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# $L_q$ NORM INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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*Abstract.* If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then it was shown by Govil, Nyuydinkong and Tameru [ Some $L_p$ inequalities for the polar derivative of a polynomial, J. Math. Anal. Appl., **254**, (2001), 618–626 ] that for every real or complex number $\alpha$ with $|\alpha| \leqslant 1$ and $q \geqslant 1$,

$$|D_\alpha P| \leqslant \left(\frac{|\alpha|+1}{\|1+z\|_q}\right)\|P\|_q$$

where $D_\alpha P(z)$ denotes the polar derivative of $P(z)$ with respect to $\alpha \in C$. Unfortnunately the proof of this result is not correct. In this paper, we prove a more general result which not only provides a correct proof of this result but also extends some known $L_q$ norm inequalities for the polar derivative of a polynomial. We also present $L_q$ norm inequality for polynomials not vanishing in $|z| > k$ where k $\leqslant 1$ .

## 1. Introduction and statements results

Let $P_n(z)$ be the class of polynomials $P(z)$ of degree at most n. For $P_n \in P$, define

$$\|P\|_q := \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^q\right\}^{1/q}, \qquad 1 \leqslant q < \infty,$$

and

$$\|P\|_\infty := \max_{|z|=1}|P(z)|\,.$$

A famous result known as Bernstein's inequality ( for reference see [13] or [16]) states that if $P_n \in P$, then

$$\|P'\|_\infty \leqslant n\,\|P\|_\infty \tag{1}$$

Inequality (1) is sharp and equality in (1) holds for $P(z) = az^n$, $a \neq 0$. Inequality (1) was extended to $L_q$-norm by Zygmund [17] who proved that if $P_n \in P$, then

$$\|P'\|_q \leqslant n\,\|P\|_q\,, \quad q \geqslant 1. \tag{2}$$

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The result is sharp and equality in (2) holds for $P(z) = az^n$, $a \neq 0$. If we let $q \to \infty$ in (2), we get inequality (1). Melas [12] showed that inequality (2) remains true for $0 < q < 1$ as well. For polynomials $P_n \in P$ which does not vanish int he unit disk, the right hand side of (2) can be improved. In fact, in this direction, it was shown by De- Bruijn [5] that if $P_n \in P$ and $P(z) \neq 0$ for $|z| < 1$, then

$$\|P'\|_q \leqslant \frac{n}{\|1+z\|_q} \|P\|_q, \quad q \geqslant 1. \tag{3}$$

The above result of De-Bruijn was extended for $q > 0$ by Rahman and Schmeisser [14]. If we let $q \to \infty$ in (3), it follows that if $P_n \in P$ and $P(z) \neq 0$ for $|z| < 1$, then

$$\|P'\|_\infty \leqslant \frac{n}{2} \|P\|_\infty. \tag{4}$$

Inequality (4) was conjectured by P. Erdös and later verified by P. D. Lax [10]. Both the estimates are sharp and equality in (3) and (4) holds for $P(z) = az^n + b, |a| = |b|$.

Malik [11] generalized inequality (4) by proving that if $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geqslant 1$, then

$$\|P'\|_\infty \leqslant \frac{n}{1+k} \|P\|_\infty. \tag{5}$$

Govil and Rahman [8] extended inequality (5) to $L_q$-norm by proving that if $P \in P_n$ and $P(z) \neq 0$ for $|z| < k$ where $k \geqslant 1$, then

$$\|P'\|_q \leqslant \frac{n}{\|k+z\|_q} \|P\|_q, q \geqslant 1. \tag{6}$$

It was shown by Gardner and Weems [7] and independently by Rather [15] that the inequality (6) remains true for $0 < q < 1$ as well. Let $D_\alpha P(z)$ denote the polar differentiation of polynomial $P(z)$ with respect to a real or complex number $\alpha$, then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

If $P \in P_n$, then $D_\alpha P \in P_{n-1}$. Furthermore, the polar derivative $D_\alpha P(z)$ generalizes the ordinary derivative $P'(z)$ in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

As an extension of (1) to the polar derivative, Aziz and Shah [4, Theorem 4 with k =1] have shown that if $P \in P_n$, then for every real or complex number $\alpha$ with $|\alpha| \geqslant 1$,

$$\|D_\alpha P\|_\infty \leqslant n|\alpha| \, \|P\|_\infty \tag{7}$$

Inequality (7) becomes equality for $P(z) = az^n, a \neq 0$. If we divide the both sides of (7) by $|\alpha|$ and let $|\alpha| \to \infty$, we get inequality (1).

It is natural to seek $L_q$ - norm analog of the inequality (7). In view of $L_q$-norm extension of (2) of inequality (2), one would expect that if $P \in P_n$, then

$$\|D_\alpha P\|_q \leqslant n|\alpha| \, \|P\|_q \tag{8}$$

to be $L_q$-norm extension of (7) analogous to (2). But unfortunately inequality (8) is not, in general, true for every real or complex number $\alpha$. To see this, we take , in particular $q = 2, P(z) = (1 - iz)^n$ and $\alpha = i\beta$ where $\beta$ is any positive real number such that

$$1 \leqslant \beta < \frac{n + \sqrt{2n(2n-1)}}{3n-2}. \tag{9}$$

Now,

$$\begin{aligned} D_\alpha P(z) &= n(1 - iz)^n - ni(\alpha - z)(1 - iz)^{n-1} \\ &= n(1 - iz)^{n-1}(1 - i\alpha) \end{aligned}$$

so that

$$\begin{aligned} \int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^q d\theta &= n^2 \left|1 - i\alpha\right|^2 \int_0^{2\pi} \left|1 - ie^{i\theta}\right|^{2(n-1)} d\theta \\ &= n^2 \left|1 - i\alpha\right|^2 \int_0^{2\pi} \left|(1 - ie^{i\theta})^{n-1}\right|^2 d\theta \\ &= n^2 \left|1 - i\alpha\right|^2 \int_0^{2\pi} |\binom{n-1}{0} - \binom{n-1}{1}(ie^{i\theta}) \\ &\qquad + \cdots + (-1)^{n-1}\binom{n-1}{n-1}(ie^{i\theta})^{n-1}|^2 d\theta \\ &= n^2 \left|1 - i\alpha\right|^2 \{\binom{n-1}{0}^2 + \binom{n-1}{1}^2 + \binom{n-1}{2}^2 \\ &\qquad + \cdots + \binom{n-1}{n-1}^2\} \\ &= 2\pi n^2 \left|1 - i\alpha\right|^2 \binom{2(n-1)}{n-1}. \end{aligned} \tag{10}$$

Also,

$$\begin{aligned} n^2 \left|\alpha\right|^2 \int_0^{2\pi} \left|P(e^{i\theta}))\right|^q d\theta &= n^2 \left|\alpha\right|^2 \int_0^{2\pi} \left|1 - ie^{i\theta})\right|^{2n} d\theta \\ &= n^2 \left|\alpha\right|^2 \int_0^{2\pi} \left|(1 - ie^{i\theta})^n\right|^2 d\theta \\ &= n^2 \left|\alpha\right|^2 \int_0^{2\pi} |\binom{n}{0} - \binom{n}{1}(ie^{i\theta}) \\ &\qquad + \cdots + (-1)^n\binom{n}{n}(ie^{i\theta})^n|^2 d\theta \\ &= n^2 \left|\alpha\right|^2 \{\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 \\ &\qquad + \cdots + \binom{n}{n}^2\} \end{aligned}$$

$$= 2\pi n^2 |\alpha|^2 \binom{2n}{n}. \tag{11}$$

Using (10) and (11) in (8), we get

$$2\pi n^2 \binom{2(n-1)}{n-1} |1 - i\alpha|^2 \leqslant 2\pi n^2 |\alpha|^2 \binom{2n}{n}.$$

This implies

$$n |1 - i\alpha|^2 \leqslant 2(2n-1) |\alpha|^2. \tag{12}$$

Setting $\alpha = i\beta$ in (12), we get

$$n(1+\beta)^2 \leqslant 2(2n-1)^2.$$

This inequality can be written as

$$\left(\beta - \frac{n + \sqrt{2n(2n-1)}}{3n-2}\right) \left(\beta - \frac{n - \sqrt{2n(2n-1)}}{3n-2}\right) \geqslant 0. \tag{13}$$

Since $\beta \geqslant 1$, we have

$$\left(\beta - \frac{n - \sqrt{2n(2n-1)}}{3n-2}\right) \geqslant \left(1 - \frac{n - \sqrt{2n(2n-1)}}{3n-2}\right)$$
$$= \left(\frac{2(n-1) + \sqrt{2n(2n-1)}}{3n-2}\right) > 0$$

and hence from (13), it follows that

$$\left(\beta - \frac{n + \sqrt{2n(2n-1)}}{3n-2}\right) \geqslant 0.$$

This gives

$$\beta \geqslant \frac{n + \sqrt{2n(2n-1)}}{3n-2},$$

which clearly contradicts (9). Hence inequality (8) is not, in general, true for all polynomials of degree n $\geqslant 1$.

A. Aziz [1] extended (4) to the polar derivative of a polynomial and proved that if $P \in P_n$ and *P(z)* does not vanish in $|z| < 1$, then for every real or complex number $\alpha$ with $|\alpha| \geqslant 1$,

$$\|D_\alpha P\|_\infty \leqslant \frac{n}{2}(|\alpha| + 1) \|P\|_\infty \tag{14}$$

The estimate is best possible and equality in (14) holds for $P(z) = z^n + 1$. If we divide both sides of (14) by $|\alpha|$ and make $|\alpha| \to \infty$, we get inequality (4) due to Lax [10]. Aziz [1] also extended inequality (5) to the polar derivatives by proving that if $P \in P_n$

and $P(z) \neq 0$ for $|z| < k$, where $k \geqslant 1$, then for every real or complex number $\alpha$ with $|\alpha| \geqslant 1$,

$$\|D_\alpha P\|_\infty \leqslant n\left(\frac{|\alpha|+k}{1+k}\right)\|P\|_\infty \tag{15}$$

The result is best possible and equality in (15) holds for $P(z) = (z+k)^n$ where $\alpha$ is any real number with $\alpha \geqslant 1$.

Whille seeking the desired extension to the $L_q$ - norm, recently Govil, Nyuydinkong and Tameru [9] have made an incomplete attempt by claiming to have proved the following generalization of (3) and (14).

THEOREM 1. *If $P \in P_n$ and P(z) does not vanish in $|z| < 1$, then for every real or complex number $\alpha$ with $|\alpha| \geqslant 1$, and $q \geqslant 1$,*

$$\|D_\alpha P\|_q \leqslant n\left(\frac{|\alpha|+1}{\|1+z\|_q}\right)\|P\|_q. \tag{16}$$

Unfotrunately the proof of of this theorem, which is the main result in [9, Theorem 1.1] given by Govil, Nyuydinkong and Tameru is not correct, because the claim made by the authors on page 624 in lines 12 to 16 by using Lemma 2,3 [9] is incorrect. The reason being that their polynomial

$$D_\alpha P_n(z) + e^{i\gamma}\left\{n\overline{\alpha}zP_n(z) + (1-\overline{\alpha}z)zP_n'(z)\right\}, \qquad z = e^{i\theta}$$

in general, does not take the form

$$\sum_{k=0}^{n} l_k a_k z^k, \qquad z = e^{i\theta}$$

where

$$P_n(z) = \sum_{k=0}^{n} l_k a_k z^k$$

and the complex numbers $l_k$ defined by them on page 624, line 10, by

$$L\left(P_n(e^{i\theta})\right) = \left[\Lambda P_n(e^{i\theta})\right]_{\theta=0} = \sum_{k=0}^{n} l_k a_k$$

along with the equation (24).

It is worthwhile to note it here that if we use the same argument as used by Govil, Nyuydinkong and Tameru [9], page 624, line 10, then it would follow from (7) that

$$\|D_\alpha P\|_q \leqslant n|\alpha|\,\|P\|_q$$

for every $q \geqslant 1$ and $\alpha$ with $|\alpha| \geqslant 1$, which is not true, in general, as shown above.

In this paper, we first present the following more general result which not only provides a correct proof of Theorem 1 but also extends inequality (15) due to the first to the $L_q$ - norm.

THEOREM 2. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$, where $k \geqslant 1$, then for every real or complex number $\alpha$ with $|\alpha| \geqslant 1$, and $q \geqslant 1$,*

$$\|D_\alpha P\|_q \leqslant n \left( \frac{|\alpha| + k}{\|k + z\|_q} \right) \|P\|_q . \tag{17}$$

In the limiting case, when $q \to \infty$, the above inequality is sharp and equality in (17) holds for $P(z) = (z+k)^n$ where $\alpha$ is any real number with $\alpha \geqslant 1$.

REMARK 1. For k =1, Theorem 2 validates Theorem 1.

REMARK 2. If we let $q \to \infty$, in (17), we get inequality (15).

REMARK 3. The result of Govil and Rahman ( inequality (6) ) follows from Theorem 2 by dividing the two sides of inequality (17) by $|\alpha|$ and then letting $|\alpha| \to \infty$.

Next we prove the following result.

THEOREM 3. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| > k$ where $k \leqslant 1$, then for every real or complex number $\alpha$ with $|\alpha| \leqslant 1$, and $q \geqslant 1$,*

$$\|D_\alpha P\|_q \leqslant n \left( \frac{|\alpha| + k}{\|k + z\|_q} \right) \|P\|_q . \tag{18}$$

In the limiting case, when $q \to \infty$, the above inequality is sharp and equality in (18) holds for $P(z) = (z+k)^n$ for any real $\alpha$ with $0 \leqslant \alpha \leqslant 1$.

The following result immediately follows from Theorem 3 by letting $q \to \infty$.

COROLLARY 1. *If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then for every real or complex number $\alpha$ with $|\alpha| \leqslant 1$,*

$$\|D_\alpha P\|_\infty \leqslant n \left( \frac{|\alpha| + k}{1 + k} \right) \|P\|_\infty . \tag{19}$$

The result is best possible and equality in (19) holds for $P(z) = (z+k)^n$ for any real $\alpha$ with $0 \leqslant \alpha \leqslant 1$ and $k \leqslant 1$.

Setting $\alpha = 0$ in (19), it follows that if $P \in P_n$ and P(z) has all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then for $|z| = 1$,

$$|nP(z) - zP'(z)| \leqslant \frac{nk}{1+k} \|P\|_\infty .$$

This gives for $|z| = 1$,

$$\begin{aligned} |P'(z)| &= |nP(z) + zP'(z) - nP(z)| \\ &\geqslant n\,|P(z)| - |nP(z) - zP'(z)| \\ &\geqslant n\,|P(z)| - \frac{nk}{1+k} \|P\|_\infty , \end{aligned}$$

which implies

$$\|P'\|_\infty \geqslant \frac{n}{1+k}\|P\|_\infty. \tag{20}$$

Inequality (20) is due to Malik [11].

## 2. Lemmas

For the proofs of these theorems, we need the following lemmas. The first lemma is due to A. Aziz.[1].

LEMMA 1. *If $P \in P_n$ and P(z) does not vanish in $|z| < k$ where $k \geqslant 1$, then for every real or complex number $\delta$ with $|\delta| \geqslant 1$,*

$$|D_{\delta k}P(z)| \leqslant k\left|D_{\delta/k}Q(z)\right| \qquad for \quad |z| = 1$$

*where $Q(z) = z^n\overline{P(1/\overline{z})}$ .*

Setting $\alpha = \delta k$ where $k \geqslant 1$ in Lemma 1, we immediately get:

LEMMA 2. *If $P \in P_n$ and P(z) does not vanish in $|z| < k$ where $k \geqslant 1$, then for every real or complex number $\alpha$ with $|\alpha| \geqslant 1$,*

$$|D_\alpha P(z)| \leqslant k\left|D_{\alpha/k^2}Q(z)\right| \qquad for \quad |z| = 1$$

*where $Q(z) = z^n\overline{P(1/\overline{z})}$ .*

LEMMA 3. *If $P \in P_n$ and P(z) $\neq 0$ in $|z| < k$ where $k \geqslant 1$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, then for $|z| = 1$,*

$$k|P'(z)| \leqslant |Q'(z)|.$$

Lemma 3 is due to Malik [ 9 ].

LEMMA 4. *If $P \in P_n$ and P(z) $\neq 0$ in $|z| < k$ where $k \geqslant 1$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, then for every real $\beta, 0 \leqslant \beta < 2\pi$,*

$$\left|k^2|P'(z)| + e^{i\beta}|Q'(z)|\right| \leqslant k\left||P'(z)| + e^{i\beta}|Q'(z)|\right| \quad for \quad |z| = 1.$$

*Proof of Lemma 4.* Since $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geqslant 1$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, by Lemma 3, we have

$$k^2|P'(z)|^2 \leqslant |Q'(z)|^2 \quad \text{for} \quad |z| = 1.$$

Multiplying both sides of this inequality by $(k^2-1)$ and rearranging the terms, we get

$$k^4|P'(z)|^2 + |Q'(z)|^2 \leqslant k^2|P'(z)|^2 + k^2|Q'(z)|^2 \quad \text{for} \quad |z| = 1. \tag{22}$$

Adding $2k^2|P'(z)||Q'(z)|\cos\beta$ to the both sides of (22), we obtain for $|z|=1$,

$$\begin{aligned}k^4|P'(z)|^2+|Q'(z)|^2+2k^2|P'(z)||Q'(z)|\cos\beta\\ \leqslant k^2|P'(z)|^2+k^2|Q'(z)|^2+2k^2|P'(z)||Q'(z)|\cos\beta,\end{aligned}$$

which implies

$$\left|k^2|P'(z)|+e^{i\beta}|Q'(z)|\right|^2\leqslant k^2\left||P'(z)|+e^{i\beta}|Q'(z)|\right|^2 \quad\text{for}\quad |z|=1$$

and hence

$$\left|k^2|P'(z)|+e^{i\beta}|Q'(z)|\right|\leqslant k\left||P'(z)|+e^{i\beta}|Q'(z)|\right| \quad\text{for}\quad |z|=1.$$

This proves Lemma 4. □

LEMMA 5. *If* $P\in P_n$ *and* $Q(z)=z^n\overline{P(1/\overline{z})}$, *then for every* $q>0$ *and* $\beta$ *real,* $0\leqslant\beta<2\pi$,

$$\int_0^{2\pi}\int_0^{2\pi}\left|P'(e^{i\theta})+e^{i\beta}Q'(e^{i\theta})\right|^q d\theta d\beta\leqslant 2\pi n^q\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta.$$

Lemma 5 is due to Aziz and Rather [3] (see also [15])

LEMMA 6. *If* $P\in P_n$ *and P(z) does not vanish in* $|z|<k$ *where* $k\geqslant 1$ *and* $Q(z)=z^n\overline{P(1/\overline{z})}$, *then for every real or complex number* $\alpha$, $q\geqslant 1$ *and* $\beta$ *real,* $0\leqslant\beta<2\pi$,

$$\int_0^{2\pi}\int_0^{2\pi}\left|D_\alpha P(e^{i\theta})+e^{i\beta}k^2D_{\alpha/k^2}Q(e^{i\theta})\right|^q d\theta d\beta\leqslant 2\pi n^q(|\alpha|+k)^q\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta.$$

*Proof of Lemma 6.*. Let r be any positive real number. We have by Minkowski's inequality for every $q\geqslant 1$ and $\beta$ real,

$$\begin{aligned}&\left\{\int_0^{2\pi}\int_0^{2\pi}\left|D_\alpha P(e^{i\theta})+e^{i\beta}r^2D_{\alpha/k^2}Q(e^{i\theta})\right|^q d\theta d\beta\right\}^{1/q}\\
&\qquad=\{\int_0^{2\pi}\int_0^{2\pi}|\left(nP(e^{i\theta})+(\alpha-e^{i\theta})P'(e^{i\theta})\right)\\
&\qquad\qquad+e^{i\beta}r^2\left(nQ(e^{i\theta})+(\frac{\alpha}{r^2}-e^{i\theta})Q'(e^{i\theta})\right)|^q d\theta d\beta\}^{1/q}\\
&\qquad=\{\int_0^{2\pi}\int_0^{2\pi}|\left(nP(e^{i\theta})-e^{i\theta}P'(e^{i\theta})\right)+e^{i\beta}r^2\left(nQ(e^{i\theta})-e^{i\theta}Q'(e^{i\theta})\right)\\
&\qquad\qquad+\alpha\left(P'(e^{i\theta})+e^{i\beta}Q'(e^{i\theta})\right)|^q d\theta d\beta\}^{1/q}\\
&\leqslant\left\{\int_0^{2\pi}\int_0^{2\pi}\left|\left(nP(e^{i\theta})-e^{i\theta}P'(e^{i\theta})\right)+e^{i\beta}r^2\left(nQ(e^{i\theta})-e^{i\theta}Q'(e^{i\theta})\right)\right|^q d\theta d\beta\right\}^{1/q}\end{aligned}$$

$$+|\alpha|\left\{\int_0^{2\pi}\int_0^{2\pi}\left|P'(e^{i\theta})+e^{i\beta}Q'(e^{i\theta})\right|^q d\theta d\beta\right\}^{1/q}. \tag{23 }$$

Since $P \in P_n$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, we have $P(z) = z^n\overline{Q(1/\overline{z})}$ and it can be easily verified that for $0 \leqslant \theta < 2\pi$,

$$nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) = e^{i(n-1)\theta}\overline{Q'(e^{i\theta})} \tag{24}$$

and

$$nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta}) = e^{i(n-1)\theta}\overline{P'(e^{i\theta})}. \tag{25}$$

Using (24) and (25) in (23), we obtain

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi}\left|D_\alpha P(e^{i\theta})+e^{i\beta}r^2D_{\alpha/k^2}Q(e^{i\theta})\right|^q d\theta d\beta\\
&\quad\leqslant\left\{\int_0^{2\pi}\int_0^{2\pi}\left|e^{i(n-1)\theta}\overline{Q'(e^{i\theta})}+e^{i\beta}r^2e^{i(n-1)\theta}\overline{P'(e^{i\theta})}\right|^q d\theta d\beta\right\}^{1/q}\\
&\qquad+|\alpha|\left\{\int_0^{2\pi}\int_0^{2\pi}\left|P'(e^{i\theta})+e^{i\beta}Q'(e^{i\theta})\right|^q d\theta d\beta\right\}^{1/q}\\
&\quad=\left\{\int_0^{2\pi}\int_0^{2\pi}\left|r^2P'(e^{i\theta})+e^{i\beta}Q'(e^{i\theta})\right|^q d\theta d\beta\right\}^{1/q}\\
&\qquad+|\alpha|\left\{\int_0^{2\pi}\int_0^{2\pi}\left|P'(e^{i\theta})+e^{i\beta}Q'(e^{i\theta})\right|^q d\theta d\beta\right\}^{1/q}.
\end{aligned}\tag{26}$$

Now for points $e^{i\theta}, 0 \leqslant \theta < 2\pi$, for which $Q'(e^{i\theta}) \neq 0$,

$$\begin{aligned}
\int_0^{2\pi}\left|r^2P'(e^{i\theta})+e^{i\beta}Q'(e^{i\theta})\right|^q d\beta &= \left|Q'(e^{i\theta})\right|^q\int_0^{2\pi}\left|\frac{r^2P'(e^{i\theta})}{Q'(e^{i\theta})}+e^{i\beta}\right|^q d\beta\\
&=\left|Q'(e^{i\theta})\right|^q\int_0^{2\pi}\left|\left|\frac{r^2P'(e^{i\theta})}{Q'(e^{i\theta})}\right|+e^{i\beta}\right|^q d\beta\\
&=\int_0^{2\pi}\left|r^2\left|P'(e^{i\theta})\right|+e^{i\beta}\left|Q'(e^{i\theta})\right|\right|^q d\beta.
\end{aligned}\tag{27}$$

Since (27) is trivially true for points $e^{i\theta}, 0 \leqslant \theta < 2\pi$, for which $Q'(e^{i\theta}) = 0$, it follows by taking $r = k \geqslant 1$,

$$\int_0^{2\pi}\left|k^2P'(e^{i\theta})+e^{i\beta}Q'(e^{i\theta})\right|^q d\beta = \int_0^{2\pi}\left|k^2\left|P'(e^{i\theta})\right|+e^{i\beta}\left|Q'(e^{i\theta})\right|\right|^q d\beta. \tag{28}$$

Integrating (28) both sides with respect to $\theta$ from 0 to $2\pi$ and then using Lemma 4, we get

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi}&\left|k^2P'(e^{i\theta})+e^{i\beta}Q'(e^{i\theta})\right|^q d\theta d\beta\\ &=\int_0^{2\pi}\int_0^{2\pi}\left|k^2\left|P'(e^{i\theta})\right|+e^{i\beta}\left|Q'(e^{i\theta})\right|\right|^q d\beta d\theta\\ &\leqslant k\left\{\int_0^{2\pi}\int_0^{2\pi}\left|\left|P'(e^{i\theta})\right|+e^{i\beta}\left|Q'(e^{i\theta})\right|\right|^q d\beta\right\}d\theta\\ &=k\int_0^{2\pi}\left\{\int_0^{2\pi}\left|P'(e^{i\theta})+e^{i\beta}Q'(e^{i\theta})\right|^q d\beta\right\}d\theta\\ &=k\int_0^{2\pi}\int_0^{2\pi}\left|P'(e^{i\theta})+e^{i\beta}Q'(e^{i\theta})\right|^q d\theta d\beta. \qquad (29)\end{aligned}$$

Setting $r=k\geqslant 1$ and using inequality (29), we obtain

$$\begin{aligned}\left\{\int_0^{2\pi}\int_0^{2\pi}\left|D_\alpha P(e^{i\theta})+e^{i\beta}k^2D_{\alpha/k^2}Q(e^{i\theta})\right|^q d\theta d\beta\right\}^{1/q}\\ \leqslant(|\alpha|+k)\left\{\int_0^{2\pi}\int_0^{2\pi}\left|P'(e^{i\theta})+e^{i\beta}Q'(e^{i\theta})\right|^q d\theta d\beta\right\}^{1/q}.\end{aligned}$$

This gives with the help of Lemma 5,

$$\int_0^{2\pi}\int_0^{2\pi}\left|D_\alpha P(e^{i\theta})+e^{i\beta}k^2D_{\alpha/k^2}Q(e^{i\theta})\right|^q d\theta d\beta\leqslant 2\pi n^q\left(|\alpha|+k\right)^q\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta,$$

and the proof of Lemma 6 is complete. □

We also need the following lemma due to A. Aziz [1].

LEMMA 2. *If* $P\in P_n$ *and* $Q(z)=z^n\overline{P(1/\overline{z})}$, *then for every real or complex number* $\delta$ *with* $|\delta|\neq 0$,

$$|D_\delta Q(z)|=|\delta|\left|D_{1/\overline{\delta}}P(z)\right| \quad for\ |z|=1.$$

## 3. Proofs of the theorems

*Proof of Theorem 2.* Since $P\in P_n$ and *P(z)* does not vanish in $|z|<k$ where $k\geqslant 1$, by Lemma 2, we have for every real or complex number $\alpha$ with $|\alpha|\geqslant 1$,

$$|D_\alpha P(z)|\leqslant k\left|D_{\alpha/k^2}Q(z)\right| \qquad \text{for} \quad |z|=1 \qquad (30)$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$. Also, by Lemma 6, for every real or complex number $\alpha, q \geqslant 1$ and $\beta$ real,

$$\int_0^{2\pi} \left\{ \int_0^{2\pi} \left| D_\alpha P(e^{i\theta}) + e^{i\beta} k^2 D_{\alpha/k^2} Q(e^{i\theta}) \right|^q d\beta \right\} d\theta \leqslant 2\pi n^q \left( |\alpha| + k \right)^q \int_0^{2\pi} \left| P(e^{i\theta}) \right|^q d\theta. \tag{31}$$

Now for every real $\beta, 0 \leqslant \theta < 2\pi$ and $t_1 \geqslant t_2 \geqslant 1,$ we have

$$\left| t_1 + e^{i\beta} \right| \geqslant \left| t_2 + e^{i\beta} \right|,$$

which implies

$$\int_0^{2\pi} \left| t_1 + e^{i\beta} \right|^q d\beta \geqslant \int_0^{2\pi} \left| t_2 + e^{i\beta} \right|^q d\beta, \quad q \geqslant 1.$$

If $D_\alpha P(e^{i\theta}) \neq 0,$ we take $t_1 = k^2 \left| D_{\alpha/k^2} Q(e^{i\theta}) \right| / \left| D_\alpha P(e^{i\theta}) \right|$ and $t_2 = k,$ then by (30), $t_1 \geqslant t_2 \geqslant 1,$ and we get

$$\begin{aligned}
\int_0^{2\pi} \left| D_\alpha P(e^{i\theta}) + e^{i\beta} k^2 D_{\alpha/k^2} Q(e^{i\theta}) \right|^q d\beta &= \left| D_\alpha P(e^{i\theta}) \right|^q \int_0^{2\pi} \left| \frac{k^2 D_{\alpha/k^2} Q(e^{i\theta})}{D_\alpha P(e^{i\theta})} e^{i\beta} + 1 \right|^q d\beta \\
&= \left| D_\alpha P(e^{i\theta}) \right|^q \int_0^{2\pi} \left| \left| \frac{k^2 D_{\alpha/k^2} Q(e^{i\theta})}{D_\alpha P(e^{i\theta})} \right| e^{i\beta} + 1 \right|^q d\beta \\
&= \left| D_\alpha P(e^{i\theta}) \right|^q \int_0^{2\pi} \left| \left| \frac{k^2 D_{\alpha/k^2} Q(e^{i\theta})}{D_\alpha P(e^{i\theta})} \right| + e^{i\beta} \right|^q d\beta \\
&\geqslant \left| D_\alpha P(e^{i\theta}) \right|^q \int_0^{2\pi} \left| k + e^{i\beta} \right|^q d\beta.
\end{aligned}$$

For $D_\alpha P(e^{i\theta}) = 0,$ this inequality is trivially true. Using this in (31), we conclude that for every real or complex number $\alpha$ with $|\alpha| \geqslant 1$ and $q \geqslant 1,$

$$\int_0^{2\pi} \left| k + e^{i\beta} \right|^q d\beta \int_0^{2\pi} \left| D_\alpha P(e^{i\theta}) \right|^q d\theta \leqslant 2\pi n^q \left( |\alpha| + k \right)^q \int_0^{2\pi} \left| P(e^{i\theta}) \right|^q d\theta,$$

which immediately leads to (17) and this completes the proof of Theorem 2. □

*Proof of Theorem 3.* Let $Q(z) = z^n\overline{P(1/\overline{z})}$. Since $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leqslant k$ where $k \leqslant 1, Q \in P_n$ and $Q(z)$ does not vanish in $|z| < (1/k)$ where $(1/k) \geqslant 1$. Applying Theorem 2 to the polynomial $Q(z)$, we get for every real or complex number $\alpha$ with $|\alpha| \leqslant 1$ (so that $|1/\alpha| \geqslant 1$ ) and $q \geqslant 1,$

$$\left\{ \int_0^{2\pi} \left| D_{1/\overline{\alpha}} Q(e^{i\theta}) \right|^q d\theta \right\}^{1/q} \leqslant n \left( \frac{\left| \frac{1}{\overline{\alpha}} \right| + \frac{1}{k}}{\left\| z + \frac{1}{k} \right\|_q} \right) \left\{ \int_0^{2\pi} \left| Q(e^{i\theta}) \right|^q d\theta \right\}^{1/q}. \tag{32}$$

Now since

$$\left|Q(e^{i\theta})\right| = \left|P(e^{i\theta})\right|, \quad 0 \leqslant \theta < 2\pi,$$

and

$$\left\|z+\frac{1}{k}\right\|_q = \frac{1}{k}\left\|z+k\right\|_q,$$

it follows that (32) is equivalent to

$$\left\{\int_0^{2\pi} |\alpha| \left|D_{1/\overline{\alpha}}Q(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leqslant n\left(\frac{|\alpha|+k}{\|z+k\|_q}\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q}. \quad (33)$$

Moreover, $Q(z) = z^n\overline{P(1/\overline{z})}$ implies $P(z) = z^n\overline{Q(1/\overline{z})}$, therefore by Lemma 7, we have for every $\alpha$ with $|\alpha| \leqslant 1$,

$$\left|D_\alpha P(e^{i\theta})\right| = |\alpha| \left|D_{1/\overline{\alpha}}Q(e^{i\theta})\right|, \quad 0 \leqslant \theta < 2\pi.$$

Using this in (33), we get

$$\left\{\int_0^{2\pi}\left|D_\alpha P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leqslant n\left(\frac{|\alpha|+k}{\|z+k\|_q}\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q}, \quad q \geqslant 1$$

and the proof of Theorem 3 is complete. □

## 4. Some concluding remarks

REMARK 4. If $P \in P_n$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, then by inequality (26) with r = 1, we have for every real or complex number $\alpha$ and $q \geqslant 1$,

$$\left\{\int_0^{2\pi}\int_0^{2\pi}\left|D_\alpha P(e^{i\theta}) + e^{i\beta}D_\alpha Q(e^{i\theta})\right|^q d\theta d\beta\right\}^{1/q}$$
$$\leqslant (|\alpha|+1)\int_0^{2\pi}\int_0^{2\pi}\left|P'(e^{i\theta}) + e^{i\beta}Q'(e^{i\theta})\right|^q d\theta d\beta.$$

Combining this inequality with lemma 6, *it follows that if* $P \in P_n$, *then for every real or complex number* $\alpha$ *and* $q \geqslant 1$,

$$\int_0^{2\pi}\int_0^{2\pi}\left|D_\alpha P(e^{i\theta}) + e^{i\beta}D_\alpha Q(e^{i\theta})\right|^q d\theta d\beta \leqslant 2\pi n^q(|\alpha|+1)^q\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta \quad (34)$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.

It is interesting to compare inequality (34) with Lemma 6 for k = 1.

REMARK 5. Using in (34), the fact that for any q $>0$,

$$\int_0^{2\pi} \left|a+be^{i\theta}\right|^q d\theta \leqslant 2\pi Max\left(|a|^q, |b|^q\right),$$

(see [5, Inequality (19)]), *it follows that if* $P \in P_n$, *then for every real or complex number* $\alpha$ *and* $q \geqslant 1$,

$$\left\{\int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leqslant n\left(|\alpha|+1\right)\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q}, \tag{35}$$

which is an interesting extension of Zygmund's inequality (2) for the polar derivative of a polynomial. If we divide both sides of (35) by $|\alpha|$ and then let $|\alpha| \to \infty$, we get inequality (2).

REMARK 6. If $P \in P_n$ is self-inversive and $Q(z) = z^n\overline{P(1/\overline{z})}$, then we have

$$P(z) = Q(z) \quad \text{for all} \quad z \in C,$$

so that

$$\left|D_\alpha Q(e^{i\theta})\right| / \left|D_\alpha P(e^{i\theta})\right| = 1.$$

Using this in (32) and proceeding similarly as in the proof of Theorem 2, *it follows that if* $P \in P_n$ *is self-inversive, then for every real or complex number* $\alpha$ *and* $q \geqslant 1$,

$$\|D_\alpha P\|_q \leqslant n\left(\frac{|\alpha|+1}{\|1+z\|_q}\right)\|P\|_q. \tag{36}$$

Inequality (36) is a generalization of a result of Dewan and Govil [6] and A. Aziz [2] for the polar derivatives. Moreover, it also extends a result due to A. Aziz [1, Theorem 2] to $L_q-$ norm.

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