved:

THEOREM B. *If $f \in C([m,M])$ is an operator convex function on $[m,M]$, then*

$$f\left(\sum_{j=1}^{k}\Phi_j(A_j)\right) \leqslant \sum_{j=1}^{k}\Phi_j\left(f(A_j)\right). \tag{1.2}$$

To obtain some further results we need several well known assertions. The first one is Löwner-Heinz inequality (see for example [2, p. 9]):

THEOREM C. *Let $A,\ B \in \mathscr{B}(H)$ be positive operators. If $A \geqslant B$, then $A^p \geqslant B^p$ for all $p \in [0,1]$.*

In [2, p. 220, 232, 250] the following theorems are also proved:

THEOREM D. *Let $A,B \in \mathscr{B}(H)$ be positive operators with $Sp(A) \subseteq [m_1,M_1]$, and $Sp(B) \subseteq [m_2,M_2]$ for some scalars $M_j > m_j > 0\ (j=1,2)$. If $A \geqslant B$, then the following inequalities hold:*
(i) for all $p>1$:

$$K(m_1,M_1,p)A^p \geqslant B^p,$$
$$K(m_2,M_2,p)A^p \geqslant B^p,$$

(ii) for all $p<-1$:

$$K(m_1,M_1,p)B^p \geqslant A^p,$$
$$K(m_2,M_2,p)B^p \geqslant A^p,$$

where a generalized Kantorovich constant $K(m,M,p)$ is defined by

$$K(m,M,p)=\frac{(mM^p-Mm^p)}{(p-1)(M-m)}\left(\frac{p-1}{p}\frac{M^p-m^p}{mM^p-Mm^p}\right)^p$$

for all $p \in \mathbb{R}$.

THEOREM E. *Let $A,B \in \mathscr{B}(H)$ be selfadjoint operators with $Sp(B) \subseteq [m,M]$ for some scalars $M>m$. If $A \geqslant B$, then*

$$S\left(e^{M-m},1\right)e^A \geqslant e^B,$$

where the Specht ratio $S(h)$ *for* $h > 0$ *is defined by* $S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \ln h}$ $(h \neq 1)$ *and* $S(1) = 1$.

In Section 2 we give the main result of our paper which is a refinement of Theorem A for operator convex functions. In Section 3 we use that result to refine monotonicity properties of power means of Mercer's type for operators. In the final section we consider related quasi-arithmetic means for operators.

2. Main result

THEOREM 1. *Let* $A_1, ..., A_k \in \mathscr{B}(H)$ *be selfadjoint operators with spectra in* $[m, M]$ *for some scalars* $m < M$ *and* $\Phi_1, ..., \Phi_k \in \mathbf{P}[\mathscr{B}(H), \mathscr{B}(K)]$ *positive linear maps with* $\sum_{j=1}^{k} \Phi_j(1_H) = 1_K$. *If* $f \in C([m, M])$ *is operator convex on* $[m, M]$, *then we have the following series of inequalities*

$$\begin{aligned} f\left(m1_K + M1_K - \sum_{j=1}^{k} \Phi_j(A_j)\right) & \\ &\leqslant \sum_{j=1}^{k} \Phi_j(f(m1_H + M1_H - A_j)) \qquad (2.1) \\ &\leqslant \frac{M1_K - \sum_{j=1}^{k} \Phi_j(A_j)}{M-m} \cdot f(M) + \frac{\sum_{j=1}^{k} \Phi_j(A_j) - m1_K}{M-m} \cdot f(m) \\ &\leqslant f(m)1_K + f(M)1_K - \sum_{j=1}^{k} \Phi_j(f(A_j)). \end{aligned}$$

If function f *is operator concave, then the inequalities* (2.1) *are reversed.*

To prove Theorem 1, we need the following Lemma:

LEMMA 1. *Let* $A_1, ..., A_k$, $\Phi_1, ..., \Phi_k$ *be as in Theorem 1 and* f *convex on* $[m, M]$. *Then*

$$\sum_{j=1}^{k} \Phi_j(f(A_j)) \leqslant \frac{M1_K - \sum_{j=1}^{k} \Phi_j(A_j)}{M-m} \cdot f(m) + \frac{\sum_{j=1}^{k} \Phi_j(A_j) - m1_K}{M-m} \cdot f(M).$$

Proof. Since f is convex,

$$f(A_j) \leqslant \frac{M1_H - A_j}{M-m} \cdot f(m) + \frac{A_j - m1_H}{M-m} \cdot f(M),$$

holds for all $j = 1, \ldots, k$. Applying linear maps Φ_j and summing, it follows that

$$\sum_{j=1}^{k} \Phi_j(f(A_j)) \leqslant \frac{M1_K - \sum_{j=1}^{k} \Phi_j(A_j)}{M-m} \cdot f(m) + \frac{\sum_{j=1}^{k} \Phi_j(A_j) - m1_K}{M-m} \cdot f(M),$$

since $\sum_{j=1}^{k} \Phi_j(1_H) = 1_K$. $\square$

Proof of Theorem 1. Since $m1_H \leqslant A_j \leqslant M1_H$ for $j = 1, \dots, k$ and $\sum_{j=1}^{k} \Phi_j(1_H) = 1_K$, applying linear maps Φ_j and summing, it follows that $m1_K \leqslant \sum_{j=1}^{k} \Phi_j(A_j) \leqslant M1_K$, and hence $m1_K \leqslant m1_K + M1_K - \sum_{j=1}^{k} \Phi_j(A_j) \leqslant M1_K$.

Since f is continuous and operator convex, the same is also true for the function $g : [m, M] \to \mathbb{R}$ defined by $g(t) = f(m + M - t),\ t \in [m, M]$. By Theorem B,

$$g\left(\sum_{j=1}^{k} \Phi_j(A_j)\right) \leqslant \sum_{j=1}^{k} \Phi_j(g(A_j))$$

i.e.,

$$f\left(m1_K + M1_K - \sum_{j=1}^{k} \Phi_j(A_j)\right) \leqslant \sum_{j=1}^{k} \Phi_j(f(m1_H + M1_H - A_j)). \tag{2.2}$$

Applying Lemma 1 to g and then to f, we have

$$\begin{aligned}
&\sum_{j=1}^{k} \Phi_j(f(m1_H + M1_H - A_j)) \\
&\quad \leqslant \frac{M1_K - \sum_{j=1}^{k} \Phi_j(A_j)}{M - m} \cdot g(m) + \frac{\sum_{j=1}^{k} \Phi_j(A_j) - m1_K}{M - m} \cdot g(M) \\
&\quad = \frac{M1_K - \sum_{j=1}^{k} \Phi_j(A_j)}{M - m} \cdot f(M) + \frac{\sum_{j=1}^{k} \Phi_j(A_j) - m1_K}{M - m} \cdot f(m) \qquad (2.3) \\
&\quad = f(m)1_K + f(M)1_K - \left[\frac{M1_K - \sum_{j=1}^{k} \Phi_j(A_j)}{M - m} \cdot f(m)\right. \\
&\qquad \left. + \frac{\sum_{j=1}^{k} \Phi_j(A_j) - m1_K}{M - m} \cdot f(M)\right] \\
&\quad \leqslant f(m)1_K + f(M)1_K - \sum_{j=1}^{k} \Phi_j(f(A_j)).
\end{aligned}$$

Using the inequalities (2.2) and (2.3), we obtain desired inequalities (2.1).

The last statement follows immediately from the fact that if f is operator concave then $-f$ is operator convex. □

3. Applications to Mercer's power means

We suppose that:

(*i*) $\mathbf{A} = (A_1, \dots, A_k)$, where $A_j \in \mathscr{B}(H)$ are positive invertible operators with $Sp(A_j) \subseteq [m, M]$ for some scalars $0 < m < M$,

(*ii*) $\mathbf{\Phi} = (\Phi_1, \dots, \Phi_k)$, where $\Phi_j \in \mathbf{P}[\mathscr{B}(H), \mathscr{B}(K)]$ are positive linear maps with $\sum_{j=1}^{k} \Phi_j(1_H) = 1_K$,

(iii) $\Delta(m,M,p) = K\left(m^p, M^p, \frac{1}{p}\right) = \frac{p\left(m^pM - M^pm\right)}{(1-p)(M^p-m^p)}\left(\frac{(1-p)(M-m)}{m^pM-M^pm}\right)^{\frac{1}{p}}$, for $0 < m < M$ and $p \in \mathbb{R}$, $p \neq 0$. Set: $\Delta(m,M,0) = \lim_{p\to 0}\Delta(m,M,p) = S(\frac{M}{m},1) = \frac{M-m}{\ln M - \ln m}\exp\left(\frac{m(1+\ln M)-M(1+\ln m)}{M-m}\right)$

In [5] the power mean of Mercer's type for operators

$$\widetilde{M}_r(\mathbf{A},\mathbf{\Phi}) = \begin{cases} \left[m^r 1_K + M^r 1_K - \sum_{j=1}^k \Phi_j\left(A_j^r\right)\right]^{\frac{1}{r}}, & r \neq 0, \\ \exp\left((\ln m)\, 1_K + (\ln M)\, 1_K - \sum_{j=1}^k \Phi_j\left(\ln\left(A_j\right)\right)\right), & r = 0\,. \end{cases}$$

is defined and the following theorem is proved.

THEOREM F. *Let* $r,s \in \mathbb{R}$, $r < s$.
(i) *If either* $r \leqslant -1$ *or* $s \geqslant 1$*, then*

$$\widetilde{M}_r(\mathbf{A},\mathbf{\Phi}) \leqslant \widetilde{M}_s(\mathbf{A},\mathbf{\Phi}).$$

(ii) *If* $-1 < r$ *and* $s < 1$*, then*

$$\widetilde{M}_r(\mathbf{A},\mathbf{\Phi}) \leqslant \Delta(m,M,s)\cdot\widetilde{M}_s(\mathbf{A},\mathbf{\Phi}).$$

Now, we define, for any $r,s \in \mathbb{R}$

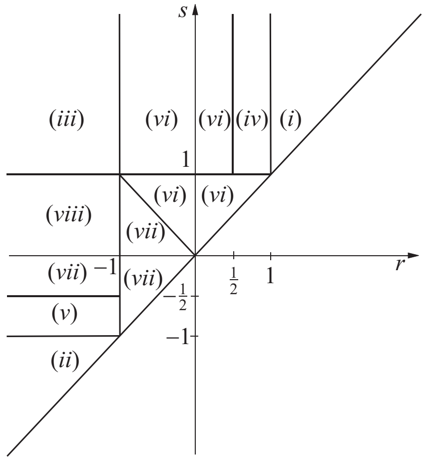
$$R(r,s,\mathbf{A},\mathbf{\Phi}) := \begin{cases} \left[\sum_{j=1}^k \Phi_j\left(\left[m^r 1_H + M^r 1_H - A_j^r\right]^{\frac{s}{r}}\right)\right]^{\frac{1}{s}}, & r \neq 0, s \neq 0, \\ \exp\left(\sum_{j=1}^k \Phi_j\left(\ln\left[m^r 1_H + M^r 1_H - A_j^r\right]^{\frac{1}{r}}\right)\right), & r \neq 0, s = 0, \\ \left[\sum_{j=1}^k \Phi_j\left(\exp s\left[(\ln m)\, 1_H + (\ln M)\, 1_H - \ln A_j\right]\right)\right]^{\frac{1}{s}}, & r = 0, s \neq 0, \end{cases}$$

$$S(r,s,\mathbf{A},\mathbf{\Phi}) := \begin{cases} \left[\frac{M^r 1_K - S_r}{M^r - m^r}\cdot M^s + \frac{S_r - m^r 1_K}{M^r - m^r}\cdot m^s\right]^{\frac{1}{s}}, & r \neq 0, s \neq 0, \\ \exp\left(\frac{M^r 1_K - S_r}{M^r - m^r}\cdot \ln M + \frac{S_r - m^r}{M^r - m^r}\cdot \ln m\right), & r \neq 0, s = 0, \\ \left[\frac{(\ln M) 1_K - S_0}{\ln M - \ln m}\cdot M^s + \frac{S_0 - (\ln m) 1_K}{\ln M - \ln m}\cdot m^s\right]^{\frac{1}{s}}, & r = 0, s \neq 0, \end{cases}$$

where $S_r = \sum_{j=1}^k \Phi_j\left(A_j^r\right)$ and $S_0 = \sum_{j=1}^k \Phi_j\left(\ln\left(A_j\right)\right)$. It is easy to see that $R(r,s,\mathbf{A},\mathbf{\Phi})$ and $S(r,s,\mathbf{A},\mathbf{\Phi})$ are well defined and also notice that $R(r,r,\mathbf{A},\mathbf{\Phi}) = S(r,r,\mathbf{A},\mathbf{\Phi}) = \widetilde{M}_r(\mathbf{A},\mathbf{\Phi})$ (including $r = 0$).

To simplify notations, in what follows we will write $\widetilde{M}_r, R(r,s), S(r,s)$ instead of $\widetilde{M}_r(\mathbf{A},\mathbf{\Phi}), R(r,s,\mathbf{A},\mathbf{\Phi}), S(r,s,\mathbf{A},\mathbf{\Phi})$, respectively.

The following figure illustrates regions $(i)-(vii)$ which determine the seven cases occurring in Theorem 2 (compare with the figure on p. 117 in [2]).



THEOREM 2. *Let* $r, s \in \mathbb{R}$, $r < s$.
(i) *If* $1 \leqslant r$, *then*

$$\widetilde{M}_r \leqslant S(s,r) \leqslant R(s,r) \leqslant \widetilde{M}_s.$$

(ii) *If* $s \leqslant -1$, *then*

$$\widetilde{M}_r \leqslant R(r,s) \leqslant S(r,s) \leqslant \widetilde{M}_s.$$

(iii) *If* $r \leqslant -1, s \geqslant 1$, *then*

$$\begin{aligned}\widetilde{M}_r &\leqslant R(r,-1) \leqslant S(r,-1) \leqslant \widetilde{M}_{-1} \leqslant S(1,-1)\\ &\leqslant R(1,-1) \leqslant \widetilde{M}_1 \leqslant S(s,1) \leqslant R(s,1) \leqslant \widetilde{M}_s.\end{aligned}$$

(iv) *If* $\frac{1}{2} < r < 1 < s$, *then*

$$\widetilde{M}_r \leqslant R(r,1) \leqslant S(r,1) \leqslant \widetilde{M}_1 \leqslant S(s,1) \leqslant R(s,1) \leqslant \widetilde{M}_s.$$

(v) *If* $r < -1 < s < -\frac{1}{2}$, *then*

$$\widetilde{M}_r \leqslant R(r,-1) \leqslant S(r,-1) \leqslant \widetilde{M}_{-1} \leqslant S(s,-1) \leqslant R(s,-1) \leqslant \widetilde{M}_s.$$

(vi) *If* $-1 < r \leqslant \frac{1}{2}, s \geqslant 1$; *or* $-s \leqslant r < s \leqslant 1$, *then*

$$\widetilde{M}_r \leqslant \Delta(m,M,r)\, S(s,r) \leqslant \Delta(m,M,r)^2\, R(s,r) \leqslant \Delta(m,M,r)^3\, \widetilde{M}_s.$$

(vii) *If* $r \leqslant -1, -\frac{1}{2} \leqslant s < 1$; *or* $-1 \leqslant r < s \leqslant -r$, *then*

$$\widetilde{M}_r \leqslant \Delta(m,M,s)\, R(r,s) \leqslant \Delta(m,M,s)^2\, S(r,s) \leqslant \Delta(m,M,s)^3\, \widetilde{M}_s.$$

Proof. (i) Suppose that $1 \leqslant r < s$.

Applying the inequalities (2.1) to the operator concave function $f(t) = t^{\frac{r}{s}}$ (note that $0 < \frac{r}{s} \leqslant 1$ here) and replacing A_j, m and M with A_j^s, m^s and M^s respectively, we have

$$\begin{aligned}\left[m^s 1_K + M^s 1_K - \sum_{j=1}^{k} \Phi_j\left(A_j^s\right)\right]^{\frac{r}{s}} &\geqslant \sum_{j=1}^{k} \Phi_j\left(\left[m^s 1_H + M^s 1_H - A_j^s\right]^{\frac{r}{s}}\right)\\ &\geqslant \frac{M^s 1_K - S_s}{M^s - m^s} \cdot M^r + \frac{S_s - m^s 1_K}{M^s - m^s} \cdot m^r\\ &\geqslant m^r 1_K + M^r 1_K - \sum_{j=1}^{k} \Phi_j\left(A_j^r\right),\end{aligned}$$

or

$$\left[\widetilde{M}_s\right]^r \geqslant [R(s,r)]^r \geqslant [S(s,r)]^r \geqslant \left[\widetilde{M}_r\right]^r. \tag{3.1}$$

Raising these inequalities to the power $\frac{1}{r}$ $\left(0 < \frac{1}{r} \leqslant 1\right)$, by Theorem C it follows that

$$\widetilde{M}_r \leqslant S(s,r) \leqslant R(s,r) \leqslant \widetilde{M}_s. \tag{3.2}$$

(ii) Suppose that $r < s \leqslant -1$.

Applying the inequalities (3.2) to $\mathbf{A}^{-1} = \left(A_1^{-1}, \ldots, A_k^{-1}\right)$ (notice that $Sp\left(A_j^{-1}\right) \subseteq \left[M^{-1}, m^{-1}\right]$ and $-s < -r$), it follows that

$$\widetilde{M}_{-s}(\mathbf{A}^{-1}, \mathbf{\Phi}) \leqslant S\left(-r, -s, \mathbf{A}^{-1}, \mathbf{\Phi}\right) \leqslant R\left(-r, -s, \mathbf{A}^{-1}, \mathbf{\Phi}\right) \leqslant \widetilde{M}_{-r}(\mathbf{A}^{-1}, \mathbf{\Phi}).$$

Observing that $\widetilde{M}_{-r}(\mathbf{A}^{-1}, \mathbf{\Phi}) = \left[\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi})\right]^{-1}$, $\widetilde{M}_{-s}(\mathbf{A}^{-1}, \mathbf{\Phi}) = \left[\widetilde{M}_s(\mathbf{A}, \mathbf{\Phi})\right]^{-1}$, $S\left(-r, -s, \mathbf{A}^{-1}, \mathbf{\Phi}\right) = \left[S\left(r, s, \mathbf{A}, \mathbf{\Phi}\right)\right]^{-1}$ and $R\left(-r, -s, \mathbf{A}^{-1}, \mathbf{\Phi}\right) = \left[R\left(r, s, \mathbf{A}, \mathbf{\Phi}\right)\right]^{-1}$, we have

$$\left[\widetilde{M}_s\right]^{-1} \leqslant \left[S\left(r,s\right)\right]^{-1} \leqslant \left[R\left(r,s\right)\right]^{-1} \leqslant \left[\widetilde{M}_r\right]^{-1}.$$

Hence,

$$\widetilde{M}_r \leqslant R(r,s) \leqslant S(r,s) \leqslant \widetilde{M}_s. \tag{3.3}$$

(iii) Suppose that $r \leqslant -1$ and $s \geqslant 1$.

Applying the inequalities (2.1) to the operator convex function $f(t) = t^{-1}$ we have

$$\begin{aligned}\left[m 1_K + M 1_K - \sum_{j=1}^{k} \Phi_j\left(A_j\right)\right]^{-1} &\leqslant \sum_{j=1}^{k} \Phi_j\left(\left[m 1_H + M 1_H - A_j\right]^{-1}\right)\\ &\leqslant \frac{M 1_K - S_1}{M - m} \cdot M^{-1} + \frac{S_1 - m 1_K}{M - m} \cdot m^{-1}\\ &\leqslant m^{-1} 1_K + M^{-1} 1_K - \sum_{j=1}^{k} \Phi_j\left(A_j^{-1}\right),\end{aligned}$$

or

$$\left[\widetilde{M}_1\right]^{-1} \leqslant [R(1,-1)]^{-1} \leqslant [S(1,-1)]^{-1} \leqslant \left[\widetilde{M}_{-1}\right]^{-1}.$$

Hence,

$$\widetilde{M}_{-1} \leqslant S(1,-1) \leqslant R(1,-1) \leqslant \widetilde{M}_1.$$

If we let $r=1$ in (3.2) and $s=-1$ in (3.3) then it follows that

$$\widetilde{M}_1 \leqslant S(s,1) \leqslant R(s,1) \leqslant \widetilde{M}_s,$$

and

$$\widetilde{M}_r \leqslant R(r,-1) \leqslant S(r,-1) \leqslant \widetilde{M}_{-1}.$$

Hence,

$$\begin{aligned}\widetilde{M}_r \leqslant R(r,-1) \leqslant S(r,-1) \leqslant \widetilde{M}_{-1} \leqslant S(1,-1)\\ \leqslant R(1,-1) \leqslant \widetilde{M}_1 \leqslant S(s,1) \leqslant R(s,1) \leqslant \widetilde{M}_s.\end{aligned}$$

(iv) Suppose that $\frac{1}{2} < r < 1 < s$.

Applying the inequalities (2.1) to the operator convex function $f(t) = t^{\frac{1}{r}}$ (note that $1 \leqslant \frac{1}{r} \leqslant 2$ here) and replacing A_j, m and M with A_j^r, m^r and M^r respectively, we have

$$\begin{aligned}\left[m^r 1_K + M^r 1_K - \sum_{j=1}^{k} \Phi_j\left(A_j^r\right)\right]^{\frac{1}{r}} &\leqslant \sum_{j=1}^{k} \Phi_j\left(\left[m^r 1_H + M^r 1_H - A_j^r\right]^{\frac{1}{r}}\right)\\ &\leqslant \frac{M^r 1_K - S_r}{M^r - m^r} \cdot M + \frac{S_r - m^r 1_K}{M^r - m^r} \cdot m\\ &\leqslant m 1_K + M 1_K - \sum_{j=1}^{k} \Phi_j\left(A_j\right),\end{aligned}$$

or

$$\widetilde{M}_r \leqslant R(r,1) \leqslant S(r,1) \leqslant \widetilde{M}_1.$$

If we let $r=1$ in (3.2) then it follows that

$$\widetilde{M}_r \leqslant R(r,1) \leqslant S(r,1) \leqslant \widetilde{M}_1 \leqslant S(s,1) \leqslant R(s,1) \leqslant \widetilde{M}_s. \tag{3.4}$$

(v) Suppose that $r < -1 < s < -\frac{1}{2}$.

Applying the inequalities (3.4) to $\mathbf{A}^{-1} = \left(A_1^{-1}, \ldots, A_k^{-1}\right)$ and following analogous arguing as in (ii), we have

$$\widetilde{M}_r \leqslant R(r,-1) \leqslant S(r,-1) \leqslant \widetilde{M}_{-1} \leqslant S(s,-1) \leqslant R(s,-1) \leqslant \widetilde{M}_s.$$

(vi) *Step 1:* Suppose that $0 < r \leqslant \frac{1}{2}, 1 \leqslant s$.

In the same way as in (i) we obtain the inequalities (3.1). Observing that all terms in (3.1) have spectra in $[m^r, M^r]$ and raising (3.1) to the power $\frac{1}{r}$ $\left(\frac{1}{r} > 1\right)$, by Theorem D (i) it follows that

$$\widetilde{M}_r \leqslant K\left(m^r, M^r, \frac{1}{r}\right) S(s,r) \leqslant K\left(m^r, M^r, \frac{1}{r}\right)^2 R(s,r) \leqslant K\left(m^r, M^r, \frac{1}{r}\right)^3 \widetilde{M}_s. \tag{3.5}$$

Step 2: Suppose that $-1 < r < 0, 1 \leqslant s$.

Applying the inequalities (2.1) to the operator convex function $f(t) = t^{\frac{r}{s}}$ (note that $-1 \leqslant \frac{r}{s} < 0$ here) and replacing A_j, m and M with A_j^s, m^s and M^s respectively, we have

$$\left[\widetilde{M}_s\right]^r \leqslant [R(s,r)]^r \leqslant [S(s,r)]^r \leqslant \left[\widetilde{M}_r\right]^r.$$

Observing that all terms have spectra in $[M^r, m^r]$ and raising these inequalities to the power $\frac{1}{r}$ $\left(\frac{1}{r} < -1\right)$, by Theorem D (ii) it follows that

$$\widetilde{M}_r \leqslant K\left(M^r, m^r, \frac{1}{r}\right) S(s,r) \leqslant K\left(M^r, m^r, \frac{1}{r}\right)^2 R(s,r) \leqslant K\left(M^r, m^r, \frac{1}{r}\right)^3 \widetilde{M}_s.$$

Since $K(M,m,p) = K(m,M,p)$ (see [2, p. 77]), we have (3.5).

Step 3: Suppose that $0 < r < s \leqslant 1$.

In the same way as in *Step 1*, we have (3.5).

Step 4: Suppose that $-1 \leqslant -s \leqslant r < 0$.

In the same way as in *Step 2*, we have (3.5).

Step 5: Suppose that $0 = r < s$.

Applying the inequalities (2.1) to the operator concave function $f(t) = \frac{1}{s}\ln t$ (note that $\frac{1}{s} > 0$ here) and replacing A_j, m and M with A_j^s, m^s and M^s respectively, we obtain

$$\begin{aligned}\frac{1}{s}\ln\left(m^s 1_K + M^s 1_K - \sum_{j=1}^k \Phi_j\left(A_j^s\right)\right) &\geqslant \sum_{j=1}^k \Phi_j\left(\frac{1}{s}\ln\left(m^s 1_H + M^s 1_H - A_j^s\right)\right) \\ &\geqslant \frac{M^s 1_K - S_s}{M^s - m^s}\cdot \ln M + \frac{S_s - m^s 1_K}{M^s - m^s}\cdot \ln m \\ &\geqslant (\ln m)\, 1_K + (\ln M)\, 1_K - \sum_{j=1}^k \Phi_j\left(\ln\left(A_j\right)\right),\end{aligned}$$

or

$$\ln \widetilde{M}_s \geqslant \ln R(s,0) \geqslant \ln S(s,0) \geqslant \ln \widetilde{M}_0.$$

Observing that all terms have spectra in $[\ln m, \ln M]$, by Theorem E it follows that

$$\widetilde{M}_0 \leqslant S\left(e^{\ln M - \ln m}, 1\right) S(s,0) \leqslant S\left(e^{\ln M - \ln m}, 1\right)^2 R(s,0) \leqslant S\left(e^{\ln M - \ln m}, 1\right)^3 \widetilde{M}_s.$$

(vii) Suppose that $r \leqslant -1, -\frac{1}{2} \leqslant s < 0$.

Applying the inequalities (3.5) to $\mathbf{A}^{-1} = \left(A_1^{-1}, \ldots, A_k^{-1}\right)$ and following analogous arguing as in (ii), we have

$$\widetilde{M}_r \leqslant K\left(M^s, m^s, \frac{1}{s}\right) R(r,s) \leqslant K\left(M^s, m^s, \frac{1}{s}\right)^2 S(r,s) \leqslant K\left(M^s, m^s, \frac{1}{s}\right)^3 \widetilde{M}_s.$$

Similarly, we obtain desired inequalities in cases when $r \leqslant -1, 0 < s < 1$; $-1 \leqslant r \leqslant -s < 0$; $-1 \leqslant r < s < 0$; $r < s = 0$. $\square$

REMARK 1. Besides these results in Theorem 2, one can prove in the same way that for $r < s < 2r, s > 1$

$$\widetilde{M}_r \leqslant R(r,s) \leqslant S(r,s) \leqslant \widetilde{M}_s,$$

and for $r < s < \frac{1}{2}r, r < -1$

$$\widetilde{M}_r \leqslant S(s,r) \leqslant R(s,r) \leqslant \widetilde{M}_s$$

also hold, but to include these cases in the figure we should compare sequences of inequalities in common regions (see Remark 2).

REMARK 2. If we define by $M_r(\mathbf{A},\mathbf{\Phi}) = \left(\sum_{j=1}^k \Phi_j(A_j^r)\right)^{\frac{1}{r}}$ (the weighted power mean), by $\widetilde{M}_r(B) = (m^r 1 + M^r 1 - B^r)^{\frac{1}{r}}$ (the Mercer mean for positive invertible operator B with $Sp(B) \subset [m,M]$, $0 < m < M$) and by $\widetilde{M}_r(\mathbf{A}) = (\widetilde{M}_r(A_1),\ldots,\widetilde{M}_r(A_k))$ (for an $k-$ tuple $\mathbf{A}$ of positive invertible operators), we can write:

$$\widetilde{M}_r(\mathbf{A},\mathbf{\Phi}) = \widetilde{M}_r(M_r(\mathbf{A},\mathbf{\Phi}))$$

$$R(r,s,\mathbf{A},\mathbf{\Phi}) = M_s(\widetilde{M}_r(\mathbf{A}),\mathbf{\Phi}),$$

so one can describe inequalities in Theorem 2 as mixed mean inequalities. We can also state the following open problem: What is the complete set of inequalities among mixed means $M_r(\widetilde{M}_s(\mathbf{A}),\mathbf{\Phi})$, $\widetilde{M}_s(M_r(\mathbf{A},\mathbf{\Phi}))$, $\widetilde{M}_r(M_s(\mathbf{A},\mathbf{\Phi}))$ and $M_s(\widetilde{M}_r(\mathbf{A}),\mathbf{\Phi})$? Some special cases are given in Theorem 2 and Remark 1. Also, it is easy to see that

$$\widetilde{M}_r(M_r(\mathbf{A},\mathbf{\Phi})) \leqslant \widetilde{M}_s(M_r(\mathbf{A},\mathbf{\Phi}))$$

reduces to monotonicity property of Mercer's means, and that in some cases,

$$\widetilde{M}_s(M_s(\mathbf{A},\mathbf{\Phi})) \leqslant \widetilde{M}_s(M_r(\mathbf{A},\mathbf{\Phi}))$$

reduces to inequalities between $(\sum_{j=1}^k \Phi_j(A_j^r))^{s/r}$ and $\sum_{j=1}^k \Phi(A_j^s)$.

4. Quasi-arithmetic means of Mercer's type

Let $\mathbf{A}$ and $\mathbf{\Phi}$ be as in the previous section. Let $\varphi, \psi \in C([m,M])$ be strictly monotonic functions on an interval $[m,M]$. We define

$$\widetilde{M}_\varphi(\mathbf{A},\mathbf{\Phi}) = \varphi^{-1}\left(\varphi(m)\,1_K + \varphi(M)\,1_K - \sum_{j=1}^k \Phi_j(\varphi(A_j))\right).$$

Observe that, since $m1_H \leqslant A_j \leqslant M1_H$, it follows that

- $\varphi(m)\,1_H \leqslant \varphi(A_j) \leqslant \varphi(M)\,1_H$ if φ is increasing,
- $\varphi(M)\,1_H \leqslant \varphi(A_j) \leqslant \varphi(m)\,1_H$ if φ is decreasing.

Applying positive linear maps Φ_j and summing, it follows that

- $\varphi(m)\,1_K \leqslant \sum_{j=1}^k \Phi_j(\varphi(A_j)) \leqslant \varphi(M)\,1_K$ if φ is increasing,

- $\varphi(M)\,1_K \leqslant \sum_{j=1}^{k}\Phi_j(\varphi(A_j)) \leqslant \varphi(m)\,1_K$ if φ is decreasing,

since $\sum_{j=1}^{k}\Phi_j(1_H) = 1_K$. Hence, $\widetilde{M}_\varphi(\mathbf{A},\mathbf{\Phi})$ is well defined.

A function $f \in C([m,M])$ is said to be *operator increasing* if f is operator monotone, i.e., if $A \leqslant B$ implies $f(A) \leqslant f(B)$, for all selfadjoint operators A and B on a Hilbert space H with $Sp(A), Sp(B) \subseteq [m,M]$. A function $f \in C([m,M])$ is said to be *operator decreasing* if $-f$ is operator monotone.

THEOREM 3. *Under the above hypotheses, we have*

(i) if either $\psi\circ\varphi^{-1}$ is operator convex and ψ^{-1} is operator increasing, or $\psi\circ\varphi^{-1}$ is operator concave and ψ^{-1} is operator decreasing, then

$$\begin{aligned}\widetilde{M}_\varphi(\mathbf{A},\mathbf{\Phi}) &\leqslant \psi^{-1}\left(\sum_{j=1}^{k}\Phi_j\left(\left(\psi\circ\varphi^{-1}\right)\left(\varphi(m)\,1_H+\varphi(M)\,1_H-\varphi(A_j)\right)\right)\right)\\ &\leqslant \psi^{-1}\left(\frac{\varphi(M)\,1_K-\sum_{j=1}^{k}\Phi_j(\varphi(A_j))}{\varphi(M)-\varphi(m)}\cdot\psi(M)\right.\\ &\qquad\left.+\frac{\sum_{j=1}^{k}\Phi_j(\varphi(A_j))-\varphi(m)\,1_K}{\varphi(M)-\varphi(m)}\cdot\psi(m)\right)\\ &\leqslant \widetilde{M}_\psi(\mathbf{A},\mathbf{\Phi}).\end{aligned} \tag{4.1}$$

(ii) if either $\psi\circ\varphi^{-1}$ is operator concave and ψ^{-1} is operator increasing, or $\psi\circ\varphi^{-1}$ is operator convex and ψ^{-1} is operator decreasing, then the reverse inequalities (4.1) *hold.*

Proof. Suppose that $\psi\circ\varphi^{-1}$ is operator convex. If in Theorem 1 we let $f = \psi\circ\varphi^{-1}$ and replace A_j, m and M with $\varphi(A_j)$, $\varphi(m)$ and $\varphi(M)$ respectively, then we obtain

$$\begin{aligned}&\left(\psi\circ\varphi^{-1}\right)\left(\varphi(m)\,1_K+\varphi(M)\,1_K-\sum_{j=1}^{k}\Phi_j(\varphi(A_j))\right)\\ &\quad\leqslant \sum_{j=1}^{k}\Phi_j\left(\left(\psi\circ\varphi^{-1}\right)\left(\varphi(m)\,1_H+\varphi(M)\,1_H-\varphi(A_j)\right)\right)\\ &\quad\leqslant \frac{\varphi(M)\,1_K-\sum_{j=1}^{k}\Phi_j(\varphi(A_j))}{\varphi(M)-\varphi(m)}\cdot\left(\psi\circ\varphi^{-1}\right)(\varphi(M))\\ &\qquad+\frac{\sum_{j=1}^{k}\Phi_j(\varphi(A_j))-\varphi(m)\,1_K}{\varphi(M)-\varphi(m)}\cdot\left(\psi\circ\varphi^{-1}\right)(\varphi(m))\\ &\quad\leqslant \left(\psi\circ\varphi^{-1}\right)(\varphi(m))\,1_K+\left(\psi\circ\varphi^{-1}\right)(\varphi(M))\,1_K-\sum_{j=1}^{k}\Phi_j\left(\left(\psi\circ\varphi^{-1}\right)(\varphi(A_j))\right).\end{aligned}$$

or

$$
\begin{aligned}
&\psi\left(\varphi^{-1}\left(\varphi(m)1_K+\varphi(M)1_K-\sum_{j=1}^{k}\Phi_j\left(\varphi\left(A_j\right)\right)\right)\right)\\
&\quad\leqslant\sum_{j=1}^{k}\Phi_j\left(\psi\left(\varphi^{-1}\left(\varphi\left(m\right)1_H+\varphi\left(M\right)1_H-\varphi\left(A_j\right)\right)\right)\right) \qquad (4.2)\\
&\quad\leqslant\frac{\varphi\left(M\right)1_K-\sum_{j=1}^{k}\Phi_j\left(\varphi\left(A_j\right)\right)}{\varphi\left(M\right)-\varphi\left(m\right)}\cdot\psi\left(M\right)+\frac{\sum_{j=1}^{k}\Phi_j\left(\varphi\left(A_j\right)\right)-\varphi\left(m\right)1_K}{\varphi\left(M\right)-\varphi\left(m\right)}\cdot\psi\left(m\right)\\
&\quad\leqslant\psi\left(m\right)1_K+\psi\left(M\right)1_K-\sum_{j=1}^{k}\Phi_j\left(\psi\left(A_j\right)\right).
\end{aligned}
$$

If $\psi\circ\varphi^{-1}$ is operator concave then we obtain the reverse of inequalities (4.2).

If ψ^{-1} is operator increasing, then (4.2) implies (4.1). If ψ^{-1} is operator decreasing, then the reverse of (4.2) implies (4.1). Analogously, we get the reverse of (4.1) in the cases when $\psi\circ\varphi^{-1}$ is operator convex and ψ^{-1} is operator decreasing, or $\psi\circ\varphi^{-1}$ is operator concave and ψ^{-1} is operator increasing. □

THEOREM 4. *Under the above hypotheses, we have*

(i) *if either φ is operator concave and φ^{-1} is operator increasing or φ is operator convex and φ^{-1} is operator decreasing, and either ψ is operator convex and ψ^{-1} is operator increasing or ψ is operator concave and ψ^{-1} is operator decreasing, then*

$$
\begin{aligned}
\widetilde{M}_{\varphi}\left(\mathbf{A},\mathbf{\Phi}\right) &\leqslant \varphi^{-1}\left(\frac{M1_K-\sum_{j=1}^{k}\Phi_j\left(A_j\right)}{M-m}\cdot\varphi(M)+\frac{\sum_{j=1}^{k}\Phi_j\left(A_j\right)-m1_K}{M-m}\cdot\varphi(m)\right)\\
&\leqslant \varphi^{-1}\left(\sum_{j=1}^{k}\Phi_j\left(\varphi\left(m1_H+M1_H-A_j\right)\right)\right)\\
&\leqslant \widetilde{M}_1\left(\mathbf{A},\mathbf{\Phi}\right) \qquad (4.3)\\
&\leqslant \psi^{-1}\left(\sum_{j=1}^{k}\Phi_j\left(\psi\left(m1_H+M1_H-A_j\right)\right)\right)\\
&\leqslant \psi^{-1}\left(\frac{M1_K-\sum_{j=1}^{k}\Phi_j\left(A_j\right)}{M-m}\cdot\psi(M)+\frac{\sum_{j=1}^{k}\Phi_j\left(A_j\right)-m1_K}{M-m}\cdot\psi(m)\right)\\
&\leqslant \widetilde{M}_{\psi}\left(\mathbf{A},\mathbf{\Phi}\right).
\end{aligned}
$$

(ii) *if either φ is operator convex and φ^{-1} is operator increasing or φ is operator concave and φ^{-1} is operator decreasing, and either ψ is operator concave and ψ^{-1} is operator increasing or ψ is operator convex and ψ^{-1} is operator decreasing, then the reverse inequalities (4.3) hold.*

Proof. Suppose that φ is operator concave and φ^{-1} is operator increasing, and ψ is operator convex and ψ^{-1} is operator increasing. By Theorem 1, we have

$$\begin{aligned}
&\varphi\left(m1_K + M1_K - \sum_{j=1}^{k}\Phi_j\left(A_j\right)\right) \\
&\quad \geqslant \sum_{j=1}^{k}\Phi_j\left(\varphi\left(m1_H + M1_H - A_j\right)\right) \\
&\quad \geqslant \frac{M1_K - \sum_{j=1}^{k}\Phi_j\left(A_j\right)}{M-m}\cdot\varphi(M) + \frac{\sum_{j=1}^{k}\Phi_j\left(A_j\right) - m1_K}{M-m}\cdot\varphi(m) \\
&\quad \geqslant \varphi\left(m\right)1_K + \varphi\left(M\right)1_K - \sum_{j=1}^{k}\Phi_j\left(\varphi\left(A_j\right)\right).
\end{aligned}$$

Since φ^{-1} is operator increasing, it follows that

$$\begin{aligned}
\widetilde{M}_{\varphi}\left(\mathbf{A},\mathbf{\Phi}\right) &\leqslant \varphi^{-1}\left(\frac{M1_K - \sum_{j=1}^{k}\Phi_j\left(A_j\right)}{M-m}\cdot\varphi(M) + \frac{\sum_{j=1}^{k}\Phi_j\left(A_j\right) - m1_K}{M-m}\cdot\varphi(m)\right) \\
&\leqslant \varphi^{-1}\left(\sum_{j=1}^{k}\Phi_j\left(\varphi\left(m1_H + M1_H - A_j\right)\right)\right) \\
&\leqslant \widetilde{M}_{1}\left(\mathbf{A},\mathbf{\Phi}\right).
\end{aligned}$$

Also, by Theorem 1, we have

$$\begin{aligned}
&\psi\left(m1_K + M1_K - \sum_{j=1}^{k}\Phi_j\left(A_j\right)\right) \\
&\quad \leqslant \sum_{j=1}^{k}\Phi_j\left(\psi\left(m1_H + M1_H - A_j\right)\right) \\
&\quad \leqslant \frac{M1_K - \sum_{j=1}^{k}\Phi_j\left(A_j\right)}{M-m}\cdot\psi(M) + \frac{\sum_{j=1}^{k}\Phi_j\left(A_j\right) - m1_K}{M-m}\cdot\psi(m) \\
&\quad \leqslant \psi\left(m\right)1_K + \psi\left(M\right)1_K - \sum_{j=1}^{k}\Phi_j\left(\psi\left(A_j\right)\right).
\end{aligned}$$

Since ψ^{-1} is operator increasing, it follows that

$$\begin{aligned}\widetilde{M}_1(\mathbf{A},\mathbf{\Phi}) &\leqslant \psi^{-1}\left(\sum_{j=1}^{k}\Phi_j\left(\psi\left(m1_H+M1_H-A_j\right)\right)\right)\\ &\leqslant \psi^{-1}\left(\frac{M1_K-\sum_{j=1}^{k}\Phi_j\left(A_j\right)}{M-m}\cdot\psi(M)+\frac{\sum_{j=1}^{k}\Phi_j\left(A_j\right)-m1_K}{M-m}\cdot\psi(m)\right)\\ &\leqslant \widetilde{M}_\psi(\mathbf{A},\mathbf{\Phi}).\end{aligned}$$

Hence, we have the inequalities (4.3). In remaining cases the proof is analogous. □

REFERENCES

[1] W. S. CHEUNG, A. MATKOVIĆ AND J. PEČARIĆ, *A variant of Jessen's Inequality and generalized means*, JIPAM, **7**(1) (2006), Article 10.
[2] T. FURUTA, J. MIĆIĆ-HOT, J. PEČARIĆ, Y. SEO, *Mond-Pečarić Method in Operator Inequalities*, Element, Zagreb, 2005.
[3] I. GAVREA, *Some considerations on the monotonicity property of power mean*, JIPAM, **5**(4) (2004), Article 93.
[4] F. HANSEN, J. PEČARIĆ AND I. PERIĆ, *A generalization of discrete Jensen's operator inequality and it's converses*, Hungarian – Croatian mathematical workshop, Debrecen, 2005.
[5] A. MATKOVIĆ, J. PEČARIĆ AND I. PERIĆ, *A variant of Jensen's inequality of Mercer's type for operators with applications*, Linear Algebra Appl, **418** (2006), 551–564.
[6] A. MCD. MERCER, *A variant of Jensen's inequality*, JIPAM, **4**(4) (2003), Article 73.

(Received May 11, 2006)

A. Matković
Faculty of Electrical Engineering
Mechanical Engineering and Naval Architecture
University of Split
Ruđera Boškovića bb
21 000 Split
Croatia
e-mail: anita@pmfst.hr

J. Pečarić
Faculty of Textile Technology
University of Zagreb
Prilaz baruna Filipovića 30
10 000 Zagreb
Croatia
e-mail: pecaric@hazu.hr

I. Perić
Faculty of Food Technology and Biotechnology
University of Zagreb
Pierottijeva 6
10 000 Zagreb
Croatia
e-mail: iperic@pbf.hr