

A NEW ELEMENTARY PROOF OF WILKER'S INEQUALITIES

LU ZHANG AND LING ZHU

(communicated by P. Bullen)

Abstract. In this note, we show a new elementary proof of Wilker's inequalities.

1. Introduction

J. B. Wilker [1] proposed two open questions and the second one was the following statement:

(A) There exists a largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x \quad (1)$$

for $0 < x < \frac{\pi}{2}$.

J. S. Sumner et al. [2] affirmed the truth of the Problem (A) and obtained a further results as follows

THEOREM. *If $0 < x < \pi/2$, then*

$$\frac{16}{\pi^4}x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 < \frac{8}{45}x^3 \tan x. \quad (2)$$

Furthermore, $16/\pi^4$ and $8/45$ are the best constants in (2).

B. N. Guo et al. [3] gave a new proof of the inequalities (2) by using the power series expansion of cotangent function and a lower bound of Bernoulli numbers. Recently, I. Pinelis [4] got other proof of inequality (2) by using L'Hospital rules for monotonicity. In this note, we show a new elementary proof of Wilker's inequalities.

2. One Lemma

LEMMA. *If $x > 0$, then*

$$\begin{aligned}
 1 - \frac{x^2}{2!} < \cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!}, \\
 x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}.
 \end{aligned}$$

Key words and phrases: Wilker's inequalities.

3. A New Elementary Proof of Theorem

Let $f(x) = \frac{\sin^2 x}{x^5 \tan x} + \frac{1}{x^4} - \frac{2}{x^3 \tan x}$, then

$$f'(x) = \frac{g(x)}{x^6 \sin^2 x},$$

where, $g(x) = 2x^3 + 6x^2 \sin x \cos x - 5 \sin^3 x \cos x - 3x \sin^2 x - 2x \sin^4 x$. Direct calculation yields

$$g'(x) = 2 \cos x h(x),$$

where,

$$h(x) = 6x^2 \cos x + 3x \sin x - 9 \sin^2 x \cos x - 4x \sin^3 x.$$

Then applying the inequalities in Lemma, we obtain

$$\begin{aligned} h(x) &< 6x^2 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right) + 3x \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) - 9 \sin^2 x \cos x - 4x \sin^3 x \\ &= 9x^2 - \frac{7}{2}x^4 + \frac{11}{40}x^6 - 9 \sin^2 x \cos x - 4x \sin^3 x = q(x), \end{aligned}$$

and

$$\begin{aligned} q'(x) &= 18x - 14x^3 + \frac{33}{20}x^5 - 18 \sin x \cos^2 x + 5 \sin^3 x - 12x \sin^2 x \cos x \\ &= 18x - 14x^3 + \frac{33}{20}x^5 - 18 \sin x + 23 \sin^3 x - 12x \sin^2 x \cos x \\ &< 18x - 14x^3 + \frac{33}{20}x^5 - 18 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right) + 23 \sin^2 x \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) \\ &\quad - 12x \sin^2 x \left(1 - \frac{x^2}{2!}\right) \\ &= -11x^3 + \frac{3}{2}x^5 + \frac{1}{280}x^7 + \left(11x + \frac{13}{6}x^3 + \frac{23}{120}x^5\right) \sin^2 x \\ &< -11x^3 + \frac{3}{2}x^5 + \frac{1}{280}x^7 + \left(11x + \frac{13}{6}x^3 + \frac{23}{120}x^5\right) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) \sin x \\ &= -11x^3 + \frac{3}{2}x^5 + \frac{1}{280}x^7 + \left(11x^2 + \frac{1}{3}x^4 - \frac{7}{90}x^6 - \frac{1}{72}x^8 + \frac{23}{14400}x^{10}\right) \sin x \\ &< -11x^3 + \frac{3}{2}x^5 + \frac{1}{280}x^7 \\ &\quad + \left(11x^2 + \frac{1}{3}x^4 - \frac{7}{90}x^6 - \frac{1}{72}x^8 + \frac{23}{14400}x^{10}\right) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) \\ &= x^2 \left(-\frac{4}{105}x^5 + \frac{1}{540}x^7 + \frac{1}{300}x^9 - \frac{17}{43200}x^{11} + \frac{1}{72000}x^{13}\right) \end{aligned}$$

$$\begin{aligned}
&< x^2 \left(-\frac{4}{105}x^5 + \frac{1}{540}x^7 + \frac{1}{300}x^9 + \frac{1}{72000}x^{13} \right) \\
&< x^7 \left(-\frac{4}{105} + \frac{1}{540} \left(\frac{\pi}{2} \right)^2 + \frac{1}{300} \left(\frac{\pi}{2} \right)^4 + \frac{1}{72000} \left(\frac{\pi}{2} \right)^8 \right) \\
&< -0.01x^7 < 0.
\end{aligned}$$

Then $q(x)$ is decreasing on $(0, \pi/2)$, and $q(x) < q(0) = 0$. So $h(x) < 0$, this leads to $g'(x) < 0$. Then $g(x)$ is decreasing on $(0, \pi/2)$. Now, $g(0) = 0$, so $g(x) < 0$. Therefore, $f(x)$ strictly decreases as x increases on $(0, \pi/2)$. At the same time, we find $\lim_{x \rightarrow 0^+} f(x) = \frac{8}{45}$ and $\lim_{x \rightarrow \frac{\pi}{2}^-} f(x) = \frac{16}{\pi^4}$, then the proof of Theorem is complete. \square

REFERENCES

- [1] J. B. WILKER, *E3306*, The American Mathematical Monthly **96**(1989), no.1, 55.
- [2] J. S. SUMNER, A. A. JAGERS, M. VOWE, AND J. ANGLESIO, *Inequalities involving trigonometric functions*, The American Mathematical Monthly **98**(1991), no.3, 264-267.
- [3] B. N. GUO, B. M. QIAO, F. QI, AND W. LI, *On new proof of Wilker's inequalities involving trigonometric functions*, Mathematical Inequalities and Applications **1**(2003), 19-22.
- [4] I. PINELIS, *L'Hospital rules for monotonicity and the Wilker-Anglesio Inequality*, The American Mathematical Monthly **111**(2004), 905-909.

(Received November 15, 2005)

Lu Zhang
Department of Mathematics
Zhejiang Gongshang University
Hangzhou 310035
P. R. China

Ling Zhu
Department of Mathematics
Zhejiang Gongshang University
Hangzhou 310035
P. R. China
e-mail: zhuling0571@163.com