Mathematical
Inequalities
& Applications
Volume 11, Number 2 (2008), 335–348

COMPACTNESS OF INTEGRAL OPERATORS IN LEBESGUE SPACES WITH MIXED NORM

ALBERTO FIORENZA, BABITA GUPTA AND PANKAJ JAIN

(*communicated by L.-E. Persson*)

Abstract. Compactness of certain double sized multidimensional integral operators is characterized in terms of the corresponding two-dimensional operators in the framework of weighted Lebesgue spaces with mixed norm.

1. Introduction

Consider the "double sized" multidimensional operators

$$(T_{E,F}f)(x,y) = \int_{b(|x|)S_M\setminus a(|x|)S_M} \int_{d(|y|)S_N\setminus c(|y|)S_N} f(s,t)dt\,ds,\ x\in E,\ y\in F \tag{1.1}$$

and

$$(H_{E,F}f)(x,y) = \int_{S_{M_x}} \int_{S_{N_y}} f(s,t)dt\,ds,\ x\in E,\ y\in F \tag{1.2}$$

where E,F are certain multidimensional cones which along with other symbols are defined in Section 2. These operators are multidimensional analogues of, respectively, the two-dimensional Hardy-Steklov operator

$$(T_2f)(x,y) = \int_{a(x)}^{b(x)} \int_{c(y)}^{d(y)} f(s,t)dt\,ds,\ x,y\in(0,\infty) \tag{1.3}$$

and the two-dimensional Hardy operator

$$(H_2f)(x,y) = \int_0^x \int_0^y f(s,t)dt\,ds,\ x,y\in(0,\infty). \tag{1.4}$$

Note that in (1.1) and (1.2), x and y are multidimensional vectors whereas in (1.3) and (1.4), they are real numbers but there should be no confusion since it is clear from the context.

In this paper, we discuss the compactness of the above operators between suitable Lebesgue spaces with mixed norm. The compactness of $T_{E,F}$ was studied in [6]. Here,

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Mathematics subject classification (2000): 46E35, 26D10.

Key words and phrases: Lebesgue space, mixed norm, compactness, Hardy Operator, Hardy-Steklov operator.

Paper MIA-11-25

we shall exhibit that the compactness of $T_{E,F}$ (respectively $H_{E,F}$) can be studied in terms of the compactness of T_2 (respectively H_2) and vice-versa.

The compactness of integral operators has applications in spectral theory. The initial results on L^p-L^q compactness (one dimensional case) are due to Stepanov [17-22] (see also [11], [12]) who dealt with the Riemann-Liouville operator with kernel $k(x,y) = (x-y)^{\alpha}$, $\alpha \geqslant 0$ and considered all choices of parameters $p,q \in (1,\infty)$. As regards the Hardy-Steklov operator $(Tf)(x) = \int_{a(x)}^{b(x)} f(t)dt$, its one-dimensional boundedness was studied by Heinig and Sinnamon [4] while the compactness was studied in [5] by Jain and Gupta. Also, a certain higher dimensional L^p-L^q compactness with usual norm (not with mixed norm) was dealt with in [7]. For up to date information about boundedness and compactness of various Hardy-type integral operators, one may refer to the very recent monograph [9].

The equivalence of higher dimensional problems and the corresponding lower dimensional ones was initiated by Sinnamon [14], [15] who did it for studying the boundedness of Hardy and Hardy-Steklov operators in Lebesgue spaces with usual norm. Here, we show that the same phenomenon works in case of compactness also studied between Lebesgue spaces with mixed norm.

The paper is organized as follows : in order not to disturb our discussions later on, we collect certain preliminaries in Section 2 required throughout the paper. Section 3 is devoted to duality where we construct the dual space to the weighted Lebesgue space with mixed norm and consequently the adjoint operators to (1.1)-(1.4) are obtained. Finally, in Section 4, we characterize the compactness of the operators (1.1) and (1.2) in terms of, respectively, (1.3) and (1.4).

2. Preliminaries

Throughout the paper, all the functions are taken to be Lebesgue measurable. Let $\Omega \subset \mathbb{R}^N$ be a non-empty and measurable set. By a weight function (or simply a weight), we mean a function which is measurable, positive and finite a.e. on Ω. For a weight w and $1 \leqslant p < \infty$, the weighted Lebesgue space $L^p(\Omega,w)$ is defined to be the space of all measurable functions f on Ω for which

$$\|f\|_{L^p(\Omega,w)} := \left(\int_{\Omega} |f(x)|^p w(x)dx \right)^{1/p} < \infty.$$

It is known that the space $L^p(\Omega,w)$ is a Banach space and that for $1<p<\infty$, it is reflexive too.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces of real measurable functions over $\Omega_1 \subset \mathbb{R}^M$ and $\Omega_2 \subset \mathbb{R}^N$, respectively. We denote by $[X,Y]$ the space of all real functions f on $\Omega_1 \times \Omega_2$ such that for a.e. $x \in \Omega_1$

(i) the function $f(x,\cdot)$ belongs to Y;

(ii) the function $g(x) = \|f(x,\cdot)\|_Y$ belongs to X.

The space $[X,Y]$ is equipped with the mixed norm

$$\|f\|_{[X,Y]} = \|g\|_X = \|x \to \|f(x,\cdot)\|_Y\|_X .$$

The mixed normed space $[L^p(\Omega), L^q(\Omega)]$ was introduced by Benedek and Panzone [2] (see also [3]). In [1], Appell and Kufner studied the boundedness of two-dimensional Hardy operator H_2 while in [8], Jain, Jain and Gupta studied the boundedness of two dimensional Hardy-Steklov operator T_2 and the corresponding geometric mean operator in the context of Lebesgue spaces with mixed norm. Let us mention that the boundedness of H_2 between Lebesgue spaces with usual norm has been studied by Sawyer in his celebrity paper [13].

Let $\sum_M$ be the unit ball in $\mathbb{R}^M$, i.e., $\sum_M = \{x \in \mathbb{R}^M : |x| = 1\}$. Let B_M be a measurable subset of $\sum_M$ and $E \subset \mathbb{R}^M$ be the corresponding spherical cone, i.e.,

$$E = \{x \in \mathbb{R}^M : x = s\sigma,\ 0 \leqslant s < \infty,\ \sigma \in B_M\}.$$

We denote by αS_M, $\alpha > 0$, the part of E with radius $\leqslant \alpha$. In particular, $|x|S_M$ is denoted by S_{M_x}. Note that $E = \bigcup_{\alpha>0} \alpha S_M$. For $x \in E \setminus \{0\}$, we denote by $|S_{M_x}|$, the volume of S_{M_x}. The symbols B_N, F, S_{N_y} and $|S_{N_y}|$ are defined similarly for an N-dimensional setting.

Suppose, u_1, v_1 are weight functions on E and u_2, v_2 are weight functions on F. Consider the weighted Lebesgue spaces $[L^{p_1}(E, v_1), L^{p_2}(F, v_2)]$ and $[L^{q_1}(E, u_1), L^{q_2}(F, u_2)]$ with mixed norm. Now, we define the operator

$$T_{E,F} : [L^{p_1}(E, v_1), L^{p_2}(F, v_2)] \to [L^{q_1}(E, u_1), L^{q_2}(F, u_2)]$$

by

$$(T_{E,F}f)(x,y) = \int_{b(|x|)S_M \setminus a(|x|)S_M} \int_{d(|y|)S_N \setminus c(|y|)S_N} f(s,t)\,dt\,ds,\ x \in E,\ y \in F$$

where a, b, c, d are strictly increasing differentiable functions on $[0, \infty]$ satisfying

$$a(0) = b(0) = 0,\ \ a(x) < b(x)\ \text{ for }\ 0 < x < \infty,\ \ a(\infty) = b(\infty) = \infty$$

and

$$c(0) = d(0) = 0,\ \ c(x) < d(x)\ \text{ for }\ 0 < x < \infty,\ \ c(\infty) = d(\infty) = \infty.$$

In Section 4, we shall characterize the compactness of $T_{E,F}$ in terms of the two-dimensional operator

$$T_2 : [L^{p_1}((0,\infty), V_1), L^{p_2}((0,\infty), V_2)] \to [L^{q_1}((0,\infty), U_1), L^{q_2}((0,\infty), U_2)]$$

defined by

$$(T_2f)(x,y) = \int_{a(x)}^{b(x)} \int_{c(y)}^{d(y)} f(s,t)\,dt\,ds,\ \ x, y \in (0,\infty)$$

U_1, U_2, V_1, V_2 being suitable weight functions defined on $(0, \infty)$.

3. Duality

In this Section, we shall construct the dual spaces $[L^{p_1}((0,\infty),V_1),L^{p_2}((0,\infty),V_2)]^*$ and $[L^{p_1}(E,v_1),L^{p_2}(F,v_2)]^*$ of $[L^{p_1}((0,\infty),V_1),L^{p_2}((0,\infty),V_2)]$ and $[L^{p_1}(E,v_1),$ $L^{p_2}(F,v_2)]$ respectively when $1<p_1,p_2<\infty$. In this direction, we prove the following

THEOREM 3.1. *Let* $1<p_1,p_2<\infty$, $g\in[L^{p_1'}((0,\infty),V_1^{1-p_1'}),L^{p_2'}((0,\infty),V_2^{1-p_2'})]$ *and denote*

$$\langle g,f\rangle=\int_0^\infty\int_0^\infty g(x,y)f(x,y)dxdy,$$

for $f\in[L^{p_1}((0,\infty),V_1),L^{p_2}((0,\infty),V_2)]$. *Then* $f\to\langle g,f\rangle\in[L^{p_1}((0,\infty),V_1),$ $L^{p_2}((0,\infty),V_2)]^*$ *and*

$$\|g\|_{[L^{p_1'}((0,\infty),V_1^{1-p_1'}),L^{p_2'}((0,\infty),V_2^{1-p_2'})]}=\sup_{\|f\|_{[L^{p_1}((0,\infty),V_1),L^{p_2}((0,\infty),V_2)]}=1}|\langle g,f\rangle|.$$

Proof. We apply Hölder's inequality twice and obtain

$$\begin{aligned}|\langle g,f\rangle|&\leqslant\int_0^\infty\int_0^\infty|g(x,y)||f(x,y)|V_2^{\frac{1}{p_2}}(y)V_2^{\frac{-1}{p_2}}(y)dxdy\\&\leqslant\int_0^\infty\left(\int_0^\infty|f(x,y)|^{p_2}V_2(y)dy\right)^{\frac{1}{p_2}}\left(\int_0^\infty|g(x,y)|^{p_2'}V_2^{1-p_2'}(y)dy\right)^{\frac{1}{p_2'}}dx\\&=\int_0^\infty\left(\int_0^\infty|f(x,y)|^{p_2}V_2(y)dy\right)^{\frac{1}{p_2}}\left(\int_0^\infty|g(x,y)|^{p_2'}V_2^{1-p_2'}(y)dy\right)^{\frac{1}{p_2'}}\\&\qquad\times V_1^{\frac{1}{p_1}}(x)V_1^{\frac{-1}{p_1}}(x)dx\\&\leqslant\left(\int_0^\infty\left(\int_0^\infty|f(x,y)|^{p_2}V_2(y)dy\right)^{\frac{p_1}{p_2}}V_1(x)dx\right)^{\frac{1}{p_1}}\\&\qquad\times\left(\int_0^\infty\left(\int_0^\infty|g(x,y)|^{p_2'}V_2^{1-p_2'}(y)dy\right)^{\frac{p_1'}{p_2'}}V_1^{1-p_1'}(x)dx\right)^{\frac{1}{p_1'}}\\&=\|f\|_{[L^{p_1}((0,\infty),V_1),L^{p_2}((0,\infty),V_2)]}\|g\|_{[L^{p_1'}((0,\infty),V_1^{1-p_1'}),L^{p_2'}((0,\infty),V_2^{1-p_2'})]}\end{aligned}$$

and we find that for $\|f\|_{[L^{p_1}((0,\infty),V_1),L^{p_2}((0,\infty),V_2)]}=1$

$$|\langle g,f\rangle|\leqslant\|g\|_{[L^{p_1'}((0,\infty),V_1^{1-p_1'}),L^{p_2'}((0,\infty),V_2^{1-p_2'})]}.$$

In fact, the equality can be attained in the last estimate. To see this, let us take

$$\begin{aligned}f(x,y)=|g(x,y)|^{p_2'-1}sgn\,g(x,y)V_2^{1-p_2'}(y)V_1^{1-p_1'}(x)\|g(x,.)\|_{p_2',V_2^{1-p_2'}}^{p_1'-p_2'}\\\times\|g\|^{1-p_1'}_{[L^{p_1'}((0,\infty),V_1^{1-p_1'}),L^{p_2'}((0,\infty),V_2^{1-p_2'})]}.\end{aligned}$$

Then $\|f\|_{[L^{p_1}((0,\infty),V_1),L^{p_2}((0,\infty),V_2)]} = 1$ and

$$\langle g,f\rangle = \|g\|_{[L^{p_1'}((0,\infty),V_1^{1-p_1'}),L^{p_2'}((0,\infty),V_2^{1-p_2'})]}.$$

Consequently, we are done. □

Thus in view of the above theorem, the space $[L^{p_1'}((0,\infty),V_1^{1-p_1'}),L^{p_2'}((0,\infty),V_2^{1-p_2'})]$ can be identified as the dual space of $[L^{p_1}((0,\infty),V_1),L^{p_2}((0,\infty),V_2)]$.

REMARK 3.2. Theorem 3.1 extends a result of Benedek and Panzone [2] (see also [3]) who proved it for the non-weighted case, i.e., when $V_1 \equiv V_2 \equiv 1$.

It can, now, be seen that the operators T_2 and

$$T_2^*:[L^{q_1'}((0,\infty),U_1^{1-q_1'}),L^{q_2'}((0,\infty),U_2^{1-q_2'})] \to [L^{p_1'}((0,\infty),V_1^{1-p_1'}),L^{p_2'}((0,\infty),V_2^{1-p_2'})]$$

defined by

$$(T_2^*g)(x,y) = \int_{b^{-1}(x)}^{a^{-1}(x)} \int_{d^{-1}(y)}^{c^{-1}(y)} g(s,t)\,dt\,ds, \quad x,y \in (0,\infty)$$

are mutually conjugate. Indeed, by applying Fubini's theorem, one can see that

$$\langle g, T_2 f\rangle = \langle f, T_2^* g\rangle.$$

Similarly, it can be shown that the dual space of $[L^{p_1}(E,v_1),L^{p_2}(F,v_2)]$ can be identified with the space $[L^{p_1'}(E,v_1^{1-p_1'}),L^{p_2'}(F,v_2^{1-p_2'})]$ and the operators $T_{E,F}$ and

$$T_{E,F}^* : [L^{q_1'}(E,u_1^{1-q_1'}),L^{q_2'}(F,u_2^{1-q_2'})] \to [L^{p_1'}(E,v_1^{1-p_1'}),L^{p_2'}(F,v_2^{1-p_2'})]$$

defined by

$$(T_{E,F}^*g)(x,y) = \int_{a^{-1}(|x|)S_M \setminus b^{-1}(|x|)S_M} \int_{c^{-1}(|y|)S_N \setminus d^{-1}(|y|)S_N} g(s,t)\,dt\,ds, \quad x \in E,\ y \in F$$

are also mutually conjugate.

4. Compactness

Let X be a normed linear space and X^* denote its conjugate space. We say that a sequence $\{x_n\}$ in X is strongly convergent (or simply convergent) to $x \in X$, written $x_n \to x$, if $\|x_n - x\| \to 0$ as $n \to \infty$. A sequence $\{x_n\}$ in X is said to converge weakly to $x \in X$, written $x_n \xrightarrow{w} x$, if $f(x_n) \to f(x)$, for each $f \in X^*$. A sequence $\{f_n\}$ in X^* is said to be weak* convergent to $f \in X^*$, written $f_n \xrightarrow{w^*} f$, if $f_n(x) \to f(x)$ for each $x \in X$. Note that the strong convergence implies the weak convergence which in turn implies the weak* convergence. The implications in the reverse direction do not hold in general. However, if X is a reflexive space then the weak* convergence implies the weak convergence.

We require certain well known assertions which we collect in the following theorem:

THEOREM A. *Let X and Y be Banach spaces.*

(a) *A bounded linear operator $A : X \to Y$ is compact if and only if its conjugate $A^* : Y^* \to X^*$ is compact.*

(b) *If $A : X \to Y$ is compact and $\{x_n\}$ is a sequence in X such that $x_n \xrightarrow{w} x$, for some $x \in X$, then $Ax_n \to Ax$.*

(c) *An operator $A : X \to Y$ is compact if $A^* : Y^* \to X^*$ is weak*-norm sequentially continuous, i.e., for each sequence $\{f_n\}$ in Y^* with $\{f_n\} \xrightarrow{w^*} f$, for some $f \in Y^*$, we have $\|A^* f_n - A^* f\| \to 0$.*

We, now, prove the following equivalence :

THEOREM 4.1. *Let $1 < p_1,\ p_2, q_1, q_2 < \infty$, u_1, v_1 be weight functions on E and u_2, v_2 be weight functions on F. Then the operator*

$$T_{E,F} : [L^{p_1}(E, v_1), L^{p_2}(F, v_2)] \to [L^{q_1}(E, u_1), L^{q_2}(F, u_2)]$$

is compact if and only if the operator

$$T_2 : [L^{p_1}((0,\infty), V_1), L^{p_2}((0,\infty), V_2)] \to [L^{q_1}((0,\infty), U_1), L^{q_2}((0,\infty), U_2)]$$

is compact with

$$U_1(x_0) = \int_{B_M} u_1(x_0 x') x_0^{M-1} dx', \quad x_0 > 0 \tag{4.1}$$

$$U_2(y_0) = \int_{B_N} u_2(y_0 y') y_0^{N-1} dy', \quad y_0 > 0 \tag{4.2}$$

$$V_1(x_0) = \left(\int_{B_M} v_1^{1-p_1'}(x_0 x') x_0^{M-1} dx' \right)^{1-p_1}, \quad x_0 > 0 \tag{4.3}$$

$$V_2(y_0) = \left(\int_{B_N} v_2^{1-p_2'}(y_0 y') y_0^{N-1} dy' \right)^{1-p_2}, \quad y_0 > 0. \tag{4.4}$$

Proof. First assume that T_2 is compact. It suffices to show that

$$T_{E,F}^* : [L^{q_1'}(E, u_1^{1-q_1'}), L^{q_2'}(F, u_2^{1-q_2'})] \to [L^{p_1'}(E, v_1^{1-p_1'}), L^{p_2'}(F, v_2^{1-p_2'})]$$

is weak*-norm sequentially continuous. Let $\{f_n\}$ be a sequence in $[L^{q_1'}(E, u_1^{1-q_1'}),\ L^{q_2'}(F, u_2^{1-q_2'})]$ such that $f_n \xrightarrow{w^*} 0$. Without any loss of generality, we may assume that each f_n is non-negative. Define

$$g_n(x_0, y_0) = \int_{B_M} \int_{B_N} f_n(x_0 x', y_0 y') x_0^{M-1} y_0^{N-1} dy' dx', \quad x_0, y_0 > 0,\ n \in \mathbb{N}. \tag{4.5}$$

Then using Hölder's inequality

$$g_n(x_0,y_0) \leqslant \int_{B_M}\left(\int_{B_N} f_n^{q_2'}(x_0x',y_0y')u_2^{1-q_2'}(y_0y')y_0^{N-1}dy'\right)^{\frac{1}{q_2'}} \times\left(\int_{B_N} u_2(y_0y')y_0^{N-1}dy'\right)^{\frac{1}{q_2}} x_0^{M-1}dx'.$$

Therefore, using (4.2), applying Minkowskii's integral inequality, Hölder's inequality and using (4.1) we have

$$\begin{aligned}
&\left(\int_0^\infty U_1^{1-q_1'}(x_0)\left(\int_0^\infty U_2^{1-q_2'}(y_0)g_n^{q_2'}(x_0,y_0)dy_0\right)^{\frac{q_1'}{q_2'}}dx_0\right)^{\frac{1}{q_1'}}\\
&\leqslant\left(\int_0^\infty U_1^{1-q_1'}(x_0)\left(\int_0^\infty U_2^{1-q_2'}(y_0)\left(\int_{B_M}\left(\int_{B_N} f_n^{q_2'}(x_0x',y_0y')u_2^{1-q_2'}(y_0y')y_0^{N-1}dy'\right)^{\frac{1}{q_2'}}\right.\right.\right.\\
&\quad\left.\left.\left.\times U_2^{\frac{1}{q_2}}(y_0)x_0^{M-1}dx'\right)^{q_2'}dy_0\right)^{\frac{q_1'}{q_2'}}dx_0\right)^{\frac{1}{q_1'}}\\
&=\left(\int_0^\infty U_1^{1-q_1'}(x_0)\left(\int_0^\infty\left(\int_{B_M}\left(\int_{B_N} f_n^{q_2'}(x_0x',y_0y')u_2^{1-q_2'}(y_0y')y_0^{N-1}dy'\right)^{\frac{1}{q_2'}}\right.\right.\right.\\
&\quad\left.\left.\left.\times x_0^{M-1}dx'\right)^{q_2'}dy_0\right)^{\frac{q_1'}{q_2'}}dx_0\right)^{\frac{1}{q_1'}}\\
&\leqslant\left(\int_0^\infty U_1^{1-q_1'}(x_0)\left(\int_{B_M}\left(\int_0^\infty\int_{B_N} f_n^{q_2'}(x_0x',y_0y')u_2^{1-q_2'}(y_0y')y_0^{N-1}dy'dy_0\right)^{\frac{1}{q_2'}}\right.\right.\\
&\quad\left.\left.\times x_0^{M-1}dx'\right)^{q_1'}dx_0\right)^{\frac{1}{q_1'}}\\
&\leqslant\left(\int_0^\infty U_1^{1-q_1'}(x_0)\left(\int_{B_M}\left(\int_F f_n^{q_2'}(x_0x',y)u_2^{1-q_2'}(y)dy\right)^{\frac{q_1'}{q_2'}}u_1^{1-q_1'}(x_0x')x_0^{M-1}dx'\right)\right.\\
&\quad\left.\times\left(\int_{B_M} u_1(x_0x')x_0^{M-1}dx'\right)^{\frac{q_1'}{q_1}}dx_0\right)^{\frac{1}{q_1'}}\\
&=\left(\int_E\left(\int_F f_n^{q_2'}(x,y)u_2^{1-q_2'}(y)dy\right)^{\frac{q_1'}{q_2'}}u_1^{1-q_1'}(x)dx\right)^{\frac{1}{q_1'}}<\infty,
\end{aligned}$$

which shows that $\{g_n\}$ is a sequence in $[L^{q_1'}((0,\infty),U_1^{1-q_1'}),L^{q_2'}((0,\infty),U_2^{1-q_2'})]$. Next, we note that if $g\in[L^{q_1}((0,\infty),U_1),L^{q_2}((0,\infty),U_2)]$ and $f:E\times F\to\mathbb{R}$ is defined by

$$f(x,y)=g(x_0,y_0),\qquad x=x_0x',\ y=y_0y'$$

then $f \in [L^{q_1}(E,u_1), L^{q_2}(F,u_2)]$ since by making change of variable and using (4.2), (4.1), we have

$$\begin{aligned}
&\left(\int_E \left(\int_F f^{q_2}(x,y)u_2(y)dy\right)^{\frac{q_1}{q_2}} u_1(x)dx\right)^{\frac{1}{q_1}} \\
&\quad = \left(\int_0^\infty \int_{B_M} u_1(x_0x')\left(\int_0^\infty \int_{B_N} f^{q_2}(x_0x', y_0y')u_2(y_0y')y_0^{N-1}dy'dy_0\right)^{\frac{q_1}{q_2}} x_0^{M-1}dx'dx_0\right)^{\frac{1}{q_1}} \\
&\quad = \left(\int_0^\infty \int_{B_M} u_1(x_0x')\left(\int_0^\infty g^{q_2}(x_0,y_0)U_2(y_0)dy_0\right)^{\frac{q_1}{q_2}} x_0^{M-1}dx'dx_0\right)^{\frac{1}{q_1}} \\
&\quad = \left(\int_0^\infty U_1(x_0)\left(\int_0^\infty g^{q_2}(x_0,y_0)U_2(y_0)dy_0\right)^{\frac{q_1}{q_2}} dx_0\right)^{\frac{1}{q_1}} < \infty .
\end{aligned}$$

Then, by using (4.5), we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty g_n(x_0,y_0)g(x_0,y_0)dy_0dx_0 \\
&\quad = \int_0^\infty \int_0^\infty \left(\int_{B_M}\int_{B_N} f_n(x_0x', y_0y')x_0^{M-1}y_0^{N-1}dy'dx'\right)g(x_0,y_0)dy_0dx_0 \\
&\quad = \int_0^\infty \int_{B_M}\left(\int_0^\infty \int_{B_N} f_n(x_0x', y_0y')f(x_0x', y_0y')y_0^{N-1}dy'dy_0\right)x_0^{M-1}dx'dx_0 \\
&\quad = \int_E\int_F f_n(x,y)f(x,y)dy\,dx \to 0 \ \text{ as } \ n\to\infty
\end{aligned}$$

i.e., $g_n \xrightarrow{w} 0$. Further, since T_2 is compact, in view of Theorem A((a) and (b)), T_2^* is also compact and therefore

$$\|T_2^* g_n\|_{[L^{p_1'}((0,\infty),V_1^{1-p_1'}),L^{p_2'}((0,\infty),V_2^{1-p_2'})]} \to 0 \ \text{ as } \ n\to\infty .$$

Now, using (4.3), (4.4) and (4.5), we get

$$\begin{aligned}
&\|T_{E,F}^* f_n\|_{[L^{p_1'}(E,v_1^{1-p_1'}),L^{p_2'}(F,v_2^{1-p_2'})]} \\
&\quad = \left(\int_E v_1^{1-p_1'}(x)\left(\int_F v_2^{1-p_2'}(y)\left(\int_{a^{-1}(|x|)S_M\setminus b^{-1}(|x|)S_M}\right.\right.\right. \\
&\qquad\qquad \left.\left.\left.\times\int_{c^{-1}(|y|)S_N\setminus d^{-1}(|y|)S_N} f_n(s,t)dt\,ds\right)^{p_2'} dy\right)^{\frac{p_1'}{p_2'}} dx\right)^{\frac{1}{p_1'}} \\
&\quad = \left(\int_0^\infty \int_{B_M} v_1^{1-p_1'}(x_0x')\left(\int_0^\infty \int_{B_N} v_2^{1-p_2'}(y_0y')\right.\right. \\
&\qquad\qquad \left.\left.\times\left(\int_{b^{-1}(x_0)}^{a^{-1}(x_0)}\int_{d^{-1}(y_0)}^{c^{-1}(y_0)}\int_{B_M}\int_{B_N} f_n(s_0s', t_0t')t_0^{N-1}s_0^{M-1}dt'\,ds'\,dt_0\,ds_0\right)^{p_2'}\right.\right.
\end{aligned}$$

$$\times y_0^{N-1}dy'dy_0\Big)^{\frac{p_1'}{p_2'}}x_0^{M-1}dx'dx_0\Big)^{\frac{1}{p_1'}}$$

$$= \left(\int_0^\infty V_1^{1-p_1'}(x_0)\left(\int_0^\infty V_2^{1-p_2'}(y_0)\left(\int_{b^{-1}(x_0)}^{a^{-1}(x_0)}\int_{d^{-1}(y_0)}^{c^{-1}(y_0)} g_n(s_0,t_0)dt_0ds_0\right)^{p_2'}dy_0\right)^{\frac{p_1'}{p_2'}}dx_0\right)^{\frac{1}{p_1'}}$$

$$= \|T_2^*g_n\|_{[L^{p_1'}((0,\infty),V_1^{1-p_1'}),L^{p_2'}((0,\infty),V_2^{1-p_2'})]}$$

and $T_{E,F}$ is compact.

Conversely, assume that

$$T_{E,F} : [L^{p_1}(E,v_1),L^{p_2}(F,v_2)] \to [L^{q_1}(E,u_1),L^{q_2}(F,u_2)]$$

is compact. Let $\{g_n\}$ be a sequence in $[L^{q_1'}((0,\infty),U_1^{1-q_1'}),L^{q_2'}((0,\infty),U_2^{1-q_2'})]$ such that $g_n \xrightarrow{w^*} 0$. Without any loss of generality, assume that each g_n is non-negative. Using the polar coordinates, as before, define

$$f_n(x_0x',y_0y') = g_n(x_0,y_0)u_2(y_0y')U_2^{-1}(y_0)u_1(x_0x')U_1^{-1}(x_0)\,, \tag{4.6}$$

Then

$$\int_{B_M}\int_{B_N} f_n(x_0x',y_0y')x_0^{M-1}y_0^{N-1}dy'dx' = g_n(x_0,y_0) \tag{4.7}$$

and consequently using (4.6), (4.1) and (4.2), we get

$$\left(\int_E\left(\int_F f_n^{q_2'}(x,y)u_2^{1-q_2'}(y)dy\right)^{\frac{q_1'}{q_2'}}u_1^{1-q_1'}(x)dx\right)^{\frac{1}{q_1'}}$$

$$= \left(\int_0^\infty\int_{B_M}\left(\int_0^\infty g_n^{q_2'}(x_0,y_0)\left(\int_{B_N}u_2(y_0y')y_0^{N-1}dy'\right)U_2^{-q_2'}(y_0)dy_0\right)^{\frac{q_1'}{q_2'}}\right.$$

$$\left.\times u_1(x_0x')x_0^{M-1}dx'U_1^{-q_1'}(x_0)dx_0\right)^{\frac{1}{q_1'}}$$

$$= \left(\int_0^\infty\left(\int_0^\infty g_n^{q_2'}(x_0,y_0)U_2^{1-q_2'}(y_0)dy_0\right)^{\frac{q_1'}{q_2'}}U_1^{1-q_1'}(x_0)dx_0\right)^{\frac{1}{q_1'}}$$

$$< \infty\,,$$

i.e., $\{f_n\}$ is a sequence in $[L^{q_1'}(E,u_1^{1-q_1'}),L^{q_2'}(F,u_2^{1-q_2'})]$.

Thus, (4.3), (4.4) and (4.7) yield

$$\|T_2^*g_n\|_{[L^{p_1'}((0,\infty),V_1^{1-p_1'}),L^{p_2'}((0,\infty),V_2^{1-p_2'})]} = \|T_{E,F}^*f_n\|_{[L^{p_1'}(E,v_1^{1-p_1'}),L^{p_2'}(F,v_2^{1-p_2'})]}.$$

We, now, show that $f_n \xrightarrow{w} 0$. For any function $f \in [L^{q_1}(E,u_1), L^{q_2}(F,u_2)]$, using (4.6), we have

$$\begin{aligned}
&\int_E \int_F f_n(x,y) f(x,y)\,dy\,dx \\
&\quad = \int_0^\infty \int_{B_M} \int_0^\infty \int_{B_N} g_n(x_0,y_0) u_2(y_0 y') U_2^{-1}(y_0) u_1(x_0 x') U_1^{-1}(x_0) \\
&\qquad \times f(x_0 x', y_0 y') y_0^{N-1} x_0^{M-1} dy'\,dx'\,dy_0\,dx_0 \\
&\quad = \int_0^\infty \int_0^\infty g_n(x_0,y_0) \left(\int_{B_M} \int_{B_N} u_2(y_0 y') u_1(x_0 x') f(x_0 x', y_0 y') y_0^{N-1} x_0^{M-1} dy'\,dx' \right) \\
&\qquad \times U_2^{-1}(y_0) U_1^{-1}(x_0)\,dy_0\,dx_0 \\
&\quad = \int_0^\infty \int_0^\infty g_n(x_0,y_0) g(x_0,y_0)\,dy_0\,dx_0 \to 0 \ \text{ as } \ n \to \infty,
\end{aligned}$$

where

$$g(x_0,y_0) = \left(\int_{B_M} \int_{B_N} u_2(y_0 y') u_1(x_0 x') f(x_0 x', y_0 y') y_0^{N-1} x_0^{M-1} dy'\,dx' \right) U_2^{-1}(y_0) U_1^{-1}(x_0), \quad x_0, y_0 > 0$$

and it can be verified that $g \in [L^{q_1}((0,\infty),U_1), L^{q_2}((0,\infty),U_2)]$. Indeed, using Hölder's inequality, (4.2), Minkowski's integral inequality, again Hölder's inequality and (4.1), we have

$$\begin{aligned}
&\left(\int_0^\infty \left(\int_0^\infty g^{q_2}(x_0,y_0) U_2(y_0)\,dy_0 \right)^{\frac{q_1}{q_2}} U_1(x_0)\,dx_0 \right)^{\frac{1}{q_1}} \\
&\qquad = \left(\int_0^\infty \left(\int_0^\infty \left(\int_{B_M} \int_{B_N} u_2(y_0 y') f(x_0 x', y_0 y') y_0^{N-1} dy'\, u_1(x_0 x') x_0^{M-1} dx' \right)^{q_2} \right.\right. \\
&\qquad\qquad \left.\left. \times U_2^{1-q_2}(y_0)\,dy_0 \right)^{\frac{q_1}{q_2}} U_1^{1-q_1}(x_0)\,dx_0 \right)^{\frac{1}{q_1}} \\
&\quad \leqslant \left(\int_0^\infty \left(\int_0^\infty \left(\int_{B_M} \left(\int_{B_N} f^{q_2}(x_0 x', y_0 y') u_2(y_0 y') y_0^{N-1} dy' \right)^{\frac{1}{q_2}} \right.\right.\right. \\
&\qquad\qquad \left. \times \left(\int_{B_N} u_2(y_0 y') y_0^{N-1} dy' \right)^{\frac{1}{q_2'}} u_1(x_0 x') x_0^{M-1} dx' \right)^{q_2} \\
&\qquad\qquad \left.\left. \times U_2^{1-q_2}(y_0)\,dy_0 \right)^{\frac{q_1}{q_2}} U_1^{1-q_1}(x_0)\,dx_0 \right)^{\frac{1}{q_1}} \\
&\quad \leqslant \left(\int_0^\infty \left(\int_{B_M} \left(\int_0^\infty \int_{B_N} f^{q_2}(x_0 x', y_0 y') u_2(y_0 y') y_0^{N-1} dy'\,dy_0 \right)^{\frac{1}{q_2}} \right.\right. \\
&\qquad\qquad \left.\left. \times u_1(x_0 x') x_0^{M-1} dx' \right)^{q_1} U_1^{1-q_1}(x_0)\,dx_0 \right)^{\frac{1}{q_1}}
\end{aligned}$$

$$\leqslant \left(\int_0^\infty \left(\int_{B_M}\left(\int_F f^{q_2}(x_0x',y)u_2(y)dy\right)^{\frac{q_1}{q_2}} u_1(x_0x')x_0^{M-1}dx'\right)\right.$$
$$\left.\times\left(\int_{B_M} u_1(x_0x')x_0^{M-1}dx'\right)^{\frac{q_1}{q_1'}} U_1^{1-q_1}(x_0)dx_0\right)^{\frac{1}{q_1}}$$
$$= \left(\int_E\left(\int_F f^{q_2}(x,y)u_2(y)dy\right)^{\frac{q_1}{q_2}} u_1(x)dx\right)^{\frac{1}{q_1}} < \infty.$$

Now, as $T_{E,F}$ is compact, using Theorem A((a) and (b)), $T^*_{E,F}$ is also compact and therefore $\|T^*_{E,F}f_n\|_{[L^{p_1'}(E,v_1^{1-p_1'}),L^{p_2'}(F,v_2^{1-p_2'})]}$ and hence $\|T_2^*g_n\|_{[L^{p_1'}((0,\infty),V_1^{1-p_1'}),L^{p_2'}((0,\infty),V_2^{1-p_2'})]}$ converges to 0 as $n\to\infty$. Thus T_2 is compact, by using Theorem A(c). □

REMARK 4.2. Theorem 4.1 establishes the compactness of the operators $T_{E,F}$ and T_2 in terms of each other. In order that this result makes sense, we need to ensure the compactness of at least one of these operators. To this end, let us mention that precise weight conditions are known (see [6, Corollary 5]) for the compactness of $T_{E,F}$. Using that result and Theorem 4.1, we can obtain the compactness conditions for the operator T_2. Precisely, the sufficient conditions are stated in the following (the necessary conditions can be written similarly)

THEOREM 4.3. *Let* $1 < p_i, q_i < \infty$, U_i, V_i, $i = 1,2$ *be weight functions on* $(0,\infty)$. *Then the operator* $T_2 : [L^{p_1}((0,\infty),V_1), L^{p_2}((0,\infty),V_2)] \to [L^{q_1}((0,\infty),U_1), L^{q_2}((0,\infty),U_2)]$ *is compact in the following cases*

Case (a)*:* $p_1\leqslant q_1$, $p_2\leqslant q_2$ *and the following conditions are satisfied*

$$\sup_{\substack{0<t\leqslant x\\ a(x)\leqslant b(t)}} B_1(x,t) = \sup_{\substack{0<t\leqslant x\\ a(x)\leqslant b(t)}} \left(\int_t^x U_1\right)^{\frac{1}{q_1}}\left(\int_{a(x)}^{b(t)} V_1^{1-p_1'}\right)^{\frac{1}{p_1'}} < \infty \tag{4.8}$$

$$\sup_{\substack{0<t\leqslant x\\ c(x)\leqslant d(t)}} B_2(x,t) = \sup_{\substack{0<t\leqslant x\\ c(x)\leqslant d(t)}} \left(\int_t^x U_2\right)^{\frac{1}{q_2}}\left(\int_{c(x)}^{d(t)} V_2^{1-p_2'}\right)^{\frac{1}{p_2'}} < \infty \tag{4.9}$$

$$\lim_{x\to t+} B_1(x,t) = \lim_{x\to a^{-1}(b(t))-} B_1(x,t) = 0 \quad \text{for every} \quad t>0 \tag{4.10}$$

$$\lim_{t\to x-} B_1(x,t) = \lim_{t\to b^{-1}(a(x))+} B_1(x,t) = 0 \quad \text{for every} \quad x>0 \tag{4.11}$$

$$\lim_{x\to t+} B_2(x,t) = \lim_{x\to c^{-1}(d(t))-} B_2(x,t) = 0 \quad \text{for every} \quad t>0 \tag{4.12}$$

and

$$\lim_{t\to x-} B_2(x,t) = \lim_{t\to d^{-1}(c(x))+} B_2(x,t) = 0 \quad \text{for every} \quad x>0 \tag{4.13}$$

Case (b)*:* $p_1 \leqslant q_1$, $q_2 < p_2$, (4.8), (4.10), (4.11) *and the following conditions are satisfied*

$$\left(\int_0^\infty \left(\int_{d^{-1}(c(t))}^t \left(\int_{c(t)}^{d(x)} V_2^{1-p_2'}\right)^{\frac{r_2}{p_2'}} \left(\int_x^t U_2\right)^{\frac{r_2}{p_2}} U_2(x)\,dx\right) \rho(t)\,dt\right)^{\frac{1}{r_2}} < \infty \quad (4.14)$$

$$\left(\int_0^\infty \left(\int_t^{c^{-1}(d(t))} \left(\int_{c(x)}^{d(t)} V_2^{1-p_2'}\right)^{\frac{r_2}{p_2'}} \left(\int_t^x U_2\right)^{\frac{r_2}{p_2}} U_2(x)\,dx\right) \rho(t)\,dt\right)^{\frac{1}{r_2}} < \infty \quad (4.15)$$

with $\dfrac{1}{r_2} = \dfrac{1}{q_2} - \dfrac{1}{p_2}$ *and the function* ρ *is defined as follows : Fix* $M_0 = d^{-1}(1)$ *and define* $M_{k+1} = c^{-1}(d(M_k))$ *if* $k \geqslant 0$ *and* $M_k = d^{-1}(c(M_{k+1}))$ *if* $k < 0$. *The function* ρ *is then defined by*

$$\rho(t) := \sum_{k\in\mathbb{Z}} \chi_{(M_k,M_{k+1})}(t)\frac{d}{dt}(d^{-1}\circ c)^k(t),$$

where $(d^{-1}\circ c)^k$ *denotes* k *times repeated composition.*
Case (c)*:* $q_1 < p_1$, $p_2 \leqslant q_2$, (4.9), (4.12), (4.13) *and the following conditions are satisfied*

$$\left(\int_0^\infty \left(\int_{b^{-1}(a(t))}^t \left(\int_{a(t)}^{b(x)} V_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}} \left(\int_x^t U_1\right)^{\frac{r_1}{p_1}} U_1(x)\,dx\right) \rho'(t)\,dt\right)^{\frac{1}{r_1}} < \infty \quad (4.16)$$

$$\left(\int_0^\infty \left(\int_t^{a^{-1}(b(t))} \left(\int_{a(x)}^{b(t)} V_1^{1-p_1'}\right)^{\frac{r_1}{p_1'}} \left(\int_t^x U_1\right)^{\frac{r_1}{p_1}} U_1(x)\,dx\right) \rho'(t)\,dt\right)^{\frac{1}{r_1}} < \infty \quad (4.17)$$

with $\dfrac{1}{r_1} = \dfrac{1}{q_1} - \dfrac{1}{p_1}$ *and the function* ρ' *is defined as follows : Fix* $N_0 = b^{-1}(1)$ *and define* $N_{k+1} = a^{-1}(b(N_k))$ *if* $k \geqslant 0$ *and* $N_k = b^{-1}(a(N_{k+1}))$ *if* $k < 0$. *The function* ρ' *is then defined by*

$$\rho'(t) := \sum_{k\in\mathbb{Z}} \chi_{(N_k,N_{k+1})}(t)\frac{d}{dt}(b^{-1}\circ a)^k(t),$$

where $(b^{-1}\circ a)^k$ *denotes* k *times repeated composition.*
Case (d)*:* $q_1 < p_1$, $q_2 < p_2$ *and the conditions* (4.14)-(4.17) *are satisfied.*

The operator $H_{E,F}$, defined in (1.2), can be studied analogously. Its dual $H_{E,F}^*$ is obtained as done in Section 3 for $T_{E,F}^*$. Also the result corresponding to Theorem 4.1 (and also to Theorem 4.3) can be obtained on parallel lines with some obvious modifications and therefore we only state the result.

THEOREM 4.4. *Let* $1 < p_1, p_2, q_1, q_2 < \infty$, u_1, v_1 *be weight functions on* E *and* u_2, v_2 *be weight functions on* F. *Then the operator* $H_{E,F} : [L^{p_1}(E, v_1), L^{p_2}(F, v_2)] \to [L^{q_1}(E, u_1), L^{q_2}(F, u_2)]$ *is compact if and only if* $H_2 : [L^{p_1}((0,\infty), V_1), L^{p_2}((0,\infty), V_2)] \to [L^{q_1}((0,\infty), U_1), L^{q_2}((0,\infty), U_2)]$ *is compact with* U_1, U_2, V_1, V_2 *given respectively by* (4.1), (4.2), (4.3) *and* (4.4).

REMARK 4.5. The operators $T_{E,F}$ and $H_{E,F}$ studied in this paper are considered over multidimensional cones. As a special case, one can derive operators taken over multidimensional balls. Precisely, if $E = \mathbb{R}^M, F = \mathbb{R}^N$, then $S_{M_x} = B_M(x)$ and $S_{N_y} = B_N(y)$, where $B_M(x)$ and $B_N(y)$ are the balls in $\mathbb{R}^M$ and $\mathbb{R}^N$, respectively, centered at the origin with radii $|x|$ and $|y|$, respectively. In that case, the operators $T_{E,F}$ and $H_{E,F}$ become, respectively

$$(T_{M,N})(x,y) = \int_{a(|x|)\leqslant|s|\leqslant b(|x|)} \int_{c(|x|)\leqslant|t|\leqslant d(|x|)} f(s,t)dtds$$

and

$$(H_{M,N})(x,y) = \int_{B_M(x)} \int_{B_N(y)} f(s,t)dtds.$$

All the results given in this paper can easily be translated for these operators $T_{M,N}$ and $H_{M,N}$.

Acknowledgement. The third author acknowledges with thanks the Department of Science and Technology (DST), New Delhi for the research grant no. SR/S4/MS:213/03.

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(Received December 26, 2006)

Alberto Fiorenza
Dipartimento di Costruzioni e Metodi Matematici in Architettura
Universitá di Napoli
via Monteoliveto, 3
I-80134 Napoli
Italy

Istituto per le Applicazioni del Calcolo "Mauro Picone"
sezione di Napoli
Consiglio Nazionale delle Ricerche
via Pietro Castellino, 111
I-80131 Napoli
Italy
e-mail: fiorenza@unina.it

Babita Gupta
Department of Mathematics
Shivaji College (University of Delhi)
Raja Garden
Delhi 110 027
India
e-mail: babita.gupta@hotmail.com

Pankaj Jain
Department of Mathematics
Deshbandhu College (University of Delhi)
Kalkaji
New Delhi 110 019
India
e-mail: pankajkrjain@hotmail.com