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ON $\mathscr{I}$–CAUCHY SEQUENCES IN 2–NORMED SPACES

MEHMET GÜRDAL AND IŞIL AÇIK

(*communicated by H. Srivastava*)

Abstract. The concept of $\mathscr{I}$-convergence is a generalization of statistical convergence and it is depended on the notion of the ideal $\mathscr{I}$ of subsets of the set $\mathbb{N}$ of positive integers. In this paper for sequences in 2-normed space the relationship between $\mathscr{I}$-convergence and usual convergence along a filter $\mathscr{F}(\mathscr{I})$ associated with an admissible ideal $\mathscr{I}$ with property (AP) is investigated. We introduce the concepts $\mathscr{I}$-Cauchy and $\mathscr{I}^*$-Cauchy sequences in 2-normed spaces and study their certain properties.

1. Introduction and Background

The notion of ideal convergence was first introduced by P. Kostyrko et al [8] as an interesting generalization of statistical convergence [1, 14].

The concept of 2-normed spaces was initially introduced by Gähler [4] in the 1960s. Since then, this concept has been studied by many authors (see for instance [5, 13]).

In this paper we investigate basic properties of $\mathscr{I}$-convergent sequences in 2-normed spaces. In section 2 we introduce the concepts of $\mathscr{I}$-Cauchy and $\mathscr{I}^*$-Cauchy sequences in 2-normed spaces and study their some properties.

Throughout this paper $\mathbb{N}$ will denote the set of positive integers. Let $(X, \|.\|)$ be a normed space. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is said to be statistically convergent to $x \in X$ if the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x, z\| \geqslant \varepsilon\}$ has natural density zero for each $\varepsilon > 0$.

A family $\mathscr{I} \subset 2^Y$ subsets of a nonempty set Y is said to be an ideal in Y if **(i)** $\emptyset \in \mathscr{I}$; **(ii)** $A, B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$; **(iii)** $A \in \mathscr{I}$, $B \subset A$ imply $B \in \mathscr{I}$, while an admissible ideal $\mathscr{I}$ of Y further satisfies $\{x\} \in \mathscr{I}$ for each $x \in Y$ [7, 10]. Let $\mathscr{I}$ be a proper ideal in Y (i.e. $Y \notin \mathscr{I}$), $Y \neq \emptyset$. Then the family of sets $\mathscr{F}(\mathscr{I}) = \{M \subset Y : \exists A \in \mathscr{I} : M = Y \backslash A\}$ is a filter in Y. It is called the filter associated with the ideal $\mathscr{I}$ [9].

Given $\mathscr{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$, the sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathscr{I}$-convergent to $x \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geqslant \varepsilon\}$ belongs to $\mathscr{I}$ [8, 9]. There are many examples of ideals $\mathscr{I} \subset 2^{\mathbb{N}}$ in [8, 9] and basic properties of $\mathscr{I}$-convergence have been studied in these works.

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An admissible ideal $\mathscr{I} \subset 2^{\mathbb{N}}$ is said to have the property (AP) if for any sequence $\{A_1, A_2, ...\}$ of mutually disjoint sets of $\mathscr{I}$ there is a sequence $\{B_1, B_2, ...\}$ of sets such that each symmetric difference $A_i \Delta B_i$ $(i = 1, 2, ...)$ is finite and $B = \bigcup_{i=1}^{\infty} B_i \in \mathscr{I}$, [8]. This definition is similar to the condition (APO) used in [2].

Let X be a real vector space of dimension d, where $2 \leqslant d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to R$ which satisfies (**i**) $\|x, y\| = 0$ if and only if x and y are linearly dependent; (**ii**) $\|x, y\| = \|y, x\|$; (**iii**) $\|\alpha x, y\| = |\alpha| \, \|x, y\|$, $\alpha \in \mathbb{R}$; (**iv**) $\|x, y + z\| \leqslant \|x, y\| + \|x, z\|$. The pair $(X, \|.,.\|)$ is then called a 2-normed space [4]. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|, \quad x = (x_1, x_2), \quad y = (y_1, y_2).$$

2. Some results of $\mathscr{I}$-Convergence in 2-normed space

In paper [8] it is proved that $\mathscr{I}$-convergence and $\mathscr{I}^*$-convergence are equivalent for admissible ideals with property (AP). The following Lemma 1 [[11], Lemma 4] regards to prove this fact by another way in 2-normed space $(X, \|.,.\|)$.

LEMMA 1. [11]. *Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N}$ such that $P_i \in \mathscr{F}(\mathscr{I})$ for each i, where $\mathscr{F}(\mathscr{I})$ is a filter associated by an admissible ideal $\mathscr{I}$ with property (AP). Then there is a set $P \subset \mathbb{N}$ such that $P \in \mathscr{F}(\mathscr{I})$ and the set $P \backslash P_i$ is finite for all i.*

Throughout the paper we assume X to be a 2-normed space having dimension d, where $2 \leqslant d < \infty$.

DEFINITION 1. Let $\mathscr{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. A sequence (x_n) of X is said to be $\mathscr{I}$-convergent to x, if for each $\varepsilon > 0$ and nonzero z in X the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x, z\| \geqslant \varepsilon\}$ belongs to $\mathscr{I}$.

If (x_n) is $\mathscr{I}$-convergent to x then we write $\mathscr{I}\text{-}\lim_{n\to\infty} \|x_n - x, z\| = 0$ or $\mathscr{I}\text{-}\lim_{n\to\infty} \|x_n, z\| = \|x, z\|$ for each nonzero z in X. The vector x is $\mathscr{I}$-limit of the sequence (x_n).

Further we will give some examples of ideals and corresponding $\mathscr{I}$-convergences.

(I) Let $\mathscr{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathscr{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathscr{I}_f$ convergence coincides with usual convergence [4].

(II) Put $\mathscr{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}$. Then $\mathscr{I}_\delta$ is an admissible ideal in $\mathbb{N}$ and $\mathscr{I}_\delta$ convergence coincides with the statistical convergence in [6].

Let $\mathscr{I} \subset 2^{\mathbb{N}}$ be an admissible ideal with property (AP) and $(X, \|.,.\|)$ be an arbitrary 2-normed space.

LEMMA 2. *If a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ of elements of X is $\mathscr{I}$-convergent to $\xi \in X$ then there exists a set $P \in \mathscr{F}(\mathscr{I})$, $P = \{p_1 < p_2 < ... < p_k < ...\}$ such that $\lim_{k\to\infty} \|x_{p_k} - \xi, z\| = 0$ for each nonzero z in X.*

Proof. Let $\mathscr{I}\text{-}\lim_{n\to\infty} \|x_n - \xi, z\| = 0$ for each nonzero z in X. Then by definition the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - \xi, z\| \geqslant \varepsilon\}$ belongs to $\mathscr{I}$ for every $\varepsilon > 0$ and nonzero z in X. Define $P_i = \left\{n : \|x_n - \xi, z\| < \frac{1}{i}\right\}$ for each $i \in \mathbb{N}$. Note that $P_i \in \mathscr{F}(\mathscr{I})$, $i \in \mathbb{N}$, since $H_i = \mathbb{N}\backslash P_i = \left\{n \in \mathbb{N} : \|x_n - \xi, z\| \geqslant \frac{1}{i}\right\} \in \mathscr{I}$ for each $i \in \mathbb{N}$ and nonzero z in X. Then applying Lemma 1 we get $P \in \mathscr{F}(\mathscr{I})$ such that $P = \{p_1 < p_2 < ... < p_k < ...\}$. Now define the sequence $y \in X$ by $y_n = x_n$ for each $n \in P$ and $y_n = \xi$ for $n \notin P$. Then $\lim_{n\to\infty} \|y_n - \xi, z\| = 0$ which implies $\lim_{k\to\infty} \left\|x_{p_k} - \xi, z\right\| = 0$ for each nonzero z in X. □

Now we introduce the concept of $\mathscr{I}^*$-convergence closely related to $\mathscr{I}$-convergence in 2-normed spaces X :

DEFINITION 2. A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ of elements of X is said to be $\mathscr{I}^*$-convergent to $\xi \in X$ if and only if there exists a set $M \in \mathscr{F}(\mathscr{I})$, $M = \{m_1 < m_2 < ... < m_k < ...\}$ such that $\lim_{k\to\infty} \left\|x_{m_k} - \xi, z\right\| = 0$ for each nonzero z in X.

Lemma 2 shows that if $\mathscr{I}$ is an admissible ideal with property (AP) then $\mathscr{I}\text{-}\lim_{n\to\infty} \|x_n - \xi, z\| = 0$ implies $\mathscr{I}^*\text{-}\lim_{n\to\infty} \|x_n - \xi, z\| = 0$ for each nonzero z in X.

From Proposition 3.2 in [8] we obtain the following lemma:

LEMMA 3. *Let $\mathscr{I} \subset 2^{\mathbb{N}}$ is an admissible ideal with property (AP) and (X, ρ) is an arbitrary 2-normed space. Then $\mathscr{I}\text{-}\lim_{n\to\infty} \|x_n - \xi, z\| = 0$ for every nonzero z in X if and only if there exists a set $P \in \mathscr{F}(\mathscr{I})$, $P = \{p_1 < p_2 < ... < p_k < ...\}$ such that $\lim_{k\to\infty} \left\|x_{p_k} - \xi, z\right\| = 0$ for every nonzero z in X.*

REMARK 1. Let $\mathscr{I} = \mathscr{I}_\delta$ and X be a 2-normed space. Consider the set $\{A \subset \mathbb{N} : d(A) = 0\}$ where $d(A)$ is the natural density of the set $A \subset \mathbb{N}$. Then Lemma 2 gives the equivalent of statistical convergence in 2-normed space.

3. $\mathscr{I}$-Cauchy sequences in 2-normed spaces

Now we introduce the concepts $\mathscr{I}$ and $\mathscr{I}^*$-Cauchy sequences in 2-normed spaces. Also, we will study the relations between these concepts.

DEFINITION 3. Let $(X, \|.,.\|)$ be a linear 2-normed space and $\mathscr{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. The sequence (x_n) in X is said to be an $\mathscr{I}$-Cauchy sequence in X, if for each $\varepsilon > 0$ and nonzero z in X there exists a number $N = N(\varepsilon, z)$ such that

$$\left\{k \in \mathbb{N} : \left\|x_k - x_{N(\varepsilon,z)}, z\right\| \geqslant \varepsilon\right\} \in \mathscr{I}.$$

DEFINITION 4. Let $(X, \|.,.\|)$ be a linear 2-normed space and $\mathscr{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. Then the sequence (x_n) in X is said to be an $\mathscr{I}^*$-Cauchy sequence if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$, $M \in \mathscr{F}(\mathscr{I})$ such that the subsequence $x_M = \left(x_{m_k}\right)$ is a Cauchy sequence on X, i.e.,

$$\lim_{k,p\to\infty} \left\|x_{m_k} - x_{m_p}, z\right\| = 0 \text{ for each nonzero } z \text{ in } X.$$

The next theorem gives that $\mathscr{I}^*$-Cauchy sequence condition implies $\mathscr{I}$-Cauchy sequence condition.

THEOREM 1. *Let* $\mathscr{I}$ *be an admissible ideal. If* $x = (x_n)$ *is an* $\mathscr{I}^*$*-Cauchy sequence then it is* $\mathscr{I}$*-Cauchy.*

Proof. Let $x = (x_n)$ be an $\mathscr{I}^*$-Cauchy sequence. Then by definition there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$, $M \in \mathscr{F}(\mathscr{I})$ such that $\left\|x_{m_k} - x_{m_p}, z\right\| < \varepsilon$ for every $\varepsilon > 0$, nonzero z in X and $k,p > k_0(\varepsilon)$

Let $N = N(\varepsilon) = m_{k_0+1}$. Then for every $\varepsilon > 0$ we have

$$\left\|x_{m_k} - x_N, z\right\| < \varepsilon, \quad \text{for every nonzero } z \text{ in } X \text{ and } k > k_0.$$

Now put $H = \mathbb{N} \backslash M$. It is clear that $H \in \mathscr{I}$ and

$$A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x_N, z\| \geqslant \varepsilon\} \subset (H \cup \{m_1 < m_2 < ... < m_{k_0}\}). \tag{3.1}$$

The right hand side of (3.1) belongs to $\mathscr{I}$. Therefore for every $\varepsilon > 0$ we can find an $N = N(\varepsilon)$ such that $A(\varepsilon) \in \mathscr{I}$, i.e. (x_n) is $\mathscr{I}$-Cauchy sequence. The theorem is proved. □

Now we will prove that $\mathscr{I}^*$-convergence implies $\mathscr{I}$-Cauchy condition in 2-normed space.

THEOREM 2. *Let* $\mathscr{I}$ *be an admissible ideal and* $\mathscr{I}^*\text{-}\lim_{n\to\infty} \|x_n - \xi, z\| = 0$ *where* $x = (x_n) \in X$ *and* $\xi \in X$. *Then* (x_n) *is an* $\mathscr{I}$*-Cauchy sequence in* 2*-normed space* $(X, \|.,.\|)$.

Proof. By assumption there exists a set $M \in \mathscr{F}(\mathscr{I})$, $M = \{m_1 < m_2 < ... < m_k < ...\}$ such that $\lim_{k\to\infty} \left\|x_{m_k} - \xi, z\right\| = 0$ for each nonzero z in X. It shows that there exists a $k_0 = k_0(\varepsilon)$ such that $\left\|x_{m_k} - \xi, z\right\| < \frac{\varepsilon}{2}$ for every $\varepsilon > 0$, nonzero z in X and $k > k_0$. Since

$$\begin{aligned} \left\|x_{m_k} - x_{m_p}, z\right\| &< \left\|x_{m_k} - \xi, z\right\| + \left\|x_{m_p} - \xi, z\right\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for every $\varepsilon > 0$, nonzero z in X and $k > k_0$, $p > k_0$ we have

$$\lim_{\substack{k\to\infty \\ p\to\infty}} \left\|x_{m_k} - x_{m_p}, z\right\| = 0,$$

i.e. (x_n) is a $\mathscr{I}^*$-Cauchy sequence in X. Then by Theorem 1 (x_n) is an $\mathscr{I}$-Cauchy sequence in X. Hence the proof is complete. □

From Theorem 2 and Lemma 3 we have

COROLLARY 1. *Let* $\mathscr{I}$ *be an admissible ideal with property (AP). Then* $\mathscr{I}\text{-}\lim_{n\to\infty} \|x_n - \xi, z\| = 0$ *for each nonzero* z *in* X *implies that* (x_n) *is an* $\mathscr{I}$*-Cauchy sequence in* X.

Finally, we will give the following theorem which states that the notion of $\mathscr{I}$-Cauchy sequence and $\mathscr{I}^*$-Cauchy sequence coincide in the case that $\mathscr{I}$ has the (AP) property.

THEOREM 3. *If $\mathscr{I}$ is an admissible ideal with property (AP) and $(X, \|.,.\|)$ is a linear 2-normed space then the concepts $\mathscr{I}$-Cauchy sequence and $\mathscr{I}^*$-Cauchy sequence coincide.*

Proof. $\mathscr{I}^*$-Cauchy sequence implies $\mathscr{I}$-Cauchy sequence by virtue of Theorem 1 (in this case $\mathscr{I}$ need not have (AP) property). So it suffices to prove $x = (x_n)$ in X is an $\mathscr{I}^*$-Cauchy sequence under the assumption that (x_n) is an $\mathscr{I}$-Cauchy sequence. Let $x = (x_n)$ in X be an $\mathscr{I}$-Cauchy sequence. Then by definition there is $N = N(\varepsilon)$ such that

$$A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x_N, z\| \geqslant \varepsilon\} \in \mathscr{I}$$

for every $\varepsilon > 0$ and nonzero z in X. Let $P_i = \left\{n \in \mathbb{N} : \|x_n - x_{m_i}, z\| < \frac{1}{i}\right\}$, $i = 1, 2, \ldots$ where $m_i = N\left(\frac{1}{i}\right)$. It is clear that $P_i \in \mathscr{F}(\mathscr{I})$, $i = 1, 2, \ldots$. Since $\mathscr{I}$ has (AP) property then by Lemma 1 there exists a set $P \subset \mathbb{N}$ such that $P \in \mathscr{F}(\mathscr{I})$ and $P \backslash P_i$ is finite for all i. Now we show that

$$\lim_{\substack{n,m \to \infty \\ m,n \in P}} \|x_n - x_m, z\| = 0 \text{ for each nonzero } z \text{ in } X.$$

For this, let $\varepsilon > 0$ and $j \in \mathbb{N}$ be such that $j > \frac{2}{\varepsilon}$. If $m, n \in P$ then $P \backslash P_j$ is a finite set, so there exists $k = k(j)$ such that $m \in P_j$ and $n \in P_j$ for all $m, n > k(j)$. Therefore, $\|x_n - x_{m_j}, z\| < \frac{1}{j}$ and $\|x_m - x_{m_j}, zt\| < \frac{1}{j}$ for all $m, n > k(j)$ and nonzero z in X. From here we find

$$\begin{aligned} \|x_n - x_m, z\| &< \|x_n - x_{m_j}, z\| + \|x_m - x_{m_j}, z\| \\ &< \varepsilon \end{aligned}$$

for $m, n > k(j)$ and each nonzero z in X. Hence, for any $\varepsilon > 0$ there exists a $k = k(\varepsilon)$ such that for $n, m > k(\varepsilon)$ and $n, m \in P \in \mathscr{F}(\mathscr{I})$

$$\|x_n - x_m, z\| < \varepsilon \text{ for every nonzero } z \in X.$$

This shows that the sequence (x_n) in X is an $\mathscr{I}^*$-Cauchy sequence in X. This completes the proof. □

Note that all of these results imply the similar theorems in 2-normed spaces for statistically Cauchy sequences which are investigated in [3, 12].

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Mehmet Gürdal
Department of Mathematics
Süleyman Demirel University
Doğu Campus
32 260 Isparta
Turkey
e-mail: gurdal@fef.sdu.edu.tr

Işıl Açık
Department of Mathematics
Süleyman Demirel University
Doğu Campus
32 260 Isparta
Turkey
e-mail: isilacik@mynet.com