Mathematical
Inequalities
& Applications
Volume 11, Number 2 (2008), 237–269

GENERAL HILBERT–TYPE INEQUALITIES WITH NON–CONJUGATE EXPONENTS

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(*communicated by N. Elezović*)

Abstract. In this paper we derive a series of new one-dimensional and multidimensional integral and discrete inequalities of the Hilbert and the Hardy-Hilbert type, with non-conjugate exponents. First, prove and discuss two equivalent general inequalities of such type, as well as their corresponding reverse inequalities. The obtained results are then applied to various settings considering homogeneous functions of a negative real degree. In particular, we prove generalizations and refinements of some recent results of Rassias et al, related to the Hilbert-type inequalities with conjugate exponents, and some new multidimensional inequalities of the Godunova type.

1. Introduction

Suppose p and q are real parameters, such that

$$p>1, \quad q>1, \quad \frac{1}{p}+\frac{1}{q} \geqslant 1, \tag{1}$$

and let $p'=\frac{p}{p-1}$ and $q'=\frac{q}{q-1}$ respectively be their conjugate exponents, that is, $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Further, define

$$\lambda=\frac{1}{p'}+\frac{1}{q'} \tag{2}$$

and observe that $0<\lambda \leqslant 1$ holds for all p and q as in (1). In particular, equality $\lambda=1$ holds in (2) if and only if $q=p'$, that is, only if p and q are mutually conjugate. Otherwise, we have $0<\lambda<1$, and such parameters p and q will be referred to as non-conjugate exponents.

Considering p, q, and λ as in (1) and (2), Hardy, Littlewood, and Pólya, [5], proved that there exists a constant $C_{p,q}$, dependent only on the parameters p and q, such that the following Hilbert-type inequality holds for all non-negative functions $f \in L^p(\mathbb{R}_+)$ and $g \in L^q(\mathbb{R}_+)$:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dxdy \leqslant C_{p,q}\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)}. \tag{3}$$

Mathematics subject classification (2000): 26D10, 26D15.

Key words and phrases: Inequalities, Hilbert's inequality, Hardy-Hilbert inequality, non-conjugate exponents, homogeneous kernel.

Paper MIA-11-18

However, the original proof did not bring any information about the value of the best possible constant $C_{p,q}$. That drawback was improved by Levin, [9], who obtained an explicit upper bound for $C_{p,q}$,

$$C_{p,q} \leqslant \left(\pi \operatorname{cosec} \frac{\pi}{\lambda p'}\right)^{\lambda}. \tag{4}$$

This was an interesting result since the right-hand side of (4) reduces to the previously known sharp constant $\pi \operatorname{cosec}\frac{\pi}{p'}$ when the exponents p and q are conjugate. A simpler proof of (4), based on a single application of Hölder's inequality, was given later by F. F. Bonsall, [2].

On the other hand, through the years, Hilbert-type inequalities with conjugate exponents were discussed by several authors, who either reproved them using various techniques, or applied and generalized them in many different ways. A comprehensive survey of the classical Hilbert and Hardy-Hilbert-type inequalities for integrals and sums, as well as their new extensions, generalizations, and refinements, can be found in the recent paper [3] of M. Gao and L C. Hsu. Here we just refer to a recent result of B. Yang and T. M. Rassias, [13]. Namely, for $p > 1$, $s > 2 - \min\{p, p'\}$, and non-negative functions f and g, they proved that the following inequalities hold and are equivalent:

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^s}\,dxdy \\ &\quad\leqslant B\left(1+\frac{s-2}{p}, 1+\frac{s-2}{p'}\right)\left[\int_0^\infty x^{1-s} f^p(x)\,dx\right]^{\frac{1}{p}} \cdot \left[\int_0^\infty y^{1-s} g^{p'}(y)\,dy\right]^{\frac{1}{p'}}\end{aligned}$$

and

$$\int_0^\infty y^{(s-1)(p-1)}\left[\int_0^\infty \frac{f(x)}{(x+y)^s}\,dx\right]^p dy \leqslant B^p\left(1+\frac{s-2}{p}, 1+\frac{s-2}{p'}\right)\int_0^\infty x^{1-s} f^p(x)\,dx,$$

where $B(\cdot,\cdot)$ is the usual Beta function. Moreover, the constant involved in the right-hand sides of both inequalities is the best possible, that is, cannot be replaced by any smaller constant.

Our aim in this paper is to provide a unified treatment of the mentioned results, and extend them to cover also the case when p and q are not conjugate exponents. In particular, we obtain a series of new one-dimensional and multidimensional integral and discrete inequalities of the Hilbert and the Hardy-Hilbert type, with homogeneous kernels of some negative real degree and with both conjugate and non-conjugate exponents.

The paper is organized in the following way: After this Introduction, in Section 2 we state and prove a pair of equivalent Hilbert and Hardy-Hilbert-type inequalities with conjugate and non-conjugate exponents p and q, related to general measure spaces X and Y with positive σ-finite measures, and to general non-negative kernels K. These relations are also discussed with respect to parameters p and q, in order to obtain the corresponding reverse inequalities. The results of this section are then applied to a range of settings considering homogeneous functions K of some negative real degree s. First, in Section 3, we derive equivalent one-dimensional integral Hilbert and Hardy-Hilbert-type inequalities related to intervals in $\mathbb{R}_+$, whose right-hand sides involve power weights with arbitrary exponents from certain intervals in $\mathbb{R}$. Especially, in the case when integrals are taken over $\mathbb{R}_+$, we obtain inequalities with explicit constant factors on their right-hand sides. On the other hand, in the case of proper intervals, we show that these relations can be strengthened. In particular, in Section 4 we perform a detailed analysis of such relations in the case when $K(x,y)=(x+y)^{-s}$ and derive new related strengthened inequalities with non-conjugate exponents. The following Section 5 is dedicated to some further generalizations of the results from Section 3, obtained by some suitable transformations of $\mathbb{R}_+^2$, namely, translations and transformations of the form $(x,y)\mapsto(Ax^{\mu},By^{\nu})$. Moreover, in Section 6 we present the corresponding discrete results, related to sequences of non-negative real numbers. Finally, in Section 7 we prove some new Godunova-type inequalities, that is, equivalent inequalities of the Hilbert and Hardy-Hilbert type, with conjugate and non-conjugate exponents, related to cells in $\mathbb{R}_+^n$ and kernels of the form $\mathbf{x}^{-\mathbf{s}}K\left(\frac{\mathbf{y}}{\mathbf{x}}\right)$.

It is important to emphasize that the results presented in this paper cover all discrete and integral results from [3], [6], [7], [8], [10], [11], and [13], and extend them to the case of non-conjugate exponents. Moreover, the technique introduced in Section 2 enables generalizations of one-dimensional integral inequalities to some multidimensional settings.

Conventions. Throughout this paper, let r' be the conjugate exponent to a positive real number $r\neq 1$, that is, $\frac{1}{r}+\frac{1}{r'}=1$, or $r'=\frac{r}{r-1}$. All measures are assumed to be positive and σ-finite, and functions to be non-negative and measurable. Expressions of the form $0\cdot\infty$, $\frac{0}{\infty}$, $\frac{\infty}{\infty}$, and $\frac{0}{0}$ are taken to be equal to zero. In addition, inequalities like (7) and (8) are interpreted to mean that if the right-hand side is finite, so is the left-hand side and the inequality holds.

2. General inequalities of the Hardy-Hilbert type

To provide a basis for our main results, in this section we first discuss two general inequalities of the Hardy-Hilbert type. These equivalent relations are stated and proved in the following theorem.

THEOREM 1. *Let real parameters p, q, and λ be as in (1) and (2), and let X and Y be measure spaces with positive σ-finite measures μ_1 and μ_2 respectively. Let K be a non-negative measurable function on $X\times Y$, φ a measurable, a.e. positive function on X, and ψ a measurable, a.e. positive function on Y. If the functions F on X and G on Y are defined by*

$$F(x)=\left[\int_Y K(x,y)\psi^{-q'}(y)\,d\mu_2(y)\right]^{\frac{1}{q'}},\ \ x\in X, \tag{5}$$

and

$$G(y)=\left[\int_X K(x,y)\varphi^{-p'}(x)\,d\mu_1(x)\right]^{\frac{1}{p'}},\ \ y\in Y, \tag{6}$$

then for all non-negative measurable functions f *on* X *and* g *on* Y *the inequalities*

$$\int_X \int_Y K^\lambda(x,y) f(x) g(y)\, d\mu_1(x) d\mu_2(y) \leqslant \|\varphi F f\|_{L^p(\mu_1)} \|\psi G g\|_{L^q(\mu_2)} \tag{7}$$

and

$$\left\{ \int_Y \left[(\psi G)^{-1}(y) \int_X K^\lambda(x,y) f(x)\, d\mu_1(x) \right]^{q'} d\mu_2(y) \right\}^{\frac{1}{q'}} \leqslant \|\varphi F f\|_{L^p(\mu_1)} \tag{8}$$

hold and are equivalent.

Proof. We prove the inequality (7) first. Let K, φ, and ψ be as in the statement of Theorem 1 and let f and g be arbitrary non-negative measurable functions on X and Y respectively. Since $\frac{1}{q'} + \frac{1}{p'} + (1-\lambda) = 1$, the left-hand side of the relation (7) can be written as

$$\int_X \int_Y K^\lambda(x,y) f(x) g(y)\, d\mu_1(x) d\mu_2(y) = \int_X \int_Y \left[K(x,y) \psi^{-q'}(y) (\varphi^p F^{p-q'} f^p)(x) \right]^{\frac{1}{q'}} \times$$
$$\times \left[K(x,y) \varphi^{-p'}(x) (\psi^q G^{q-p'} g^q)(y) \right]^{\frac{1}{p'}} \left[(\varphi F f)^p(x) \, (\psi G g)^q(y) \right]^{1-\lambda} d\mu_1(x) d\mu_2(y). \tag{9}$$

Now, by using Hölder's inequality, either with the parameters $q', p', \frac{1}{1-\lambda} > 1$ in the case of non-conjugate exponents p and q, or with the parameters p and p' when $q' = p$ (that is, when $\lambda = 1$), and then applying Fubini's theorem, we obtain that the right-hand side of (9) is not greater than

$$\left\{ \int_X \left[\int_Y K(x,y) \psi^{-q'}(y)\, d\mu_2(y) \right] (\varphi^p F^{p-q'} f^p)(x)\, d\mu_1(x) \right\}^{\frac{1}{q'}} \times$$
$$\times \left\{ \int_Y \left[\int_X K(x,y) \varphi^{-p'}(x)\, d\mu_1(x) \right] (\psi^q G^{q-p'} g^q)(y)\, d\mu_2(y) \right\}^{\frac{1}{p'}} \times$$
$$\times \left[\int_X (\varphi F f)^p(x)\, d\mu_1(x) \right]^{1-\lambda} \cdot \left[\int_Y (\psi G g)^q(y)\, d\mu_2(y) \right]^{1-\lambda}$$
$$= \left[\int_X (\varphi F f)^p(x)\, d\mu_1(x) \right]^{\frac{1}{q'}+1-\lambda} \cdot \left[\int_Y (\psi G g)^q(y)\, d\mu_2(y) \right]^{\frac{1}{p'}+1-\lambda}$$
$$= \|\varphi F f\|_{L^p(\mu_1)} \|\psi G g\|_{L^q(\mu_2)},$$

so (7) is proved. The further step is to prove that (7) implies (8) to hold for all non-negative measurable functions f on X. In particular, for any such f and the function g defined by

$$g(y) = (\psi G)^{-q'}(y) \left[\int_X K^\lambda(x,y) f(x)\, d\mu_1(x) \right]^{\frac{q'}{q}}, \quad y \in Y,$$

applying Fubini's theorem, the left-hand side of (7) becomes

$$\begin{aligned} L &= \int_X \int_Y K^\lambda(x,y) f(x) (\psi G)^{-q'}(y) \left[\int_X K^\lambda(x,y) f(x)\, d\mu_1(x) \right]^{\frac{q'}{q}} d\mu_1(x) d\mu_2(y) \\ &= \int_Y \left[(\psi G)^{-1}(y) \int_X K^\lambda(x,y) f(x)\, d\mu_1(x) \right]^{q'} d\mu_2(y), \end{aligned}$$

that is, the integral on the left-hand side of (8), while on the right-hand side of (7) we have

$$\begin{aligned} R &= \|\varphi F f\|_{L^p(\mu_1)} \left\{ \int_Y (\psi G)^{q(1-q')}(y) \left[\int_X K^\lambda(x,y) f(x)\, d\mu_1(x) \right]^{q'} d\mu_2(y) \right\}^{\frac{1}{q}} \\ &= \|\varphi F f\|_{L^p(\mu_1)} L^{\frac{1}{q}}. \end{aligned}$$

Hence,

$$L \leqslant \|\varphi F f\|_{L^p(\mu_1)} L^{\frac{1}{q}},$$

which directly yields (8), so the implication (7) $\Rightarrow$ (8) is proved. Conversly, by using Hölder's inequality for the conjugate exponents q and q', together with the relation (8), for arbitrary $f, g \geqslant 0$ we have

$$\begin{aligned} &\int_X \int_Y K^\lambda(x,y) f(x) g(y)\, d\mu_1(x) d\mu_2(y) \\ &\quad = \int_Y (\psi G g)(y) \left[(\psi G)^{-1}(y) \int_X K^\lambda(x,y) f(x)\, d\mu_1(x) \right] d\mu_2(y) \\ &\quad \leqslant \|\psi G g\|_{L^q(\mu_2)} \left\{ \int_Y \left[(\psi G)^{-1}(y) \int_X K^\lambda(x,y) f(x)\, d\mu_1(x) \right]^{q'} d\mu_2(y) \right\}^{\frac{1}{q'}} \\ &\quad \leqslant \|\varphi F f\|_{L^p(\mu_1)} \|\psi G g\|_{L^q(\mu_2)}. \end{aligned}$$

Thus, (8) implies (7), so these inequalities are equivalent. The proof of Theorem 1 is now completed. □

REMARK 1. Observe that the sign of inequality in (7) depends only on the parameters p', q', and λ, since the crucial step in proving this relation was in applying Hölder's inequality. Therefore, besides $p', q' > 1$ and $\lambda \in \langle 0, 1]$, as in (1) and (2), we can consider exponents which provide the reversed sign of inequality in (7). Especially, if the parameters p and q from Theorem 1 are such that

$$p < 0, \quad q \in \langle 0, 1 \rangle, \quad \frac{1}{p} + \frac{1}{q} \leqslant 1, \tag{10}$$

and λ is defined by (2), we have $p' \in \langle 0, 1 \rangle$, $q' < 0$, and $1 - \lambda \leqslant 0$, so the sign of inequality in (7) is reversed as a direct consequence of the so-called reversed Hölder's inequality (for details, see e.g. [12, Chapter V]). The same result is achieved also with the parameters p and q satisfying

$$p \in \langle 0,1 \rangle, \quad q < 0, \quad \frac{1}{p} + \frac{1}{q} \leqslant 1, \tag{11}$$

since from (11) we obtain $p' < 0$, $q' \in \langle 0,1 \rangle$, and $1-\lambda \leqslant 0$. Moreover, by using the same arguments, $p,q \in \langle 0,1 \rangle$ give another sufficient condition for the reversed inequality sign in (7). In that case we have $p',q' < 0$, and $1-\lambda > 0$. Finally, it is obvious that for all cases of the parameters p', q', and λ, the relations (7) and (8) hold with the same sign of inequality.

REMARK 2. Note that equality in (7) holds if and only if it holds in Hölder's inequality, that is, if and only if the functions $K\psi^{-q'}\varphi^p F^{p-q'} f^p$, $K\varphi^{-p'}\psi^q G^{q-p'} g^q$, and $(\varphi F f)^p(\psi G g)^q$ are effectively proportional on $X \times Y$. Of course, this trivially happens if at least one of the functions involved in the left-hand side of (7) is a zero-function. To discuss other non-trivial cases of equality in (7), we can without loss of generality assume that the functions K, f, and g are positive. Otherwise, instead of $X \times Y$, we consider the set $S = \{(x,y) \in X \times Y : K(x,y)f(x)g(y) > 0\}$, which has a positive measure. Under such assumptions, equality in (7) occurs if and only if there exist positive real constants α_1, β_1, and γ_1, such that the relations

$$\begin{aligned}\alpha_1 K(x,y)\psi^{-q'}(y)(\varphi^p F^{p-q'} f^p)(x) &= \beta_1 K(x,y)\varphi^{-p'}(x)(\psi^q G^{q-p'} g^q)(y)\\ &= \gamma_1 (\varphi F f)^p(x)(\psi G g)^q(y)\end{aligned}$$

hold for a.e. $(x,y) \in X \times Y$. Further, these equalities can be written in a more suitable, equivalent form, as

$$\alpha_1(\varphi^{p+p'} F^{p-q'} f^p)(x) = \beta_1(\psi^{q+q'} G^{q-p'} g^q)(y), \quad \text{for a.e. } (x,y) \in X \times Y, \tag{12}$$

and

$$\alpha_1 K(x,y) = \gamma_1 F^{q'}(x)(\psi^{q+q'} G^q g^q)(y), \quad \text{for a.e. } (x,y) \in X \times Y. \tag{13}$$

Since the left-hand side of (12) depends only on $x \in X$, while the right-hand side of this relation is a single-variable function of $y \in Y$, (12) holds only if

$$\varphi^{p+p'} F^{p-q'} f^p = \alpha^p = const. \quad \text{a.e. on } X$$

and

$$\psi^{q+q'} G^{q-p'} g^q = \beta^p = const. \quad \text{a.e. on } Y,$$

for some positive real constants α and β. Considering $1 + \frac{p'}{p} = p'$ and $1 + \frac{q'}{q} = q'$, these identities can be finally transformed to

$$f = \alpha\varphi^{-p'} F^{\frac{q'}{p}-1} \quad \text{a.e. on } X \text{ and } g = \beta\psi^{-q'} G^{\frac{p'}{q}-1} \quad \text{a.e. on } Y. \tag{14}$$

Moreover, combining (14) with (13), we obtain

$$K = \gamma F^{q'} G^{p'} \quad \text{a.e. on } X \times Y, \tag{15}$$

for some positive real constant γ. Therefore, we proved that the conditions (14) and (15) are necessary and sufficient for equality in (7). Moreover, it is clear from the proof of Theorem 1 that the equality in (8) holds only if it holds in (7).

As an example of the function K which fulfills (15), here we mention

$$K(x,y)=\frac{\varphi^{p'}(x)\psi^{q'}(y)}{\mu_1(X)\mu_2(Y)},\quad (x,y)\in X\times Y,$$

where the sets X and Y are such that $\mu_1(X),\mu_2(Y)<\infty$ and the functions φ and ψ are arbitrary, as in Theorem 1. In particular, in this setting we have

$$F=\mu_1(X)^{-\frac{1}{q'}}\varphi^{\frac{p'}{q'}}\quad\text{and}\quad G=\mu_2(Y)^{-\frac{1}{p'}}\psi^{\frac{q'}{p'}},$$

so K fulfills (15) with $\gamma=1$. Equality in (7) is attained for $f=\alpha\varphi^{-1-\frac{p'}{q'}}$ and $g=\beta\psi^{-1-\frac{q'}{p'}}$, where α and β are positive constants.

In the case of conjugate exponents, that is, when $q=p'$ and $\lambda=1$, Theorem 1 reduces to the following corollary.

COROLLARY 1. *Suppose $p>1$ and X and Y are measure spaces with positive σ-finite measures μ_1 and μ_2 respectively. If K is a non-negative measurable function on $X\times Y$, φ is a measurable, a.e. positive function on X, ψ is a measurable, a.e. positive function on Y, the function F on X is defined by*

$$F(x)=\left[\int_Y K(x,y)\psi^{-p}(y)\,d\mu_2(y)\right]^{\frac{1}{p}},\ \ x\in X,$$

and the function G on Y is given by (6)*, then for all non-negative measurable functions f on X and g on Y the inequalities*

$$\int_X\int_Y K(x,y)f(x)g(y)\,d\mu_1(x)d\mu_2(y)\leqslant\|\varphi Ff\|_{L^p(\mu_1)}\|\psi Gg\|_{L^{p'}(\mu_2)}\tag{16}$$

and

$$\left\{\int_Y\left[(\psi G)^{-1}(y)\int_X K(x,y)f(x)\,d\mu_1(x)\right]^p d\mu_2(y)\right\}^{\frac{1}{p}}\leqslant\|\varphi Ff\|_{L^p(\mu_1)}\tag{17}$$

hold and are equivalent. Moreover, if $0\neq p<1$, the sign of inequality in (16) *and* (17) *is reversed.*

REMARK 3. Corollary 1 is a slight generalization of Theorem 1 from [7], where M. Krnić and J. Pečarić considered only the case of conjugate parameters. Thus, our Theorem 1 may be regarded as an extension of the mentioned result to non-conjugate exponents.

3. The case of homogeneous functions K

In this section, we apply general results from Theorem 1 to non-negative monotone homogeneous functions K on $\mathbb{R}_+^2$ of some negative real degree. More precisely, we consider a non-negative function $K:\mathbb{R}_+^2\to\mathbb{R}$ with the following properties:

(i) K is strictly decreasing in each argument, that is, for all $x_1, x_2, y \in \mathbb{R}_+$, $x_1 < x_2$, we have

$$K(x_1, y) > K(x_2, y) \quad \text{and} \quad K(y, x_1) > K(y, x_2);$$

(ii) K is homogeneous of degree $-s$ for some $s \in \mathbb{R}_+$, that is, the identity $K(tx, ty) = t^{-s}K(x, y)$ holds for all $t, x, y \in \mathbb{R}_+$;

(iii) K is such that $k(\alpha) < \infty$ holds for all $\alpha \in \langle 1-s, 1\rangle$, where we denote

$$k(\alpha) = \int_0^\infty K(1, u)u^{-\alpha}du, \quad \alpha \in \mathbb{R}.$$

Observe that the condition (ii) implies the following sequence of identities:

$$k(\alpha) = \int_0^\infty K\left(\frac{1}{u}, 1\right) u^{-s-\alpha}du = \int_0^\infty K(u, 1)u^{s+\alpha-2}du,$$

while from (i) we obtain that K is strictly positive on $\mathbb{R}^2_+$. In particular, for $\alpha \geqslant 1$, monotonicity of K in the second argument and the fact that $K(1, 1) > 0$ yield

$$k(\alpha) = \int_0^\infty K(1, u)u^{-\alpha}du \geqslant \int_0^1 K(1, u)u^{-\alpha}du \geqslant K(1, 1)\int_0^1 u^{-\alpha}du = \infty.$$

Analogous result holds also for $\alpha \leqslant 1-s$, since

$$\begin{aligned} k(\alpha) &= \int_0^\infty K(u, 1)u^{s+\alpha-2}du \geqslant \int_0^1 K(u, 1)u^{s+\alpha-2}du \\ &\geqslant K(1, 1)\int_0^1 u^{s+\alpha-2}du = \infty. \end{aligned}$$

Therefore, the interval $\langle 1-s, 1\rangle$, considered in (iii), covers all arguments α for which $k(\alpha)$ may converge. The same conclusion on convergence of $k(\alpha)$ can be drawn if in (i) we consider a function K decreasing in each argument and such that $K(1, 1) > 0$.

Now, we can state our first result concerning homogeneous functions.

THEOREM 2. *Let p, q, and λ be as in (1) and (2), and let $0 \leqslant a, b, c, d \leqslant \infty$ be such that $a < b$ and $c < d$. Suppose K is a non-negative measurable function on $\mathbb{R}^2_+$, fulfilling the conditions* (i), (ii), *and* (iii), *and the function H is, for $\alpha \in \langle 1-s, 1\rangle$, $0 \leqslant e < f \leqslant \infty$, and $t \in \mathbb{R}_+$ defined by*

$$H(\alpha, e, f; t) = \int_{\frac{e}{t}}^{\frac{f}{t}} K(1, u)u^{-\alpha}du. \tag{18}$$

Then for all real parameters A_1 and A_2, such that $A_1p', A_2q' \in \langle 1-s, 1\rangle$, and for all *non-negative measurable functions* f *on* $\langle a,b\rangle$ *and* g *on* $\langle c,d\rangle$ *, the inequalities*

$$\begin{aligned}&\int_a^b\int_c^d K^{\lambda}(x,y)f(x)g(y)\,dx\,dy\\&\quad\leqslant\left[\int_a^b x^{(A_1-A_2)p+(1-s)\frac{p}{q'}}H^{\frac{p}{q'}}(A_2q',c,d;x)f^p(x)\,dx\right]^{\frac{1}{p}}\times\\&\qquad\times\left[\int_c^d y^{(A_2-A_1)q+(1-s)\frac{q}{p'}}H^{\frac{q}{p'}}\left(2-A_1p'-s,b^{-1},a^{-1};y^{-1}\right)g^q(y)\,dy\right]^{\frac{1}{q}}\end{aligned}\tag{19}$$

and

$$\begin{aligned}&\left\{\int_c^d y^{(A_1-A_2)q'+(s-1)\frac{q'}{p'}}H^{-\frac{q'}{p'}}\left(2-A_1p'-s,b^{-1},a^{-1};y^{-1}\right)\left[\int_a^b K^{\lambda}(x,y)f(x)\,dx\right]^{q'}dy\right\}^{\frac{1}{q'}}\\&\quad\leqslant\left[\int_a^b x^{(A_1-A_2)p+(1-s)\frac{p}{q'}}H^{\frac{p}{q'}}(A_2q',c,d;x)f^p(x)\,dx\right]^{\frac{1}{p}}\end{aligned}\tag{20}$$

hold and are equivalent. Moreover, if p *and* q *are as in* (10)*,* (11)*, or* $p,q\in\langle 0,1\rangle$ *, the signs of inequality in* (19) *and* (20) *are reversed.*

Proof. Suppose that in Theorem 1 we have $X=\langle a,b\rangle$, $Y=\langle c,d\rangle$, $\varphi(x)=x^{A_1}$, $\psi(y)=y^{A_2}$, and Lebesgue measures $d\mu_1(x)=dx$, $d\mu_2(y)=dy$. In this setting, by using homogenity of the function K and some suitable substitutions, (5) and (6) respectively read

$$\begin{aligned}F(x)&=\left[\int_c^d K(x,y)y^{-A_2q'}dy\right]^{\frac{1}{q'}}=x^{-\frac{s}{q'}}\left[\int_c^d K\left(1,\frac{y}{x}\right)y^{-A_2q'}dy\right]^{\frac{1}{q'}}\\&=x^{\frac{1-s}{q'}-A_2}H^{\frac{1}{q'}}(A_2q',c,d;x),\quad x\in\langle a,b\rangle,\end{aligned}$$

and

$$\begin{aligned}G(y)&=\left[\int_a^b K(x,y)x^{-A_1p'}dx\right]^{\frac{1}{p'}}=y^{-\frac{s}{p'}}\left[\int_a^b K\left(\frac{x}{y},1\right)x^{-A_1p'}dx\right]^{\frac{1}{p'}}\\&=y^{\frac{1-s}{p'}-A_1}\left[\int_{\frac{a}{y}}^{\frac{b}{y}}K(u,1)u^{-A_1p'}du\right]^{\frac{1}{p'}}\\&=y^{\frac{1-s}{p'}-A_1}H^{\frac{1}{p'}}\left(2-A_1p'-s,b^{-1},a^{-1};y^{-1}\right),\quad y\in\langle c,d\rangle,\end{aligned}$$

so (19) and (20) hold directly from Theorem 1 by inserting the obtained expressions for $F(x)$ and $G(y)$ in the relations (7) and (8). The conditions for the reverse inequalities are already discussed in Remark 1. Note that the function H is well-defined since we have $A_1p',A_2q'\in\langle 1-s,1\rangle$. □

Evidently, the results of Theorem 2 follow mainly from the fact that K is homogeneous function of degree $-s$, that is, from the property (ii), while other two properties of K, namely (i) and (iii), serve only to ensure that $H\left(2-A_1p'-s, b^{-1}, a^{-1}; \cdot\right)$ and $H(A_2q', c, d; \cdot)$ are well-defined, that is, that $k(2-A_1p'-s)$ and $k(A_2q')$ converge. Therefore, in the statement of Theorem 2 the conditions (i) and (iii) can be replaced with the condition that A_1 and A_2 are such that $k(2-A_1p'-s) < \infty$ and $k(A_2q') < \infty$.

Of course, the most important case of Theorem 2 is with integrals over $\mathbb{R}_+$, that is, when $a = c = 0$ and $b = d = \infty$. The corresponding equivalent Hardy-Hilbert-type inequalities are given in the following corollary.

COROLLARY 2. *If p, q, and λ are as in* (1) *and* (2), *and K is a non-negative measurable function on $\mathbb{R}_+^2$, fulfilling* (i), (ii), *and* (iii), *then the inequalities*

$$\int_0^\infty \int_0^\infty K^\lambda(x,y) f(x) g(y)\, dx dy \leqslant k^{\frac{1}{p'}}(2-A_1p'-s) k^{\frac{1}{q'}}(A_2q') \times \\ \times \left[\int_0^\infty x^{(A_1-A_2)p+(1-s)\frac{p}{q'}} f^p(x)\, dx\right]^{\frac{1}{p}} \cdot \left[\int_0^\infty y^{(A_2-A_1)q+(1-s)\frac{q}{p'}} g^q(y)\, dy\right]^{\frac{1}{q}} \tag{21}$$

and

$$\left\{\int_0^\infty y^{(A_1-A_2)q'+(s-1)\frac{q'}{p'}} \left[\int_0^\infty K^\lambda(x,y) f(x)\, dx\right]^{q'} dy\right\}^{\frac{1}{q'}} \\ \leqslant k^{\frac{1}{p'}}(2-A_1p'-s) k^{\frac{1}{q'}}(A_2q') \left[\int_0^\infty x^{(A_1-A_2)p+(1-s)\frac{p}{q'}} f^p(x)\, dx\right]^{\frac{1}{p}} \tag{22}$$

hold for all real parameters A_1 and A_2, such that $A_1p', A_2q' \in \langle 1-s, 1\rangle$, and for all non-negative measurable functions f and g on $\mathbb{R}_+$. Moreover, these inequalities are equivalent. If p and q are as in (10), (11), *or $p, q \in \langle 0, 1\rangle$, the inequalities* (21) *and* (22) *are reversed.*

REMARK 4. If the parameters p and q are conjugate, that is, if $\lambda = 1$, Corollary 2 reduces to Corollary 4 from [7]. Thus, the relations (21) and (22) can be seen as a generalization of the mentioned result of M. Krnić and J. Pečarić.

REMARK 5. Obviously, the function $K(x,y) = (x+y)^{-s}$, where $s > 0$, fulfils the conditions of Theorem 2 and Corollary 2. In this case, $k(\alpha)$ converges for all $\alpha \in \langle 1-s, 1\rangle$ and we have

$$k(\alpha) = \int_0^\infty (1+u)^{-s} u^{-\alpha} du = B(1-\alpha, s+\alpha-1) = B(s+\alpha-1, 1-\alpha) = k(2-s-\alpha),$$

where B is the usual Beta function. Hence, the constant on the right-hand sides of (21) and (22) becomes $C = B(1-A_1p', A_1p'+s-1)^{\frac{1}{p'}} B(1-A_2q', A_2q'+s-1)^{\frac{1}{q'}}$. Especially, for $s = 1$ and $A = A_1 = A_2 \in \left\langle 0, \min\{\frac{1}{p'}, \frac{1}{q'}\}\right\rangle$, we have

$$C_A = C = \pi^\lambda \operatorname{cosec}^{\frac{1}{p'}}(\pi p'A) \operatorname{cosec}^{\frac{1}{q'}}(\pi q'A)$$

and (21) reads

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy \leqslant C_A\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)}.$$

Since

$$\inf_A C_A = \pi^\lambda \operatorname{cosec}^{\frac{1}{p'}}\frac{\pi}{\lambda q'}\operatorname{cosec}^{\frac{1}{q'}}\frac{\pi}{\lambda p'} = \left(\pi \operatorname{cosec}\frac{\pi}{\lambda p'}\right)^\lambda,$$

we obtained (3). On the other hand, in the case of conjugate parameters p and q we get Theorem 1 in [6], so our result extends the one of I. Brnetić and J. Pečarić. Moreover, for $p > 1$, $\lambda = 1$, $s > \max\{2-p, 2-p', 0\}$, and $A_1 = A_2 = \frac{2-s}{pp'}$, we obtain $C = B\left(\frac{p+s-2}{p}, \frac{p'+s-2}{p'}\right)$, that is, the inequalities from [13, Theorem 4.1], also stated in our Introduction.

REMARK 6. In Theorem 2 and Corollary 2 we can also consider the function K defined on $\mathbb{R}_+^2$ by $K(x,y) = \frac{\ln y - \ln x}{y-x}$. Evidently, it is homogeneous of degree -1 and decreasing in both arguments, $k(\alpha)$ converges for all $\alpha \in \langle 0,1\rangle$, and we have

$$k(\alpha) = \int_0^\infty \frac{\ln u}{u-1}u^{-\alpha}du = \int_{-\infty}^\infty \frac{te^{(1-\alpha)t}}{e^t-1}dt = \psi'(\alpha)+\psi'(1-\alpha) = \frac{\pi^2}{\sin^2 \pi\alpha},$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, $x > 0$, is the Digamma function and we used the well-known identity $\psi(1-x) = \psi(x) + \pi\cot \pi x$, $x \in \langle 0,1\rangle$ (for details on ψ see [1]). Therefore, (21) and (22) hold with the constant $\pi^{2\lambda}\sin^{-\frac{2}{p'}} A_1p' \sin^{-\frac{2}{q'}} A_2q'$ on their right-hand sides.

REMARK 7. For the function K given on $\mathbb{R}_+^2$ by $K(x,y) = \max\{x,y\}^{-s}$, where $s > 0$, we have

$$k(\alpha) = \frac{s}{(1-\alpha)(s+\alpha-1)}, \quad \alpha \in \langle 1-s, 1\rangle,$$

so the constant factor on the right-hand sides of (21) and (22) in this setting becomes $s^\lambda[(1-A_1p')(s+A_1p'-1)]^{-\frac{1}{p'}}[(1-A_2q')(s+A_2q'-1)]^{-\frac{1}{q'}}$.

Note that for all $t \in \mathbb{R}_+$ and $\alpha \in \langle 1-s,1\rangle$ we have $H(\alpha,0,\infty;t) = k(\alpha) < \infty$, that is, in the case when $a = c = 0$ and $b = d = \infty$, (19) and (20) hold with an explicit constant factor on their right-hand sides. Therefore, our next step is to consider other cases of the intervals $\langle a,b\rangle$ and $\langle c,d\rangle$. In the sequel, we obtain upper bounds for H which bring some new interesting inequalities of the Hardy-Hilbert type.

LEMMA 1. *Let K be a non-negative measurable function on $\mathbb{R}_+^2$, fulfilling* (i), (ii), *and* (iii). *If $\alpha \in \langle 1-s,1\rangle$, $0 \leqslant e < f \leqslant \infty$, $0 \leqslant m < t < M \leqslant \infty$, and the function H is defined by* (18), *then*

$$H(\alpha,e,f;t) \leqslant k(\alpha) - E_{m,M}(\alpha,e,f;t), \tag{23}$$

where

$$E_{m,M}(\alpha,e,f;t) = \left(\frac{m}{t}\right)^{1-\alpha}\int_0^{\frac{e}{m}} K(1,u)u^{-\alpha}du + \left(\frac{t}{M}\right)^{s+\alpha-1}\int_0^{\frac{M}{f}} K(u,1)u^{s+\alpha-2}du. \quad (24)$$

Proof. Starting from (18) and then applying (i), (ii), (iii), and some suitable substitutions in integrals, we obtain

$$\begin{aligned} H(\alpha,e,f;t) &= k(\alpha) - \int_0^{\frac{e}{t}} K(1,u)u^{-\alpha}du - \int_{\frac{f}{t}}^{\infty} K(1,u)u^{-\alpha}du \\ &= k(\alpha) - \left(\frac{e}{t}\right)^{1-\alpha}\int_0^1 K\left(1,\frac{e}{t}u\right)u^{-\alpha}du - \left(\frac{t}{f}\right)^{s+\alpha-1}\int_0^1 K\left(\frac{t}{f}u,1\right)u^{s+\alpha-2}du \\ &\leqslant k(\alpha) - \left(\frac{e}{t}\right)^{1-\alpha}\int_0^1 K\left(1,\frac{e}{m}u\right)u^{-\alpha}du - \left(\frac{t}{f}\right)^{s+\alpha-1}\int_0^1 K\left(\frac{M}{f}u,1\right)u^{s+\alpha-2}du \\ &= k(\alpha) - E_{m,M}(\alpha,e,f;t), \end{aligned}$$

so the proof is completed. □

REMARK 8. Since

$$E_{m,M}(\alpha,0,f;t) = \left(\frac{t}{M}\right)^{s+\alpha-1}\int_0^{\frac{M}{f}} K(u,1)u^{s+\alpha-2}du, \quad t \in \langle 0,M\rangle,$$

for $e=0$ the estimate (23) does not depend on m. Similarly, for $f=\infty$ we have

$$E_{m,M}(\alpha,e,\infty;t) = \left(\frac{m}{t}\right)^{1-\alpha}\int_0^{\frac{e}{m}} K(1,u)u^{-\alpha}du, \quad t > m.$$

REMARK 9. Besides functions decreasing in both arguments, we also consider non-negative functions $K:\mathbb{R}_+^2 \to \mathbb{R}$ which are homogeneous of degree $-s$ for some $s>0$, strictly decreasing in the first argument, strictly increasing in the second argument, and such that $k(\alpha)<\infty$ holds for all $\alpha>1$. As it was shown in an analysis presented at the beginning of this section, it can be easily obtained that such functions are positive on $\mathbb{R}_+$ and that $k(\alpha)$ diverges for all $\alpha \leqslant 1$. Moreover, for $\alpha>1$, $e\in[0,\infty\rangle$, $f\in\langle 0,\infty]$, and $0\leqslant m<t<M\leqslant\infty$, in this setting we have

$$\begin{aligned} H(\alpha,e,\infty;t) &= k(\alpha) - \int_0^{\frac{e}{t}} K(1,u)u^{-\alpha}du = k(\alpha) - \left(\frac{e}{t}\right)^{1-\alpha}\int_0^1 K\left(1,\frac{e}{t}u\right)u^{-\alpha}du \\ &\geqslant k(\alpha) - \left(\frac{e}{t}\right)^{1-\alpha}\int_0^1 K\left(1,\frac{e}{m}u\right)u^{-\alpha}du = k(\alpha) - E_{m,M}(\alpha,e,\infty;t) \quad (25) \end{aligned}$$

and

$$
\begin{aligned}
H(\alpha,0,f;t) &= k(\alpha) - \int_{\frac{f}{t}}^{\infty} K(1,u)u^{-\alpha}du \\
&= k(\alpha) - \left(\frac{t}{f}\right)^{s+\alpha-1}\int_0^1 K\left(\frac{t}{f}u,1\right)u^{s+\alpha-2}du \\
&\leqslant k(\alpha) - \left(\frac{t}{f}\right)^{s+\alpha-1}\int_0^1 K\left(\frac{M}{f}u,1\right)u^{s+\alpha-2}du = k(\alpha) - E_{m,M}(\alpha,0,f;t), \quad (26)
\end{aligned}
$$

where $E_{m,M}(\alpha,e,\infty;t)$ and $E_{m,M}(\alpha,0,f;t)$ are defined as in Remark 8.

Analogously, for a non-negative function $K:\mathbb{R}_+^2 \to \mathbb{R}$, strictly increasing in the first argument, strictly decreasing in the second argument, which fulfils (ii), and such that for all $\alpha < 1-s$ we have $k(\alpha) < \infty$, the relations (25) and (26) again hold for $\alpha < 1-s$, $e \in [0,\infty\rangle$, $f \in \langle 0,\infty]$, and $0 \leqslant m < t < M \leqslant \infty$, but the signs of inequality are reversed.

Lemma 1 provides estimates which yield further pairs of equivalent Hardy-Hilbert-type inequalities.

THEOREM 3. *Let p, q, and λ be as in* (1) *and* (2)*, and let $0 \leqslant a,b,c,d \leqslant \infty$ be such that $a<b$ and $c<d$. If K is a non-negative measurable function on $\mathbb{R}_+^2$, fulfilling the conditions* (i), (ii), *and* (iii)*, and the function E is defined by* (24)*, then the inequalities*

$$
\begin{aligned}
&\int_a^b\int_c^d K^{\lambda}(x,y)f(x)g(y)\,dx\,dy \\
&\leqslant \left\{\int_a^b x^{(A_1-A_2)p+(1-s)\frac{p}{q'}}\left[k(A_2q') - E_{a,b}(A_2q',c,d;x)\right]^{\frac{p}{q'}} f^p(x)\,dx\right\}^{\frac{1}{p}} \times \\
&\times \left\{\int_c^d y^{(A_2-A_1)q+(1-s)\frac{q}{p'}} \times \right. \\
&\left. \times \left[k(2-A_1p'-s) - E_{d^{-1},c^{-1}}\left(2-A_1p'-s,b^{-1},a^{-1};y^{-1}\right)\right]^{\frac{q}{p'}} g^q(y)\,dy\right\}^{\frac{1}{q}} \quad (27)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\{\int_c^d y^{(A_1-A_2)q'+(s-1)\frac{q'}{p'}}\left[\int_a^b K^{\lambda}(x,y)f(x)\,dx\right]^{q'} \times \right. \\
&\qquad \left. \times \left[k(2-A_1p'-s) - E_{d^{-1},c^{-1}}\left(2-A_1p'-s,b^{-1},a^{-1};y^{-1}\right)\right]^{-\frac{q'}{p'}} dy\right\}^{\frac{1}{q'}} \\
&\leqslant \left[\int_a^b x^{(A_1-A_2)p+(1-s)\frac{p}{q'}}\left[k(A_2q') - E_{a,b}(A_2q',c,d;x)\right]^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}} \quad (28)
\end{aligned}
$$

hold for all real parameters A_1 *and* A_2, *such that* $A_1p', A_2q' \in \langle 1-s, 1\rangle$, *and for all non-negative measurable functions* f *on* $\langle a,b\rangle$ *and* g *on* $\langle c,d\rangle$. *Moreover, these inequalities are equivalent. If* $p,q \in \langle 0,1\rangle$, *the relations* (27) *and* (28) *hold with the reversed sign of inequality.*

Proof. Note that the relations (27) and (28) follow directly from Theorem 2 and Lemma 1. To prove their equivalence, we consider X, Y, φ, ψ from the proof of Theorem 2, the function

$$g(y) = \left[k(2-A_1p'-s) - E_{d^{-1},c^{-1}}\left(2-A_1p'-s, b^{-1}, a^{-1}; y^{-1}\right)\right]^{-\frac{q'}{p'}} \times \\ \times\, y^{(A_1-A_2)q'+(s-1)\frac{q'}{p'}} \left[\int_a^b K^{\lambda}(x,y) f(x)\,dx\right]^{\frac{q'}{q}}, \quad y \in \langle c,d\rangle,$$

and use the same technique as in the proof of Theorem 1. □

REMARK 10. For $a=c=0$ and $b=d=\infty$ Theorem 3 reduces to Corollary 2, since $E_{m,M}(\alpha,0,\infty;t) \equiv 0$.

REMARK 11. Suppose p and q are as in (10), $0=a=c<b,d \leqslant \infty$, and a non-negative function $K:\mathbb{R}_+ \to \mathbb{R}$ is homogeneous of degree $-s$ for some $s>0$, strictly decreasing in the first argument, strictly increasing in the second argument, and such that $k(\alpha)<\infty$ holds for all $\alpha>1$. If $A_1p'>1$ and $A_2q'<1-s$, then (27) and (28) hold with the reversed sign of inequality as a consequence of Remark 1. and Remark 9. For such p and q, the same conclusion holds also if $0 \leqslant a,c < b=d=\infty$, $A_1p'<1-s$, $A_2q'>1$, and K is a non-negative homogeneous function of degree $-s$ for some $s>0$, strictly increasing in the first argument, strictly decreasing in the second argument, and such that $k(\alpha)<\infty$ holds for all $\alpha<1-s$. Note that the case when p and q are as in (11) can be analyzed similarly.

REMARK 12. For $a=c$, $b=d$, and $\lambda=1$ in Theorem 3 and Remark 11 we obtain Theorem 5 from [7], so our results generalize those of M. Krnić and J. Pečarić.

4. Symmetric functions K and one important example

To complete our analysis, we consider a class of non-negative symmetric functions K on $\mathbb{R}_+^2$, satisfying (i), (ii), and (iii). Since $K(x,y)=K(y,x)$, $x,y\in\mathbb{R}_+$, for such functions and parameters as in Lemma 1 we have

$$k(\alpha) = k(2-\alpha-s), \quad H(\alpha,e,f;t) = H(2-\alpha-s, f^{-1}, e^{-1}; t^{-1}),$$

and

$$E_{m,M}(\alpha,e,f;t) = E_{M^{-1},m^{-1}}(2-\alpha-s, f^{-1}, e^{-1}; t^{-1}).$$

Therefore, in the case when $a=c$ and $b=d$, the relations (19) and (27) from Theorem 2 and Theorem 3 together read

$$\begin{aligned}
&\int_a^b\int_a^b K^{\lambda}(x,y)f(x)g(y)\,dx\,dy\\
&\quad\leqslant\left[\int_a^b x^{(A_1-A_2)p+(1-s)\frac{p}{q'}}H^{\frac{p}{q'}}(A_2q',a,b;x)f^p(x)\,dx\right]^{\frac{1}{p}}\times\\
&\quad\quad\times\left[\int_a^b y^{(A_2-A_1)q+(1-s)\frac{q}{p'}}H^{\frac{q}{p'}}(A_1p',a,b;y)g^q(y)\,dy\right]^{\frac{1}{q}} \qquad (29)\\
&\quad\leqslant\left\{\int_a^b x^{(A_1-A_2)p+(1-s)\frac{p}{q'}}\left[k(A_2q')-E_{a,b}(A_2q',a,b;x)\right]^{\frac{p}{q'}}f^p(x)\,dx\right\}^{\frac{1}{p}}\times\\
&\quad\quad\times\left\{\int_a^b y^{(A_2-A_1)q+(1-s)\frac{q}{p'}}\left[k(A_1p')-E_{a,b}(A_1p',a,b;y)\right]^{\frac{q}{p'}}g^q(y)\,dy\right\}^{\frac{1}{q}},
\end{aligned}$$

while (20) and (28) become

$$\begin{aligned}
&\left\{\int_a^b y^{(A_1-A_2)q'+(s-1)\frac{q'}{p'}}\times\right.\\
&\quad\left.\times\left[k(A_1p')-E_{a,b}(A_1p',a,b;y)\right]^{-\frac{q'}{p'}}\left[\int_a^b K^{\lambda}(x,y)f(x)\,dx\right]^{q'}dy\right\}^{\frac{1}{q'}}\\
&\quad\leqslant\left\{\int_a^b y^{(A_1-A_2)q'+(s-1)\frac{q'}{p'}}H^{-\frac{q'}{p'}}(A_1p',a,b;y)\left[\int_a^b K^{\lambda}(x,y)f(x)\,dx\right]^{q'}dy\right\}^{\frac{1}{q'}}\\
&\quad\leqslant\left[\int_a^b x^{(A_1-A_2)p+(1-s)\frac{p}{q'}}H^{\frac{p}{q'}}(A_2q',a,b;x)f^p(x)\,dx\right]^{\frac{1}{p}}\\
&\quad\leqslant\left\{\int_a^b x^{(A_1-A_2)p+(1-s)\frac{p}{q'}}\left[k(A_2q')-E_{a,b}(A_2q',a,b;x)\right]^{\frac{p}{q'}}f^p(x)\,dx\right\}^{\frac{1}{p}}. \qquad (30)
\end{aligned}$$

Especially, for $\lambda=1$, $s>\max\left\{\frac{1}{p},\frac{1}{p'}\right\}$, and $A_1=A_2=\frac{1}{pp'}$, our Theorem 3 reduces to Theorem 2 from [8]. Of course, a discussion related to reverse inequalities to (29) and (30) remains the same as in the general case.

The results obtained for symmetrical functions can be applied to the function K defined on $\mathbb{R}_+^2$ by $K(x,y)=(x+y)^{-s}$, where $s>0$. First, we consider some special choices of parameters s, A_1, and A_2.

THEOREM 4. *Let p, q, and λ be as in* (1) *and* (2)*, and let* $0 \leqslant a < b \leqslant \infty$. *Then the inequalities*

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^\lambda}dxdy$$
$$\leqslant \left[\int_a^b \left(\pi - 2\arctan\sqrt{\frac{a}{x}} - 2\arctan\sqrt{\frac{x}{b}}\right)^{\frac{p}{q'}} x^{p-1-\frac{p}{2}\lambda} f^p(x)\,dx\right]^{\frac{1}{p}} \times$$
$$\times \left[\int_a^b \left(\pi - 2\arctan\sqrt{\frac{a}{y}} - 2\arctan\sqrt{\frac{y}{b}}\right)^{\frac{q}{p'}} y^{q-1-\frac{q}{2}\lambda} g^q(y)\,dy\right]^{\frac{1}{q}} \quad (31)$$

and

$$\left\{\int_a^b \left(\pi - 2\arctan\sqrt{\frac{a}{y}} - 2\arctan\sqrt{\frac{y}{b}}\right)^{-\frac{q'}{p'}} y^{\frac{\lambda q'}{2}-1} \left[\int_a^b \frac{f(x)}{(x+y)^\lambda}dx\right]^{q'} dy\right\}^{\frac{1}{q'}}$$
$$\leqslant \left[\int_a^b \left(\pi - 2\arctan\sqrt{\frac{a}{x}} - 2\arctan\sqrt{\frac{x}{b}}\right)^{\frac{p}{q'}} x^{p-1-\frac{p}{2}\lambda} f^p(x)\,dx\right]^{\frac{1}{p}} \quad (32)$$

hold for all non-negative measurable functions f and g on $\langle a,b\rangle$ and are equivalent. Moreover, if p and q are as in (10), (11)*, or $p,q \in \langle 0,1\rangle$, the signs of inequality in* (31) *and* (32) *are reversed.*

Proof. Follows directly from Theorem 2, (29), and (30), considering the function $K(x,y) = (x+y)^{-1}$, that is, $s = 1$, and the parameters $c = a$, $d = b$, $A_1 = \frac{1}{2p'}$, $A_2 = \frac{1}{2q'}$. In this case, for $t \in \langle a,b\rangle$ and $\alpha = \frac{1}{2}$, we have

$$H\left(\frac{1}{2},a,b;t\right) = \int_{\frac{a}{t}}^{\frac{b}{t}} \frac{du}{(1+u)\sqrt{u}} = \pi - 2\arctan\sqrt{\frac{a}{t}} - 2\arctan\sqrt{\frac{t}{b}},$$

so the function H can be calculated explicitly. □

Since an elementary calculus yields

$$H\left(\frac{1}{2},a,b;t\right) \leqslant \sup_{t>0} H\left(\frac{1}{2},a,b;t\right) = H\left(\frac{1}{2},a,b;\sqrt{ab}\right) = \pi - 4\arctan\sqrt[4]{\frac{a}{b}},$$

we proved the following corollary.

COROLLARY 3. *If p, q, and λ are as in* (1) *and* (2)*, and* $0 \leqslant a < b \leqslant \infty$*, then the inequalities*

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^\lambda}dxdy$$
$$\leqslant \left(\pi - 4\arctan\sqrt[4]{\frac{a}{b}}\right)^\lambda \left[\int_a^b x^{p-1-\frac{p}{2}\lambda} f^p(x)\,dx\right]^{\frac{1}{p}} \cdot \left[\int_a^b y^{q-1-\frac{q}{2}\lambda} g^q(y)\,dy\right]^{\frac{1}{q}} \quad (33)$$

and

$$\left\{\int_a^b y^{\frac{\lambda q'}{2}-1}\left[\int_a^b \frac{f(x)}{(x+y)^\lambda}dx\right]^{q'}dy\right\}^{\frac{1}{q'}} \leqslant \left(\pi - 4\arctan\sqrt[4]{\frac{a}{b}}\right)^\lambda \left[\int_a^b x^{p-1-\frac{p}{2}\lambda} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{34}$$

hold for all non-negative measurable functions f and g on $\langle a,b\rangle$ and are equivalent. Moreover, if $p,q\in\langle 0,1\rangle$, the signs of inequality in these relations are reversed.

REMARK 13. Inequality (34) for $\lambda = 1$ can be found in [10], while B. Yang and T. M. Rassias, [13], proved a particular case of (33), with $\lambda = 1$ and $p = 2$.

Our further step is to consider the relations (29) and (30) with an arbitrary $s > 0$, $K(x,y) = (x+y)^{-s}$, $A_1 = \frac{2-s}{2p'}$, and $A_2 = \frac{2-s}{2q'}$. The case $s = 1$ is already described in Theorem 4 and Corollary 3. Since $A_1p' = A_2q' = 1-\frac{s}{2}$, here we need to use

$$H\left(1-\frac{s}{2},a,b;t\right) = \int_{\frac{a}{t}}^{\frac{b}{t}} (1+u)^{-s} u^{\frac{s}{2}-1}du,$$

which, in general, cannot be calculated easily. On the other hand, in this setting we can take advantage of the estimates from Lemma 1 and Theorem 3.

THEOREM 5. *Suppose p, q, and λ are as in* (1) *and* (2), *$s>0$, and $0\leqslant a < b \leqslant \infty$. Then the inequalities*

$$\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^{s\lambda}}dxdy \leqslant B^\lambda\left(\frac{s}{2},\frac{s}{2}\right)\left\{\int_a^b\left[1-\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{x}{b}\right)^{\frac{s}{2}}\right]^{\frac{p}{q'}} x^{p-1-\frac{s}{2}p\lambda}f^p(x)\,dx\right\}^{\frac{1}{p}} \times \left\{\int_a^b\left[1-\frac{1}{2}\left(\frac{a}{y}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{y}{b}\right)^{\frac{s}{2}}\right]^{\frac{q}{p'}} y^{q-1-\frac{s}{2}q\lambda}g^q(y)\,dy\right\}^{\frac{1}{q}} \tag{35}$$

and

$$\left\{\int_a^b\left[1-\frac{1}{2}\left(\frac{a}{y}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{y}{b}\right)^{\frac{s}{2}}\right]^{-\frac{q'}{p'}} y^{\frac{s}{2}q'\lambda-1}\left[\int_a^b\frac{f(x)}{(x+y)^{s\lambda}}dx\right]^{q'}dy\right\}^{\frac{1}{q'}} \leqslant B^\lambda\left(\frac{s}{2},\frac{s}{2}\right)\left\{\int_a^b\left[1-\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{x}{b}\right)^{\frac{s}{2}}\right]^{\frac{p}{q'}} x^{p-1-\frac{s}{2}p\lambda}f^p(x)\,dx\right\}^{\frac{1}{p}} \tag{36}$$

hold for all non-negative measurable functions f and g on $\langle a,b\rangle$ and are equivalent. Moreover, if $p,q\in\langle 0,1\rangle$, the relations (35) *and* (36) *hold with the reversed sign of inequality.*

Proof. We apply Theorem 3 and Remark 5 with parameters described before the statement of Theorem 5, considering (29), (30), and that

$$\begin{aligned} E_{a,b}\left(1-\frac{s}{2},a,b;t\right) &= \left[\left(\frac{a}{t}\right)^{\frac{s}{2}}+\left(\frac{t}{b}\right)^{\frac{s}{2}}\right]\int_0^1 (1+u)^{-s}u^{\frac{s}{2}-1}du \\ &= \frac{1}{2}B\left(\frac{s}{2},\frac{s}{2}\right)\left[\left(\frac{a}{t}\right)^{\frac{s}{2}}+\left(\frac{t}{b}\right)^{\frac{s}{2}}\right] \end{aligned}$$

holds for all $t\in\langle a,b\rangle$. □

COROLLARY 4. *If p, q, and λ are as in* (1) *and* (2), *$s>0$, and $0\leqslant a<b\leqslant\infty$, then the inequalities*

$$\begin{aligned} &\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^{s\lambda}}dxdy \\ &\leqslant B^\lambda\left(\frac{s}{2},\frac{s}{2}\right)\left[1-\left(\frac{a}{b}\right)^{\frac{s}{4}}\right]^\lambda\left\{\int_a^b x^{p-1-\frac{s}{2}p\lambda}f^p(x)dx\right\}^{\frac{1}{p}}\cdot\left\{\int_a^b y^{q-1-\frac{s}{2}q\lambda}g^q(y)dy\right\}^{\frac{1}{q}} \end{aligned}$$

and

$$\begin{aligned} &\left\{\int_a^b y^{\frac{s}{2}q'\lambda-1}\left[\int_a^b\frac{f(x)}{(x+y)^{s\lambda}}dx\right]^{q'}dy\right\}^{\frac{1}{q'}} \\ &\qquad\leqslant B^\lambda\left(\frac{s}{2},\frac{s}{2}\right)\left[1-\left(\frac{a}{b}\right)^{\frac{s}{4}}\right]^\lambda\left\{\int_a^b x^{p-1-\frac{s}{2}p\lambda}f^p(x)dx\right\}^{\frac{1}{p}} \end{aligned}$$

hold for all non-negative measurable functions f and g on $\langle a,b\rangle$ and are equivalent. Moreover, if $p,q\in\langle 0,1\rangle$, the signs of inequality in (35) *and* (36) *are reversed.*

Proof. Note that AG-inequality implies

$$\frac{1}{2}\left(\frac{a}{t}\right)^{\frac{s}{2}}+\frac{1}{2}\left(\frac{t}{b}\right)^{\frac{s}{2}}\geqslant\left[\left(\frac{a}{t}\right)^{\frac{s}{2}}\left(\frac{t}{b}\right)^{\frac{s}{2}}\right]^{\frac{1}{2}}=\left(\frac{a}{b}\right)^{\frac{s}{4}},$$

so we have the first inequality. The proof that the relations from the statement of Corollary 4 are equivalent follows the same lines as the proof of Theorem 1. □

REMARK 14. For $\lambda=1$, the results presented in Theorem 5 and Corollary 4 can be found in the papers [7] and [10], while, in particular, for $\lambda=1$ and $p=2$ we obtain Theorem 2.2 in [13].

Finally, we shall use the idea from Corollary 3 to consider (29) and (30) with $s>0$, $K(x,y)=(x+y)^{-s}$, and general parameters A_1 and A_2. Here we have

$$H(\alpha,a,b;t)=\int_{\frac{a}{t}}^{\frac{b}{t}}(1+u)^{-s}u^{-\alpha}du,\quad t>0,\quad \alpha\in\langle 1-s,1\rangle. \tag{37}$$

Since

$$H'(\alpha,a,b;t)=t^{s+\alpha-2}\left[a^{1-\alpha}(a+t)^{-s}-b^{1-\alpha}(b+t)^{-s}\right],$$

it is not hard to convince oneself that $H(\alpha,a,b;\cdot)$ attains its maximal value on $\mathbb{R}_+$ at the unique point

$$t_\alpha=\frac{a^{\frac{1-\alpha}{s}}b-ab^{\frac{1-\alpha}{s}}}{b^{\frac{1-\alpha}{s}}-a^{\frac{1-\alpha}{s}}}. \tag{38}$$

Hence,

$$H(\alpha,a,b;t)\leqslant H(\alpha,a,b;t_\alpha),\quad t>0, \tag{39}$$

and some related results are given in the following theorem.

THEOREM 6. *Let p, q, and λ be as in (1) and (2), $s>0$, $0\leqslant a<b\leqslant\infty$, and $A_1p',A_2q'\in\langle 1-s,1\rangle$. If H is defined by (37) and t_α by (38), then the inequalities*

$$\int_a^b\int_a^b\frac{f(x)g(y)}{(x+y)^{s\lambda}}dxdy\leqslant H^{\frac{1}{p'}}(A_1p',a,b;t_{A_1p'})H^{\frac{1}{q'}}(A_2q',a,b;t_{A_2q'})\times$$
$$\times\left[\int_a^b x^{(A_1-A_2)p+(1-s)\frac{p}{q'}}f^p(x)dx\right]^{\frac{1}{p}}\cdot\left[\int_a^b y^{(A_2-A_1)q+(1-s)\frac{q}{p'}}g^q(y)dy\right]^{\frac{1}{q}}$$

and

$$\left\{\int_a^b y^{(A_1-A_2)q'+(s-1)\frac{q'}{p'}}\left[\int_a^b\frac{f(x)}{(x+y)^{s\lambda}}dx\right]^{q'}dy\right\}^{\frac{1}{q'}}$$
$$\leqslant H^{\frac{1}{p'}}(A_1p',a,b;t_{A_1p'})H^{\frac{1}{q'}}(A_2q',a,b;t_{A_2q'})\left[\int_a^b x^{(A_1-A_2)p+(1-s)\frac{p}{q'}}f^p(x)dx\right]^{\frac{1}{p}}$$

hold for all non-negative measurable functions f and g on $\langle a,b\rangle$ and are equivalent. Moreover, if $p,q\in\langle 0,1\rangle$, the obtained relations hold with the reversed sign of inequality.

Proof. The first inequality follows from (39), while the rest of the proof is analogous to the proof of Theorem 1 and Theorem 3. □

REMARK 15. For $A_1=\frac{2-s}{2p'}$ and $A_2=\frac{2-s}{2q'}$, the constant on the right-hand sides of the inequalities from Theorem 6 becomes

$$H^\lambda\left(1-\frac{s}{2},a,b;\sqrt{ab}\right)=\int_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}}(1+u)^{-s}u^{\frac{s}{2}-1}du.$$

Such parameters A_1 and A_2 in the case of conjugate parameters give the central result from [11].

5. Some further generalizations

Like in previous sections, here we continue to analyze homogeneous functions and related inequalities of the Hardy-Hilbert type. Starting from Theorem 2 and Theorem 3 and applying suitable transformations of integration domains, we obtain some new interesting results.

First, we consider translations in $\mathbb{R}_+^2$. The following theorem gives a generalization of Theorem 2.

THEOREM 7. *Let p, q, and λ be as in* (1) *and* (2)*, $0 \leqslant a,b,c,d \leqslant \infty$ be such that $a<b$ and $c<d$, and let $\mu > -a$, $\nu > -c$. If K is a non-negative measurable function on $\mathbb{R}_+^2$, fulfilling* (i)*,* (ii)*, and* (iii)*, and the function H is defined by* (18)*, then for all real parameters A_1 and A_2, such that $A_1p', A_2q' \in \langle 1-s, 1\rangle$, and for all non-negative measurable functions f on $\langle a,b\rangle$ and g on $\langle c,d\rangle$, the inequalities*

$$
\begin{aligned}
&\int_a^b\int_c^d K^\lambda(x+\mu,y+\nu)f(x)g(y)\,dx\,dy\\
&\quad\leqslant\left[\int_a^b (x+\mu)^{(A_1-A_2)p+(1-s)\frac{p}{q'}}H^{\frac{p}{q'}}(A_2q',c+\nu,d+\nu;x+\mu)f^p(x)\,dx\right]^{\frac{1}{p}}\times\\
&\quad\times\left[\int_c^d (y+\nu)^{(A_2-A_1)q+(1-s)\frac{q}{p'}}H^{\frac{q}{p'}}\left(2-A_1p'-s,\tfrac{1}{b+\mu},\tfrac{1}{a+\mu};\tfrac{1}{y+\nu}\right)g^q(y)\,dy\right]^{\frac{1}{q}}
\end{aligned}
\tag{40}
$$

and

$$
\begin{aligned}
&\left\{\int_c^d (y+\nu)^{(A_1-A_2)q'+(s-1)\frac{q'}{p'}}H^{-\frac{q'}{p'}}\left(2-A_1p'-s,\tfrac{1}{b+\mu},\tfrac{1}{a+\mu};\tfrac{1}{y+\nu}\right)\times\right.\\
&\qquad\left.\times\left[\int_a^b K^\lambda(x+\mu,y+\nu)f(x)\,dx\right]^{q'}dy\right\}^{\frac{1}{q'}}\\
&\quad\leqslant\left[\int_a^b (x+\mu)^{(A_1-A_2)p+(1-s)\frac{p}{q'}}H^{\frac{p}{q'}}(A_2q',c+\nu,d+\nu;x+\mu)f^p(x)\,dx\right]^{\frac{1}{p}}
\end{aligned}
\tag{41}
$$

hold and are equivalent. Moreover, if p and q are as in (10)*,* (11)*, or $p,q\in\langle 0,1\rangle$, the signs of inequality in* (40) *and* (41) *are reversed.*

Proof. Theorem 7 follows directly from Theorem 2, rewritten with $a+\mu$, $b+\mu$, $c+\nu$, and $d+\nu$, instead of a, b, c, and d, and the functions $\tilde{f}:\langle a+\mu,b+\mu\rangle\to\mathbb{R}$, $\tilde{f}(u)=f(u-\mu)$, and $\tilde{g}:\langle c+\nu,d+\nu\rangle\to\mathbb{R}$, $\tilde{g}(v)=f(v-\nu)$, instead of f and g. Note that

$$H(A_2q',c+\nu,d+\nu;x+\mu)=\int_{\frac{c+\nu}{x+\mu}}^{\frac{d+\nu}{x+\mu}}K(1,u)u^{-A_2q'}\,du$$

and

$$H\left(2-A_1p'-s,\, \tfrac{1}{b+\mu},\, \tfrac{1}{a+\mu};\, \tfrac{1}{y+\nu}\right) = \int_{\frac{y+\nu}{b+\mu}}^{\frac{y+\nu}{a+\mu}} K(1,u)u^{A_1p'+s-2}du,$$

which makes the statement of Theorem 7 more clear. □

If the same procedure is applied to Theorem 3, we get the following result.

THEOREM 8. *Let p, q, and λ be as in* (1) *and* (2), $0 \leqslant a,b,c,d \leqslant \infty$ *be such that $a < b$ and $c < d$, and let $\mu > -a$, $\nu > -c$. If K is a non-negative measurable function on $\mathbb{R}_+^2$, fulfilling* (i), (ii), *and* (iii), *and the function E is defined by* (24), *then the inequalities*

$$\int_a^b \int_c^d K^\lambda(x+\mu, y+\nu) f(x)g(y)\,dx\,dy \leqslant \left\{ \int_a^b (x+\mu)^{(A_1-A_2)p+(1-s)\frac{p}{q'}} \times \right.$$
$$\left. \times \left[k(A_2q') - E_{a+\mu,b+\mu}(A_2q', c+\nu, d+\nu; x+\mu)\right]^{\frac{p}{q'}} f^p(x)\,dx \right\}^{\frac{1}{p}} \times$$
$$\times \left\{ \int_c^d (y+\nu)^{(A_2-A_1)q+(1-s)\frac{q}{p'}} \times \right.$$
$$\left. \times \left[k(2-A_1p'-s) - E_{\frac{1}{d+\nu},\frac{1}{c+\nu}}\left(2-A_1p'-s,\, \tfrac{1}{b+\mu},\, \tfrac{1}{a+\mu};\, \tfrac{1}{y+\nu}\right)\right]^{\frac{q}{p'}} g^q(y)\,dy \right\}^{\frac{1}{q}} \quad (42)$$

and

$$\left\{ \int_c^d (y+\nu)^{(A_1-A_2)q'+(s-1)\frac{q'}{p'}} \left[\int_a^b K^\lambda(x+\mu, y+\nu) f(x)\,dx\right]^{q'} \times \right.$$
$$\left. \times \left[k(2-A_1p'-s) - E_{\frac{1}{d+\nu},\frac{1}{c+\nu}}\left(2-A_1p'-s,\, \tfrac{1}{b+\mu},\, \tfrac{1}{a+\mu};\, \tfrac{1}{y+\nu}\right)\right]^{-\frac{q'}{p'}} dy \right\}^{\frac{1}{q'}}$$
$$\leqslant \left[\int_a^b (x+\mu)^{(A_1-A_2)p+(1-s)\frac{p}{q'}} \times \right.$$
$$\left. \times \left[k(A_2q') - E_{a+\mu,b+\mu}(A_2q', c+\nu, d+\nu; x+\mu)\right]^{\frac{p}{q'}} f^p(x)\,dx \right]^{\frac{1}{p}} \quad (43)$$

hold for all real parameters A_1 and A_2, such that $A_1p', A_2q' \in \langle 1-s, 1\rangle$, and for all non-negative measurable functions f on $\langle a,b\rangle$ and g on $\langle c,d\rangle$. Moreover, these inequalities are equivalent. If $p,q \in \langle 0,1\rangle$, the relations (42) *and* (43) *hold with the reversed sign of inequality.*

Note that in (42) and (43) we have

$$E_{a+\mu,b+\mu}(A_2q', c+\nu, d+\nu; x+\mu) = \left(\frac{a+\mu}{x+\mu}\right)^{1-A_2q'} \int_0^{\frac{c+\nu}{a+\mu}} K(1,u)u^{-A_2q'}du$$
$$+ \left(\frac{x+\mu}{b+\mu}\right)^{s+A_2q'-1} \int_0^{\frac{b+\mu}{d+\nu}} K(u,1)u^{s+A_2q'-2}du$$

and

$$E_{\frac{1}{d+\nu},\frac{1}{c+\nu}}\left(2-A_1p'-s,\,\tfrac{1}{b+\mu},\,\tfrac{1}{a+\mu};\,\tfrac{1}{y+\nu}\right)=\left(\frac{y+\nu}{d+\nu}\right)^{A_1p'+s-1}\int_0^{\frac{d+\nu}{b+\mu}}K(1,u)u^{A_1p'+s-2}du$$
$$+\left(\frac{c+\nu}{y+\nu}\right)^{1-A_1p'}\int_0^{\frac{a+\mu}{c+\nu}}K(u,1)u^{-A_1p'}du.$$

REMARK 16. The case $\lambda=1$, with $a=c$, $b=d$, and $\mu=\nu$, was already discussed in [7, Theorem 6]. Moreover, if additionaly K is symmetric, $p>1$, $\mu\in\left\langle\max\left\{\frac{1-s}{p'},-a\right\},\frac{p}{2}\right\rangle$, and $A_1=A_2=\frac{2\mu}{pp'}$, Theorem 7 reduces to Theorem 1 from [8].

REMARK 17. Note that for $\mu=\nu=0$ Theorem 7 and Theorem 8 respectively become Theorem 2 and Theorem 3.

We continue with another interesting transformation of integration domain, namely, with $(x,y)\mapsto(Ax^{\mu},By^{\nu})$. The corresponding results are given in the sequel.

THEOREM 9. *Let p, q, and λ be as in* (1) *and* (2), *$0\leqslant a,b,c,d\leqslant\infty$ be such that $a<b$ and $c<d$, and let $A,B,\mu,\nu>0$. If K is a non-negative measurable function on $\mathbb{R}^2_+$, fulfilling* (i), (ii), *and* (iii), *and the function H is defined by* (18), *then for all real parameters A_1 and A_2, such that $A_1p',A_2q'\in\langle 1-s,1\rangle$, and for all non-negative measurable functions f on $\langle a,b\rangle$ and g on $\langle c,d\rangle$, the inequalities*

$$\int_a^b\int_c^d K^{\lambda}(Ax^{\mu},By^{\nu})f(x)g(y)\,dxdy$$
$$\leqslant C\left[\int_a^b x^{\left(A_1-A_2+\frac{1-s}{q'}\right)p\mu+(p-1)(1-\mu)}H^{\frac{p}{q'}}(A_2q',Bc^{\nu},Bd^{\nu};Ax^{\mu})f^p(x)\,dx\right]^{\frac{1}{p}}\times$$
$$\times\left[\int_c^d y^{\left(A_2-A_1+\frac{1-s}{p'}\right)q\nu+(q-1)(1-\nu)}H^{\frac{q}{p'}}\left(2-A_1p'-s,\,\tfrac{1}{Ab^{\mu}},\,\tfrac{1}{Aa^{\mu}};\,\tfrac{1}{By^{\nu}}\right)g^q(y)\,dy\right]^{\frac{1}{q}}\quad(44)$$

and

$$\left\{\int_c^d y^{\left(A_1-A_2+\frac{s-1}{p'}\right)q'\nu+\nu-1}\left[\int_a^b K^{\lambda}(Ax^{\mu},By^{\nu})f(x)\,dx\right]^{q'}\times\right.$$
$$\left.\times H^{-\frac{q'}{p'}}\left(2-A_1p'-s,\,\tfrac{1}{Ab^{\mu}},\,\tfrac{1}{Aa^{\mu}};\,\tfrac{1}{By^{\nu}}\right)dy\right\}^{\frac{1}{q'}}$$
$$\leqslant C\left[\int_a^b x^{\left(A_1-A_2+\frac{1-s}{q'}\right)p\mu+(p-1)(1-\mu)}H^{\frac{p}{q'}}(A_2q',Bc^{\nu},Bd^{\nu};Ax^{\mu})f^p(x)\,dx\right]^{\frac{1}{p}},\quad(45)$$

where

$$C=\mu^{-\frac{1}{p'}}\nu^{-\frac{1}{q'}}A^{A_1-A_2+\frac{1-s}{q'}-\frac{1}{p'}}B^{A_2-A_1+\frac{1-s}{p'}-\frac{1}{q'}},\quad(46)$$

hold and are equivalent. Moreover, if p *and* q *are as in* (10), (11), *or* $p,q \in \langle 0,1\rangle$, *then* (44) *and* (45) *hold with the reversed sign of inequality.*

Proof. This time we apply Theorem 2 to Aa^{μ}, Ab^{μ}, Bc^{ν}, Bd^{ν}, instead of a, b, c, d, and to the functions $\tilde{f}: \langle Aa^{\mu}, Ab^{\mu}\rangle \to \mathbb{R}$ and $\tilde{g}: \langle Bc^{\nu}, Bd^{\nu}\rangle \to \mathbb{R}$,

$$\tilde{f}(u) = \left(\frac{u}{A}\right)^{\frac{1-\mu}{\mu}} f\left(\left(\frac{u}{A}\right)^{\frac{1}{\mu}}\right), \quad \tilde{g}(v) = \left(\frac{v}{B}\right)^{\frac{1-\nu}{\nu}} g\left(\left(\frac{v}{B}\right)^{\frac{1}{\nu}}\right),$$

instead of f and g. Moreover, in this setting we have

$$H(A_2q', Bc^{\nu}, Bd^{\nu}; Ax^{\mu}) = \int_{\frac{Bc^{\nu}}{Ax^{\mu}}}^{\frac{Bd^{\nu}}{Ax^{\mu}}} K(1,u)u^{-A_2q'}du$$

and

$$H\left(2-A_1p'-s, \tfrac{1}{Ab^{\mu}}, \tfrac{1}{Aa^{\mu}}; \tfrac{1}{By^{\nu}}\right) = \int_{\frac{By^{\nu}}{Ab^{\mu}}}^{\frac{By^{\nu}}{Aa^{\mu}}} K(1,u)u^{A_1p'+s-2}du. \quad \square$$

In particular, for $a = c = 0$ and $b = d = \infty$ we get the following general Hilbert-type inequalities.

COROLLARY 5. *If* p, q, *and* λ *are as in* (1) *and* (2), $A,B,\mu,\nu > 0$, *and* K *is a non-negative measurable function on* $\mathbb{R}_+^2$, *fulfilling* (i), (ii), *and* (iii), *then the inequalities*

$$\begin{aligned}\int_0^{\infty}\int_0^{\infty} & K^{\lambda}(Ax^{\mu}, By^{\nu})f(x)g(y)\,dx\,dy \\ & \leqslant D\left[\int_0^{\infty} x^{\left(A_1-A_2+\frac{1-s}{q'}\right)p\mu+(p-1)(1-\mu)} f^p(x)\,dx\right]^{\frac{1}{p}} \times \\ & \quad \times \left[\int_0^{\infty} y^{\left(A_2-A_1+\frac{1-s}{p'}\right)q\nu+(q-1)(1-\nu)} g^q(y)\,dy\right]^{\frac{1}{q}} \end{aligned} \tag{47}$$

and

$$\begin{aligned}\Bigg\{\int_0^{\infty} y^{\left(A_1-A_2+\frac{s-1}{p'}\right)q'\nu+\nu-1} & \left[\int_0^{\infty} K^{\lambda}(Ax^{\mu}, By^{\nu})f(x)\,dx\right]^{q'} dy\Bigg\}^{\frac{1}{q'}} \\ & \leqslant D\left[\int_0^{\infty} x^{\left(A_1-A_2+\frac{1-s}{q'}\right)p\mu+(p-1)(1-\mu)} f^p(x)\,dx\right]^{\frac{1}{p}}, \end{aligned} \tag{48}$$

where $D = Ck^{\frac{1}{p'}}(2-A_1p'-s)k^{\frac{1}{q'}}(A_2q')$ *and* C *is defined by* (46), *hold for all real parameters* A_1 *and* A_2, *such that* $A_1p', A_2q' \in \langle 1-s, 1\rangle$, *and for all non-negative measurable functions* f *and* g *on* $\mathbb{R}_+$. *Moreover, these inequalities are equivalent. If* p *and* q *are as in* (10), (11), *or* $p,q \in \langle 0,1\rangle$, *the inequalities* (47) *and* (48) *are reversed.*

Finally, we state a generalization of Theorem 3 related to the mentioned transformation. If E is as in (24), observe that

$$E_{Aa^\mu,Ab^\mu}(A_2q',Bc^\nu,Bd^\nu;Ax^\mu)=\left(\frac{a}{x}\right)^{\mu(1-A_2q')}\int_0^{\frac{Bc^\nu}{Aa^\mu}}K(1,u)u^{-A_2q'}du + \left(\frac{x}{b}\right)^{\mu(s+A_2q'-1)}\int_0^{\frac{Ab^\mu}{Bd^\nu}}K(u,1)u^{s+A_2q'-2}du$$

and

$$E_{\frac{1}{Bd^\nu},\frac{1}{Bc^\nu}}\left(2-A_1p'-s,\frac{1}{Ab^\mu},\frac{1}{Aa^\mu};\frac{1}{By^\nu}\right)=\left(\frac{y}{d}\right)^{\nu(A_1p'+s-1)}\int_0^{\frac{Bd^\nu}{Ab^\mu}}K(1,u)u^{A_1p'+s-2}du + \left(\frac{c}{y}\right)^{\nu(1-A_1p')}\int_0^{\frac{Aa^\mu}{Bc^\nu}}K(u,1)u^{-A_1p'}du.$$

THEOREM 10. *Suppose p, q, and λ are as in* (1) *and* (2)*, $A,B,\mu,\nu>0$, and $0\leqslant a,b,c,d\leqslant\infty$ are such that $a<b$ and $c<d$. If K is a non-negative measurable function on $\mathbb{R}_+^2$, fulfilling* (i)*,* (ii)*, and* (iii)*, and the function E is defined by* (24)*, then for all real parameters A_1 and A_2, such that $A_1p',A_2q'\in\langle 1-s,1\rangle$, and for all non-negative measurable functions f on $\langle a,b\rangle$ and g on $\langle c,d\rangle$, the inequalities*

$$\int_a^b\int_c^d K^\lambda(Ax^\mu,By^\nu)f(x)g(y)\,dx\,dy\leqslant C\left\{\int_a^b x^{\left(A_1-A_2+\frac{1-s}{q'}\right)p\mu+(p-1)(1-\mu)}\times\left[k(A_2q')-E_{Aa^\mu,Ab^\mu}(A_2q',Bc^\nu,Bd^\nu;Ax^\mu)\right]^{\frac{p}{q'}}f^p(x)\,dx\right\}^{\frac{1}{p}}\times$$
$$\times\left\{\int_c^d y^{\left(A_2-A_1+\frac{1-s}{p'}\right)q\nu+(q-1)(1-\nu)}\times\left[k(2-A_1p'-s)-E_{\frac{1}{Bd^\nu},\frac{1}{Bc^\nu}}\left(2-A_1p'-s,\frac{1}{Ab^\mu},\frac{1}{Aa^\mu};\frac{1}{By^\nu}\right)\right]^{\frac{q}{p'}}g^q(y)\,dy\right\}^{\frac{1}{q}}$$

and

$$\left\{\int_c^d y^{\left(A_1-A_2+\frac{s-1}{p'}\right)q'\nu+\nu-1}\left[\int_a^b K^\lambda(Ax^\mu,By^\nu)f(x)\,dx\right]^{q'}\times\left[k(2-A_1p'-s)-E_{\frac{1}{Bd^\nu},\frac{1}{Bc^\nu}}\left(2-A_1p'-s,\frac{1}{Ab^\mu},\frac{1}{Aa^\mu};\frac{1}{By^\nu}\right)\right]^{-\frac{q'}{p'}}dy\right\}^{\frac{1}{q'}}$$
$$\leqslant C\left\{\int_a^b x^{\left(A_1-A_2+\frac{1-s}{q'}\right)p\mu+(p-1)(1-\mu)}\times\left[k(A_2q')-E_{Aa^\mu,Ab^\mu}(A_2q',Bc^\nu,Bd^\nu;Ax^\mu)\right]^{\frac{p}{q'}}f^p(x)\,dx\right\}^{\frac{1}{p}},$$

where the constant C *is defined by* (46)*, hold and are equivalent. Moreover, for* $p,q \in \langle 0,1\rangle$ *, the inequality sign in these relations is reversed.*

REMARK 18. For $\lambda = 1$, $a = c$, and $b = d$, the results from Theorem 10 and Corollary 5 reduce to Theorem 7 and Theorem 8 from [7]. On the other hand, for $A = B = \mu = \nu = 1$, Theorem 9, Corollary 5, and Theorem 10, respectively become Theorem 2, Corollary 2, and Theorem 3 from our Section 2.

To conclude this section, we emphasize that an analysis from Remark 11 can be applied to both presented transformations.

6. Discrete Hardy-Hilbert-type inequalities with non-conjugate exponents

General results from Section 2, rewritten with the counting measure on $\mathbb{N}$, lead to some interesting inequalities of the Hardy-Hilbert type related to sequences of non-negative real numbers. As in previous sections, we consider non-negative functions K on $\mathbb{R}_+^2$, homogeneous of degree $-s$, where $s > 0$, and strictly decreasing in each argument. Our first result is a discrete analogue of Theorem 9.

THEOREM 11. *Let* p*,* q*, and* λ *be as in* (1) *and* (2)*,* $A, B, \mu, \nu > 0$*, and let* $m, M, n, N \in \mathbb{N}$ *be such that* $m < M$ *and* $n < N$*. If* K *is a non-negative measurable function on* $\mathbb{R}_+^2$*, fulfilling the conditions* (i)*,* (ii)*, and* (iii)*, and the function* H *is defined by* (18)*, then the inequalities*

$$\sum_{i=m}^{M}\sum_{j=n}^{N} K^{\lambda}(Ai^{\mu}, Bj^{\nu})\, a_i b_j$$
$$\leqslant C\left[\sum_{i=m}^{M} i^{\left(A_1-A_2+\frac{1-s}{q'}\right)p\mu+(p-1)(1-\mu)} H^{\frac{p}{q'}}\left(A_2q', B(n-1)^{\nu}, BN^{\nu}; Ai^{\mu}\right) a_i^p\right]^{\frac{1}{p}} \times$$
$$\times\left[\sum_{j=n}^{N} j^{\left(A_2-A_1+\frac{1-s}{p'}\right)q\nu+(q-1)(1-\nu)} H^{\frac{q}{p'}}\left(2-A_1p'-s, \tfrac{1}{AM^{\mu}}, \tfrac{1}{A(m-1)^{\mu}}; \tfrac{1}{Bj^{\nu}}\right) b_j^q\right]^{\frac{1}{q}} \quad (49)$$

and

$$\left\{\sum_{j=n}^{N} j^{\left(A_1-A_2+\frac{s-1}{p'}\right)q'\nu+\nu-1}\left[\sum_{i=m}^{M} K^{\lambda}(Ai^{\mu}, Bj^{\nu})\, a_i\right]^{q'} \times\right.$$
$$\left.\times\, H^{-\frac{q'}{p'}}\left(2-A_1p'-s, \tfrac{1}{AM^{\mu}}, \tfrac{1}{A(m-1)^{\mu}}; \tfrac{1}{Bj^{\nu}}\right)\right\}^{\frac{1}{q'}}$$
$$\leqslant C\left[\sum_{i=m}^{M} i^{\left(A_1-A_2+\frac{1-s}{q'}\right)p\mu+(p-1)(1-\mu)} H^{\frac{p}{q'}}\left(A_2q', B(n-1)^{\nu}, BN^{\nu}; Ai^{\mu}\right) a_i^p\right]^{\frac{1}{p}}, \quad (50)$$

where the constant C *is defined by* (46)*, hold for all real parameters* A_1 *and* A_2*, such that* $A_1p' \in \left\langle \max\left\{1-s, 1-\frac{1}{\mu}\right\}, 1\right\rangle$ *and* $A_2q' \in \left\langle \max\left\{1-s, 1-\frac{1}{\nu}\right\}, 1\right\rangle$*, and for* *all sequences* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *of non-negative real numbers. Moreover, these inequalities are equivalent. If* $p,q\in\langle 0,1\rangle$ *, the signs of inequality in* (49) *and* (50) *are reversed.*

Proof. Rewrite Theorem 1 for the counting measure on $\mathbb{N}$, $K_{i,j}=K(Ai^{\mu},Bj^{\nu})$, $\varphi_i=(Ai^{\mu})^{A_1+\frac{1-\mu}{p'\mu}}$, $\psi_j=(Bj^{\nu})^{A_2+\frac{1-\nu}{q'\nu}}$, and the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$. In this setting, (7) becomes

$$\sum_{i=m}^{M}\sum_{j=n}^{N}K^{\lambda}(Ai^{\mu},Bj^{\nu})a_ib_j \leqslant \left[\sum_{i=m}^{M}(Ai^{\mu})^{A_1p+(p-1)\frac{1-\mu}{\mu}}F_i^pa_i^p\right]^{\frac{1}{p}}\cdot\left[\sum_{j=n}^{N}(Bj^{\nu})^{A_2q+(q-1)\frac{1-\nu}{\nu}}G_j^qb_j^q\right]^{\frac{1}{q}}, \qquad (51)$$

where

$$F_i=\left[\sum_{j=n}^{N}K^{\lambda}(Ai^{\mu},Bj^{\nu})(Bj^{\nu})^{1-\frac{1}{\nu}-A_2q'}\right]^{\frac{1}{q'}} \qquad (52)$$

and

$$G_j=\left[\sum_{i=m}^{M}K^{\lambda}(Ai^{\mu},Bj^{\nu})(Ai^{\mu})^{1-\frac{1}{\mu}-A_1p'}\right]^{\frac{1}{p'}}. \qquad (53)$$

Since $1-\frac{1}{\nu}-A_2q'<0$ and K fulfils (i) and (ii), we have

$$\begin{aligned}F_i^{q'} &= (Ai^{\mu})^{-s}\sum_{j=n}^{N}K^{\lambda}\left(1,\frac{Bj^{\nu}}{Ai^{\mu}}\right)(Bj^{\nu})^{1-\frac{1}{\nu}-A_2q'}\\ &\leqslant (Ai^{\mu})^{-s}\int_{n-1}^{N}K^{\lambda}\left(1,\frac{By^{\nu}}{Ai^{\mu}}\right)(By^{\nu})^{1-\frac{1}{\nu}-A_2q'}dy\\ &= \nu^{-1}B^{-\frac{1}{\nu}}(Ai^{\mu})^{1-s-A_2q'}\int_{\frac{B(n-1)^{\nu}}{Ai^{\mu}}}^{\frac{BN^{\nu}}{Ai^{\mu}}}K(1,u)u^{-A_2q'}du\\ &= \nu^{-1}B^{-\frac{1}{\nu}}(Ai^{\mu})^{1-s-A_2q'}H\left(A_2q',B(n-1)^{\nu},BN^{\nu};Ai^{\mu}\right) \qquad (54)\end{aligned}$$

and, by similar arguments,

$$\begin{aligned}G_j^{p'} &= (Bj^{\nu})^{-s}\sum_{i=m}^{M}K^{\lambda}\left(\frac{Ai^{\mu}}{Bj^{\nu}},1\right)(Ai^{\mu})^{1-\frac{1}{\mu}-A_1p'}\\ &\leqslant (Bj^{\nu})^{-s}\int_{m-1}^{M}K^{\lambda}\left(\frac{Ax^{\mu}}{Bj^{\nu}},1\right)(Ax^{\mu})^{1-\frac{1}{\mu}-A_1p'}dy\\ &= \mu^{-1}A^{-\frac{1}{\mu}}(Bj^{\nu})^{1-s-A_1p'}H\Big(2-A_1p'-s,\tfrac{1}{AM^{\mu}},\tfrac{1}{A(m-1)^{\mu}};\tfrac{1}{Bj^{\nu}}\Big). \qquad (55)\end{aligned}$$

Hence, (49) holds by combining (51), (52), (53), (54), and (55). The proof that the relations (49) and (50) are equivalent follows the idea presented in the proofs of

Theorem 1 and Theorem 3. In particular, to prove that (49) implies (50), we use the sequence $(b_n)_{n\in\mathbb{N}}$ given by

$$b_j = H^{-\frac{q'}{p'}}\left(2-A_1p'-s, \frac{1}{AM^{\mu}}, \frac{1}{A(m-1)^{\mu}}; \frac{1}{Bj^{\nu}}\right) j^{\left(A_1-A_2+\frac{s-1}{p'}\right)q'\nu+\nu-1}\left[\sum_{i=m}^{M} K^{\lambda}(Ai^{\mu}, Bj^{\nu})\, a_i\right]^{\frac{q'}{q}}.$$

A discussion concerning reverse inequalities is as in the proof of Theorem 3. □

An important consequence of Theorem 11 is the following corollary for infinite series.

COROLLARY 6. *Suppose p, q, and λ are as in* (1) *and* (2) *and $A,B,\mu,\nu > 0$. If K is a non-negative measurable function on $\mathbb{R}^2_+$, fulfilling the conditions* (i), (ii), *and* (iii)*, then the inequalities*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} K^{\lambda}(Ai^{\mu}, Bj^{\nu})\, a_i b_j \leqslant D\left[\sum_{i=1}^{\infty} i^{\left(A_1-A_2+\frac{1-s}{q'}\right)p\mu+(p-1)(1-\mu)} a_i^p\right]^{\frac{1}{p}} \times$$
$$\times\left[\sum_{j=1}^{\infty} j^{\left(A_2-A_1+\frac{1-s}{p'}\right)q\nu+(q-1)(1-\nu)} b_j^q\right]^{\frac{1}{q}}$$

and

$$\left\{\sum_{j=1}^{\infty} j^{\left(A_1-A_2+\frac{s-1}{p'}\right)q'\nu+\nu-1}\left[\sum_{i=1}^{\infty} K^{\lambda}(Ai^{\mu}, Bj^{\nu})\, a_i\right]^{q'}\right\}^{\frac{1}{q'}}$$
$$\leqslant D\left[\sum_{i=1}^{\infty} i^{\left(A_1-A_2+\frac{1-s}{q'}\right)p\mu+(p-1)(1-\mu)} a_i^p\right]^{\frac{1}{p}},$$

where $D = Ck^{\frac{1}{p'}}(2-A_1p'-s)k^{\frac{1}{q'}}(A_2q')$ and the constant C is defined by (46)*, hold for all real parameters A_1 and A_2, such that $A_1p' \in \left\langle \max\left\{1-s, 1-\frac{1}{\mu}\right\}, 1\right\rangle$ and $A_2q' \in \left\langle \max\left\{1-s, 1-\frac{1}{\nu}\right\}, 1\right\rangle$, and for all sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ of non-negative real numbers. Moreover, these inequalities are equivalent. If $p,q \in \langle 0,1\rangle$, the signs of inequality in both relations are reversed.*

Finally, applying the estimate from Lemma 1 in (54) and (55), we obtain the following result.

THEOREM 12. *Let p, q, and λ be as in* (1) *and* (2)*, $A,B,\mu,\nu > 0$, and let $m,M,n,N \in \mathbb{N}$, $m < M$, $n < N$. If K is a non-negative measurable function on $\mathbb{R}^2_+$, fulfilling the conditions* (i), (ii), *and* (iii)*, and the function E is defined by* (24)*, then* *the inequalities*

$$\sum_{i=m}^{M}\sum_{j=n}^{N} K^{\lambda}(Ai^{\mu},Bj^{\nu})a_i b_j \leqslant C\left\{\sum_{i=m}^{M} i^{\left(A_1-A_2+\frac{1-s}{q'}\right)p\mu+(p-1)(1-\mu)}\times\right.$$
$$\left.\times\left[k(A_2q')-E_{Am^{\mu},AM^{\mu}}\left(A_2q',B(n-1)^{\nu},BN^{\nu};Ai^{\mu}\right)\right]^{\frac{p}{q'}} a_i^p\right\}^{\frac{1}{p}}\times$$
$$\times\left\{\sum_{j=n}^{N} j^{\left(A_2-A_1+\frac{1-s}{p'}\right)q\nu+(q-1)(1-\nu)}\times\right.$$
$$\left.\times\left[k(2-A_1p'-s)-E_{\frac{1}{BN^{\nu}},\frac{1}{Bn^{\nu}}}\left(2-A_1p'-s,\frac{1}{AM^{\mu}},\frac{1}{A(m-1)^{\mu}};\frac{1}{Bj^{\nu}}\right)\right]^{\frac{q}{p'}} b_j^q\right\}^{\frac{1}{q}}$$

and

$$\left\{\sum_{j=n}^{N} j^{\left(A_1-A_2+\frac{s-1}{p'}\right)q'\nu+\nu-1}\left[\sum_{i=m}^{M}K^{\lambda}(Ai^{\mu},Bj^{\nu})a_i\right]^{q'}\times\right.$$
$$\left.\times\left[k(2-A_1p'-s)-E_{\frac{1}{BN^{\nu}},\frac{1}{Bn^{\nu}}}\left(2-A_1p'-s,\frac{1}{AM^{\mu}},\frac{1}{A(m-1)^{\mu}};\frac{1}{Bj^{\nu}}\right)\right]^{-\frac{q'}{p'}}\right\}^{\frac{1}{q'}}$$
$$\leqslant C\left\{\sum_{i=m}^{M} i^{\left(A_1-A_2+\frac{1-s}{q'}\right)p\mu+(p-1)(1-\mu)}\times\right.$$
$$\left.\times\left[k(A_2q')-E_{Am^{\mu},AM^{\mu}}\left(A_2q',B(n-1)^{\nu},BN^{\nu};Ai^{\mu}\right)\right]^{\frac{p}{q'}} a_i^p\right\}^{\frac{1}{p}},$$

where the constant C is defined by (46)*, hold for all real parameters A_1 and A_2, such that $A_1p'\in\left\langle\max\left\{1-s,1-\frac{1}{\mu}\right\},1\right\rangle$ and $A_2q'\in\left\langle\max\left\{1-s,1-\frac{1}{\nu}\right\},1\right\rangle$, and for all sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ of non-negative real numbers. Moreover, these inequalities are equivalent. If $p,q\in\langle 0,1\rangle$, both relations hold with the reversed sign of inequality.*

REMARK 19. For $A=B=\mu=\nu=1$ we obtain discrete analogues of Theorem 2, Corollary 2, and Theorem 3.

REMARK 20. The results for the case $\lambda=1$ are given in [7, Theorem 9].

7. Godunova-type inequalities

In the previous sections, we considered only integrals and sums over some subsets of $\mathbb{R}_+$, that is, one-dimensional situations. Since Theorem 1 covers more general settings, to conclude this paper, we apply that result to n-dimensional cells in $\mathbb{R}_+^n$.

Before presenting our idea, it is necessary to introduce some notation. For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, we define

$$\frac{\mathbf{u}}{\mathbf{v}} = \left(\frac{u_1}{v_1}, \frac{u_2}{v_2}, \ldots, \frac{u_n}{v_n}\right) \quad \text{and} \quad \mathbf{u}^{\mathbf{v}} = u_1^{v_1} u_2^{v_2} \cdot \ldots \cdot u_n^{v_n}.$$

Especially, $\mathbf{u}^{\mathbf{1}} = \prod_{i=1}^n u_i$ and $\mathbf{u}^{-\mathbf{1}} = (\prod_{i=1}^n u_i)^{-1}$, where $\mathbf{1} = (1, 1, \ldots, 1)$. We shall also write $\mathbf{u} < \mathbf{v}$ if $u_i < v_i$, $i = 1, \ldots, n$. For such $\mathbf{u}$ and $\mathbf{v}$, by $C_{\mathbf{u},\mathbf{v}}$ we denote the following n-dimensional cell in $\mathbb{R}^n$:

$$C_{\mathbf{u},\mathbf{v}} = \langle u_1, v_1 \rangle \times \cdots \times \langle u_n, v_n \rangle = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \in \langle u_i, v_i \rangle, i = 1, \ldots, n\}.$$

Now, we are ready to state and prove an n-dimensional analogue of Theorem 2.

THEOREM 13. *Let p, q, and λ be as in* (1) *and* (2)*, and let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{s} \in \mathbb{R}^n_+$ be such that $\mathbf{a} < \mathbf{b}$ and $\mathbf{c} < \mathbf{d}$. Suppose K is a non-negative measurable function on $\mathbb{R}^n_+$, parameters $\mathbf{A_1}, \mathbf{A_2} \in \mathbb{R}^n$ are such that*

$$k(2\mathbf{1} - p'\mathbf{A_1} - \mathbf{s}) = \int_{\mathbb{R}^n_+} K(\mathbf{u}) \mathbf{u}^{p'\mathbf{A_1} + \mathbf{s} - 2\mathbf{1}} d\mathbf{u} < \infty \tag{56}$$

and

$$k(q'\mathbf{A_2}) = \int_{\mathbb{R}^n_+} K(\mathbf{u}) \mathbf{u}^{-q'\mathbf{A_2}} d\mathbf{u} < \infty, \tag{57}$$

and the function H is, for $\mathbf{e}, \mathbf{f} \in \mathbb{R}^n_+$, $\mathbf{e} < \mathbf{f}$, $\boldsymbol{\alpha} \in \{2\mathbf{1} - p'\mathbf{A_1} - \mathbf{s}, q'\mathbf{A_2}\}$, and $\mathbf{t} \in \mathbb{R}^n_+$ defined by

$$H(\boldsymbol{\alpha}, \mathbf{e}, \mathbf{f}; \mathbf{t}) = \int_{C_{\frac{\mathbf{e}}{\mathbf{t}}, \frac{\mathbf{f}}{\mathbf{t}}}} K(\mathbf{u}) u^{-\boldsymbol{\alpha}} d\mathbf{u}.$$

Then for all non-negative measurable functions f on $C_{\mathbf{a},\mathbf{b}}$ and g on $C_{\mathbf{c},\mathbf{d}}$, the inequalities

$$\begin{aligned} &\int_{C_{\mathbf{a},\mathbf{b}}} \int_{C_{\mathbf{c},\mathbf{d}}} \mathbf{x}^{-\lambda\mathbf{s}} K^{\lambda}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{x}) g(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \\ &\quad \leqslant \left[\int_{C_{\mathbf{a},\mathbf{b}}} \mathbf{x}^{p(\mathbf{A_1} - \mathbf{A_2}) + \frac{p}{q'}(\mathbf{1} - \mathbf{s})} H^{\frac{p}{q'}}(q'\mathbf{A_2}, \mathbf{c}, \mathbf{d}; \mathbf{x}) f^p(\mathbf{x})\, d\mathbf{x}\right]^{\frac{1}{p}} \times \\ &\quad \times \left[\int_{C_{\mathbf{c},\mathbf{d}}} \mathbf{y}^{q(\mathbf{A_2} - \mathbf{A_1}) + \frac{q}{p'}(\mathbf{1} - \mathbf{s})} H^{\frac{q}{p'}}\left(2\mathbf{1} - p'\mathbf{A_1} - \mathbf{s}, \frac{\mathbf{1}}{\mathbf{b}}, \frac{\mathbf{1}}{\mathbf{a}}; \frac{\mathbf{1}}{\mathbf{y}}\right) g^q(\mathbf{y})\, d\mathbf{y}\right]^{\frac{1}{q}} \end{aligned}$$

and

$$\left\{\int_{C_{\mathbf{c},\mathbf{d}}} \mathbf{y}^{q'(\mathbf{A_1}-\mathbf{A_2})+\frac{q'}{p'}(\mathbf{s}-\mathbf{1})} H^{-\frac{q'}{p'}}\left(2\mathbf{1}-p'\mathbf{A_1}-\mathbf{s}, \tfrac{\mathbf{1}}{\mathbf{b}}, \tfrac{\mathbf{1}}{\mathbf{a}}; \tfrac{\mathbf{1}}{\mathbf{y}}\right) \times \right.$$
$$\left. \times \left[\int_{C_{\mathbf{a},\mathbf{b}}} \mathbf{x}^{-\lambda\mathbf{s}} K^{\lambda}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{x})\, d\mathbf{x}\right]^{q'} d\mathbf{y}\right\}^{\frac{1}{q'}}$$
$$\leqslant \left[\int_{C_{\mathbf{a},\mathbf{b}}} \mathbf{x}^{p(\mathbf{A_1}-\mathbf{A_2})+\frac{p}{q'}(\mathbf{1}-\mathbf{s})} H^{\frac{p}{q'}}(q'\mathbf{A_2}, \mathbf{c}, \mathbf{d}; \mathbf{x}) f^{p}(\mathbf{x})\, d\mathbf{x}\right]^{\frac{1}{p}}$$

hold and are equivalent. Moreover, if p and q are as in (10), (11), *or $p,q \in \langle 0,1\rangle$, the signs of inequality in both relations are reversed.*

Proof. We apply Theorem 1 to $X = C_{\mathbf{a},\mathbf{b}}$, $Y = C_{\mathbf{c},\mathbf{d}}$, the functions $\varphi(\mathbf{x}) = \mathbf{x}^{\mathbf{A_1}}$ and $\psi(\mathbf{y}) = \mathbf{y}^{\mathbf{A_2}}$ on $\mathbb{R}^n_+$, the non-negative function $\tilde{K} : \mathbb{R}^{2n}_+ \to \mathbb{R}$, $\tilde{K}(\mathbf{x},\mathbf{y}) = \mathbf{x}^{-\mathbf{s}} K\left(\frac{\mathbf{y}}{\mathbf{x}}\right)$, and Lebesgue measures $d\mu_1(\mathbf{x}) = d\mathbf{x}$, $d\mu_2(\mathbf{y}) = d\mathbf{y}$. By using the substitution $\mathbf{u} = \frac{\mathbf{y}}{\mathbf{x}}$, relations (5) and (6) become

$$\begin{aligned} F(\mathbf{x}) &= \left[\int_{C_{\mathbf{c},\mathbf{d}}} \mathbf{x}^{-\mathbf{s}} K\left(\frac{\mathbf{y}}{\mathbf{x}}\right) \mathbf{y}^{-q'\mathbf{A_2}}\, d\mathbf{y}\right]^{\frac{1}{q'}} = \mathbf{x}^{-\frac{1}{q'}\mathbf{s}} \left[\int_{C_{\mathbf{c},\mathbf{d}}} K\left(\frac{\mathbf{y}}{\mathbf{x}}\right) \mathbf{y}^{-q'\mathbf{A_2}}\, d\mathbf{y}\right]^{\frac{1}{q'}} \\ &= \mathbf{x}^{-\frac{1}{q'}(\mathbf{1}-\mathbf{s})-\mathbf{A_2}} H^{\frac{1}{q'}}(q'\mathbf{A_2}, \mathbf{c}, \mathbf{d}; \mathbf{x}), \quad \mathbf{x} \in C_{\mathbf{a},\mathbf{b}}, \end{aligned}$$

and

$$\begin{aligned} G(\mathbf{y}) &= \left[\int_{C_{\mathbf{a},\mathbf{b}}} K\left(\frac{\mathbf{y}}{\mathbf{x}}\right) \mathbf{x}^{-p'\mathbf{A_1}-\mathbf{s}}\, d\mathbf{x}\right]^{\frac{1}{p'}} \\ &= \mathbf{y}^{-\frac{1}{p'}(\mathbf{1}-\mathbf{s})-\mathbf{A_1}} H^{\frac{1}{p'}}\left(2\mathbf{1}-p'\mathbf{A_1}-\mathbf{s}, \tfrac{\mathbf{1}}{\mathbf{b}}, \tfrac{\mathbf{1}}{\mathbf{a}}; \tfrac{\mathbf{1}}{\mathbf{y}}\right), \quad \mathbf{y} \in C_{\mathbf{c},\mathbf{d}}, \end{aligned}$$

so the inequalities from the statement of Theorem 13 follow directly from (7) and (8). The reverse inequalities hold due to Remark 1. □

As a direct consequence of Theorem 13, we obtain the following Hardy-Hilbert-type inequalities.

COROLLARY 7. *Suppose p, q, and λ are as in* (1) *and* (2). *If $\mathbf{s} \in \mathbb{R}^n_+$, K is a non-negative measurable function on $\mathbb{R}^n_+$, parameters $\mathbf{A_1}, \mathbf{A_2} \in \mathbb{R}^n$ fulfil* (56) *and* (57), *and $C = k^{\frac{1}{p'}}(2\mathbf{1}-p'\mathbf{A_1}-\mathbf{s}) k^{\frac{1}{q'}}(q'\mathbf{A_2})$, then the inequalities*

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \mathbf{x}^{-\lambda\mathbf{s}} K^{\lambda}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{x}) g(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}$$
$$\leqslant C\left[\int_{\mathbb{R}^n_+} \mathbf{x}^{p(\mathbf{A_1}-\mathbf{A_2})+\frac{p}{q'}(\mathbf{1}-\mathbf{s})} f^{p}(\mathbf{x})\, d\mathbf{x}\right]^{\frac{1}{p}} \cdot \left[\int_{\mathbb{R}^n_+} \mathbf{y}^{q(\mathbf{A_2}-\mathbf{A_1})+\frac{q}{p'}(\mathbf{1}-\mathbf{s})} g^{q}(\mathbf{y})\, d\mathbf{y}\right]^{\frac{1}{q}}$$

and

$$\left\{\int_{\mathbb{R}_+^n} \mathbf{y}^{q'(\mathbf{A_1}-\mathbf{A_2})+\frac{q'}{p'}(\mathbf{s}-\mathbf{1})} \left[\int_{\mathbb{R}_+^n} \mathbf{x}^{-\lambda \mathbf{s}} K^{\lambda}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{x})\, d\mathbf{x}\right]^{q'} d\mathbf{y}\right\}^{\frac{1}{q'}} \leqslant C\left[\int_{\mathbb{R}_+^n} \mathbf{x}^{p(\mathbf{A_1}-\mathbf{A_2})+\frac{p}{q'}(\mathbf{1}-\mathbf{s})} f^p(\mathbf{x})\, d\mathbf{x}\right]^{\frac{1}{p}}$$

hold for all non-negative measurable functions f *and* g *on* $\mathbb{R}_+^n$ *and are equivalent. If* p *and* q *are as in* (10)*,* (11)*, or* $p, q \in \langle 0, 1\rangle$ *, then both relations hold with the reversed sign of inequality.*

REMARK 21. For $n = 1$ we have $\tilde{K}(tx, ty) = t^{-s}\tilde{K}(x, y)$, for all $t, x, y \in \mathbb{R}_+$, so the function $\tilde{K}$ is homogeneous of degree $-s$. Thus, Theorem 13 and Corollary 7 may be regarded as n-dimensional generalizations of Theorem 2 and Corollary 2.

Finally, we explicitly state two particular cases of Corollary 7, obtained for some special choices of parameters. The first one considers $\mathbf{s} = s\mathbf{1}$, $\mathbf{A_1} = A_1\mathbf{1}$, and $\mathbf{A_2} = A_2\mathbf{1}$, where s, A_1, A_2 are real numbers.

COROLLARY 8. *Let* p*,* q*, and* λ *be as in* (1) *and* (2)*. If* $s > 0$*,* K *is a non-negative measurable function on* $\mathbb{R}_+^n$*, real parameters* A_1, A_2 *are such that* $\mathbf{A_1} = A_1\mathbf{1}$ *and* $\mathbf{A_2} = A_2\mathbf{1}$ *fulfil* (56) *and* (57)*, and the constant* C *is defined by*

$$C = k^{\frac{1}{p'}}((2 - A_1p' - s)\mathbf{1})k^{\frac{1}{q'}}(q'A_2\mathbf{1}),$$

then the inequalities

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \mathbf{x}^{-\lambda s\mathbf{1}} K^{\lambda}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{x}) g(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \leqslant C\left[\int_{\mathbb{R}_+^n} \mathbf{x}^{\left[p(A_1-A_2)+\frac{p}{q'}(1-s)\right]\mathbf{1}} f^p(\mathbf{x})\, d\mathbf{x}\right]^{\frac{1}{p}} \cdot \left[\int_{\mathbb{R}_+^n} \mathbf{y}^{\left[q(A_2-A_1)+\frac{q}{p'}(1-s)\right]\mathbf{1}} g^q(\mathbf{y})\, d\mathbf{y}\right]^{\frac{1}{q}}$$

and

$$\left\{\int_{\mathbb{R}_+^n} \mathbf{y}^{\left[q'(A_1-A_2)+\frac{q'}{p'}(s-1)\right]\mathbf{1}} \left[\int_{\mathbb{R}_+^n} \mathbf{x}^{-\lambda s\mathbf{1}} K^{\lambda}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{x})\, d\mathbf{x}\right]^{q'} d\mathbf{y}\right\}^{\frac{1}{q'}} \leqslant C\left[\int_{\mathbb{R}_+^n} \mathbf{x}^{\left[p(A_1-A_2)+\frac{p}{q'}(1-s)\right]\mathbf{1}} f^p(\mathbf{x})\, d\mathbf{x}\right]^{\frac{1}{p}}$$

hold for all non-negative measurable functions f *and* g *on* $\mathbb{R}_+^n$ *and are equivalent. If* p *and* q *are as in* (10)*,* (11)*, or* $p, q \in \langle 0, 1\rangle$ *, then both relations hold with the reversed sign of inequality.*

The second special case of Corollary 7, and also the concluding result in this paper, presents an inequality of E. K. Godunova from [4].

COROLLARY 9. *Let p, q, and λ be as in* (1) *and* (2). *If $s>0$ and L is a non-negative measurable function on $\mathbb{R}_+^n$, then the inequalities*

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n}\mathbf{x}^{-\left(\frac{1}{p'}+\lambda\right)\mathbf{1}}\mathbf{y}^{\frac{1}{p'}\mathbf{1}}L\left(\frac{\mathbf{y}}{\mathbf{x}}\right)f(\mathbf{x})g(\mathbf{y})\,d\mathbf{x}d\mathbf{y}\leqslant\|L\|_{L^{\frac{1}{\lambda}}(\mathbb{R}_+^n)}\|f\|_{L^{p'}(\mathbb{R}_+^n)}\|g\|_{L^{q'}(\mathbb{R}_+^n)}$$

and

$$\left\{\int_{\mathbb{R}_+^n}\mathbf{y}^{\frac{q'}{p'}\mathbf{1}}\left[\int_{\mathbb{R}_+^n}\mathbf{x}^{-\left(\frac{1}{p'}+\lambda\right)\mathbf{1}}L\left(\frac{\mathbf{y}}{\mathbf{x}}\right)f(\mathbf{x})\,d\mathbf{x}\right]^{q'}d\mathbf{y}\right\}^{\frac{1}{q'}}\leqslant\|L\|_{L^{\frac{1}{\lambda}}(\mathbb{R}_+^n)}\|f\|_{L^{p'}(\mathbb{R}_+^n)}$$

hold for all non-negative measurable functions f and g on $\mathbb{R}_+^n$ and are equivalent. If p and q are as in (10), (11), *or $p,q\in\langle 0,1\rangle$, then the signs of inequality in both relations are reversed.*

Proof. Corollary 9 follows from Corollary 8 if we put $A_1=\frac{2-s}{p'}$, $A_2=0$, $K=L^{\frac{1}{\lambda}}$, and consider the functions $\tilde{f},\tilde{g}:\mathbb{R}_+^n\to\mathbb{R}$,

$$\tilde{f}(\mathbf{x})=\mathbf{x}^{\left[(s-1)\lambda-\frac{1}{p'}\right]\mathbf{1}}f(\mathbf{x}),\quad \tilde{g}(\mathbf{y})=\mathbf{y}^{\frac{1}{p'}\mathbf{1}}g(\mathbf{y}),$$

instead of f and g. Note that in this setting we have

$$k((2-p'A_1-s)\mathbf{1})=k(q'A_2\mathbf{1})=\int_{\mathbb{R}_+^n}L^{\frac{1}{\lambda}}(\mathbf{u})\,d\mathbf{u}=\|L\|_{L^{\frac{1}{\lambda}}(\mathbb{R}_+^n)}^{\frac{1}{\lambda}},$$

so $C=\|L\|_{L^{\frac{1}{\lambda}}(\mathbb{R}_+^n)}$. □

Since the first inequality in Corollary 9 was proved by E. K. Godunova in [4], all inequalities obtained in this section will be called the Godunova-type inequalities.

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(Received October 30, 2006)

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