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A NEW SYSTEM OF GENERALIZED NONLINEAR MIXED VARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce and study a new system of generalized nonlinear mixed variational inequalities which contains some classes of variational inequalities and systems of variational inequalities in the literature as special cases. We prove the existence and uniqueness of solution and the convergence of some new n-step iterative algorithms with mixed errors for this system of generalized nonlinear mixed variational inequalities and its special cases. The results in this paper unify, extend and improve some known results in the literature.

1. Introduction

Variational inequality problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium, engineering science. For the past years, many existence results and iterative algorithms for various variational inequality problems have been studied. For details, please see [1–6, 8–31, 33–49, 51, 52] and the references therein.

Recently, some new and interesting problems, which are called to be system of variational inequality problems were introduced and studied. Pang [37], Cohen and Chaplais [14], Bianchi [6] and Ansari and Yao [4] considered a system of scalar variational inequalities and Pang showed that the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium, and the general equilibrium programming problem can be modeled as a system of variational inequalities. Ansari et al. [5] introduced and studied a system of vector equilibrium problems and a system of vector variational inequalities by a fixed point theorem. Allevi et al. [1] considered a system of generalized vector variational inequalities and established some existence results with relative pseudomonotonicity. Kassay and Kolumbán [23] introduced a system of variational inequalities and proved an existence theorem by the Ky Fan lemma. Kassay, Kolumbán and Páles [24] studied Minty and Stampacchia variational inequality systems with the help of the Kakutani-Fan-Glicksberg fixed point theorem. Peng

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[38, 39] introduced a system of quasi-variational inequality problems and proved its existence theorem by maximal element theorems. Verma [41–45] introduced and studied some systems of variational inequalities and developed some iterative algorithms for approximating the solutions of system of variational inequalities in Hilbert spaces. Kim and Kim [25] introduced a new system of generalized nonlinear quasi-variational inequalities and obtained some existence and uniqueness results of solution for this system of generalized nonlinear quasi-variational inequalities in Hilbert spaces. Cho et al. [13] introduced and studied a new system of nonlinear variational inequalities in Hilbert spaces. They proved some existence and uniqueness theorems of solutions for the system of nonlinear variational inequalities.

Inspired and motivated by the above works, in this paper, we introduce and study a new system of generalized nonlinear mixed variational inequalities in Hilbert spaces which contains the mathematical models in [41–44, 25] as special cases. We prove the existence and uniqueness of solution for this new system of generalized nonlinear mixed variational inequalities and its special cases. We also give the convergence of some n-step iterative sequences with mixed errors for this system of generalized nonlinear mixed variational inequalities and its special cases. The results obtained in this paper unify, extend and improve those results in [41–44, 25] and the reference therein.

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space with the inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$, we recall some definitions and lemmas needed later.

DEFINITION 2.1. Let $T : H \longrightarrow H$ be a single-valued operator. T is said to be
(i) monotone if

$$\langle Tu - Tv, u - v\rangle \geqslant 0,\ \forall u, v \in H;$$

(ii) strictly monotone if T is monotone and

$$\langle Tu - Tv, u - v\rangle = 0 \quad \text{if and only if} \quad u = v;$$

(iii) strongly monotone if there exists a constant $r > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geqslant r\|u - v\|^2,\ \forall u, v \in H.$$

(iv) Lipschitz continuous if there exists a constant $s > 0$ such that

$$\|Tu - Tv\| \leqslant s\|u - v\|,\ \forall u, v \in H.$$

LEMMA 2.1. ([7, 50]) *For any given $u \in H$, the point $z \in H$ satisfies the following inequality*

$$\langle u - z, v - u\rangle \geqslant \rho\phi(u) - \rho\phi(v),\ \forall v \in H$$

if and only if

$$u = J^{\rho}_{\partial\phi}(z),$$

where $J^{\rho}_{\partial\phi} = (I + \rho\partial\phi)^{-1}$ and $\partial\phi$ is the subdifferential of a proper convex lower semicontinuous function ϕ.

LEMMA 2.2. [32] *Let* $\{a_n\}, \{b_n\}$ *and* $\{c_n\}$ *be three sequences of nonnegative numbers satisfying the following conditions: there exists* n_0 *such that*

$$a_{n+1} \leqslant (1-t_n)a_n + b_n t_n + c_n, \ \forall n \geqslant n_0,$$

where

$$t_n \in [0,1], \quad \sum_{n=0}^{\infty} t_n = \infty, \quad \lim_{n\to\infty} b_n = 0, \quad \sum_{n=0}^{\infty} c_n < +\infty.$$

Then, $a_n \longrightarrow 0$ *as* $n \to \infty$.

3. System of generalized nonlinear mixed variational inequalities

In this section, we will introduce a new system of generalized nonlinear mixed variational inequalities and construct a new n-step iterative algorithm for solving this system of generalized nonlinear mixed variational inequalities. In what follows, unless other specified, for each $i = 1,2,...,n$, we always suppose that $T_i : H \longrightarrow H$ is a single-valued mapping, $\phi_i : H \to R \cup \{+\infty\}$ is a proper convex lower semi-continuous function. We consider the following problem:

find $(x_1^*, x_2^*, ..., x_n^*) \in H^n$ such that

$$\begin{cases} \langle \rho_1 T_1 x_2^* + \rho_1 S_1 x_2^* + x_1^* - x_2^*, x - x_1^* \rangle \geqslant \rho_1 \phi_1(x_1^*) - \rho_1 \phi_1(x), \ \forall x \in H, \\ \langle \rho_2 T_2 x_3^* + \rho_2 S_2 x_3^* + x_2^* - x_3^*, x - x_2^* \rangle \geqslant \rho_2 \phi_2(x_2^*) - \rho_2 \phi_2(x), \ \forall x \in H, \\ \cdots, \\ \langle \rho_{n-1} T_{n-1} x_n^* + \rho_{n-1} S_{n-1} x_n^* + x_{n-1}^* - x_n^*, x - x_{n-1}^* \rangle \\ \qquad \geqslant \rho_{n-1} \phi_{n-1}(x_{n-1}^*) - \rho_{n-1} \phi_{n-1}(x), \ \forall x \in H, \\ \langle \rho_n T_n x_1^* + \rho_n S_n x_1^* + x_n^* - x_1^*, x - x_n^* \rangle \geqslant \rho_n \phi_n(x_n^*) - \rho_n \phi_n(x), \ \forall x \in H, \end{cases} \tag{3.1}$$

which is called the system of generalized nonlinear mixed variational inequalities, where $\rho_i > 0 \ (i = 1,2,\cdots,n)$ are constants.

Below are some special cases of the problem (3.1).

(1) If $n = 2$, then the problem (3.1) reduces to the problem of finding $(x_1^*, x_2^*) \in H \times H$ such that

$$\begin{cases} \langle \rho_1 T_1 x_2^* + \rho_1 S_1 x_2^* + x_1^* - x_2^*, x - x_1^* \rangle \geqslant \rho_1 \phi_1(x_1^*) - \rho_1 \phi_1(x), \ \forall x \in H, \\ \langle \rho_2 T_2 x_1^* + \rho_2 S_2 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geqslant \rho_2 \phi_2(x_2^*) - \rho_2 \phi_2(x), \ \forall x \in H. \end{cases} \tag{3.2}$$

Moreover, If $\phi_1 = \phi_2 = \varphi$, then problem (3.2) becomes the system of generalized nonlinear mixed variational inequalities introduced and studied by Kim and Kim in [45].

If $\phi_1 = \phi_2 = \varphi$ and $T_1 = T_2 = 0$, then problem (3.2) becomes the system of nonlinear mixed variational inequalities in [43].

(2) For $i = 1,2,...,n$, if $\phi_i = \delta_{k_i}$ (the indicator function of a nonempty closed convex subset $K_i \subset H$), then the problem (3.1) reduces to the problem of finding

$(x_1^*, x_2^*, ..., x_n^*) \in \prod_{i=1}^n K_i$, such that

$$\begin{cases} \langle \rho_1 T_1 x_2^* + \rho_1 S_1 x_2^* + x_1^* - x_2^*, x - x_1^* \rangle \geqslant 0, \ \forall x \in K_1, \\ \langle \rho_2 T_2 x_3^* + \rho_2 S_2 x_3^* + x_2^* - x_3^*, x - x_2^* \rangle \geqslant 0, \ \forall x \in K_2, \\ \cdots, \\ \langle \rho_{n-1} T_{n-1} x_n^* + \rho_{n-1} S_{n-1} x_n^* + x_{n-1}^* - x_n^*, x - x_{n-1}^* \rangle \geqslant 0, \ \forall x \in K_{n-1}, \\ \langle \rho_n T_n x_1^* + \rho_n S_n x_1^* + x_n^* - x_1^*, x - x_n^* \rangle \geqslant 0, \ \forall x \in K_n, \end{cases} \tag{3.3}$$

(3) For $i = 1, 2, ..., n$, if $T_i = 0$, then the problem (3.3) reduces to the problem of finding $(x_1^*, x_2^*, ..., x_n^*) \in \prod_{i=1}^n K_i$, such that

$$\begin{cases} \langle \rho_1 S_1 x_2^* + x_1^* - x_2^*, x - x_1^* \rangle \geqslant 0, \ \forall x \in K_1, \\ \langle \rho_2 S_2 x_3^* + x_2^* - x_3^*, x - x_2^* \rangle \geqslant 0, \ \forall x \in K_1, \\ \cdots, \\ \langle \rho_{n-1} S_{n-1} x_n^* + x_{n-1}^* - x_n^*, x - x_{n-1}^* \rangle \geqslant 0, \ \forall x \in K_{n-1}, \\ \langle \rho_n S_n x_1^* + x_n^* - x_1^*, x - x_n^* \rangle \geqslant 0, \ \forall x \in K_n, \end{cases} \tag{3.4}$$

Both problem (3.3) and (3.4) are called the system of nonlinear variational inequalities. Moreover, if $n = 2$, then problem (3.4) reduces to the following system of nonlinear variational inequalities, which is to find $(x_1^*, x_2^*) \in K_1 \times K_2$ such that

$$\begin{cases} \langle \rho_1 S_1 x_2^* + x_1^* - x_2^*, x - x_1^* \rangle \geqslant 0, \ \forall x \in K_1, \\ \langle \rho_2 S_2 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geqslant 0, \ \forall x \in K_2. \end{cases} \tag{3.5}$$

If $S_1 = S_2$ and $K_1 = K_2 = K$, then problem (3.5) reduces to the problem introduced and researched by Verma [40–42].

LEMMA 3.1. *For any given* $x_i^* \in H (i = 1, 2, \cdots, n)$, $(x_1^*, x_2^*, ..., x_n^*)$ *is a solution of the problem (3.1) if and only if*

$$\begin{cases} x_1^* = J_{\partial\phi_1}^{\rho_1}[x_2^* - \rho_1(T_1 x_2^* + S_1 x_2^*)], \\ x_2^* = J_{\partial\phi_2}^{\rho_2}[x_3^* - \rho_2(T_2 x_3^* + S_2 x_3^*)], \\ \cdots, \\ x_{n-1}^* = J_{\partial\phi_{n-1}}^{\rho_{n-1}}[x_n^* - \rho_{n-1}(T_{n-1} x_n^* + S_{n-1} x_n^*)], \\ x_n^* = J_{\partial\phi_n}^{\rho_n}[x_1^* - \rho_n(T_n x_1^* + S_n x_1^*)], \end{cases} \tag{3.6}$$

where $J_{\partial\phi_i}^{\rho_i} = (I + \rho_i \partial\phi_i)^{-1}$ *is the resolvent operators of* $\partial\phi_i$ *for* $i = 1, 2, \cdots, n$.

Proof. It is easy to know that Lemma 3.1 follows from Lemma 2.1 and so the proof is omitted. □

4. Existence and uniqueness

In this section, we will show the existence and uniqueness of solution for problems (3.1) and its special cases.

THEOREM 4.1. *For $i = 1, 2, \cdots, n$, let $S_i : H \longrightarrow H$ be strongly monotone and Lipschitz continuous with constants k_i and μ_i, respectively, $T_i : H \longrightarrow H$ be Lipschitz continuous with constant ν_i. If for each $i = 1, 2, \cdots, n$,*

$$0 < \rho_i < \min\left\{ \frac{2(k_i - \nu_i)}{\mu_i^2 - \nu_i^2}, \frac{1}{\nu_i} \right\}, \nu_i < k_i. \tag{4.1}$$

then problem (3.1) has a unique solution $(x_1^*, x_2^*, ..., x_n^*) \in H^n$.

Proof. First, we prove the existence of the solution. Define a mapping $F : H \longrightarrow H$ as follows:

$$\begin{cases} F(x) = J_{\partial\phi_1}^{\rho_1}[x_2 - \rho_1(T_1x_2 + S_1x_2)], \\ x_2 = J_{\partial\phi_2}^{\rho_2}[x_3 - \rho_2(T_2x_3 + S_2x_3)], \\ \qquad \dots, \\ x_{n-1} = J_{\partial\phi_{n-1}}^{\rho_{n-1}}[x_n - \rho_{n-1}(T_{n-1}x_n + S_{n-1}x_n)], \\ x_n = J_{\partial\phi_n}^{\rho_n}[x - \rho_n(T_nx + S_nx)]. \end{cases} \tag{4.2}$$

For $i = 1, 2, \cdots, n$, since $J_{\partial\phi_i}^{\rho_i}$ is nonexpansive mapping, S_i is strongly monotone and Lipschitz continuous with constants k_i and μ_i, respectively, and T_i is Lipschitz continuous with constant ν_i, for any $x, y \in H$, we have:

$$\begin{aligned}
&\|F(x) - F(y)\| \\
&= \|J_{\partial\phi_1}^{\rho_1}[x_2 - \rho_1(T_1x_2 + S_1x_2)] - J_{\partial\phi_1}^{\rho_1}[y_2 - \rho_1(T_1y_2 + S_1y_2)]\| \\
&\leqslant \|(x_2 - y_2) - \rho_1((T_1x_2 + S_1x_2) - (T_1y_2 + S_1y_2))\| \\
&\leqslant \|(x_2 - y_2) - \rho_1(S_1x_2 - S_1y_2)\| + \rho_1\|T_1x_2 - T_1y_2\| \\
&\leqslant \sqrt{\|x_2 - y_2\|^2 - 2\rho_1\langle S_1x_2 - S_1y_2, x_2 - y_2\rangle + \rho_1^2\|S_1x_2 - S_1y_2\|^2} + \rho_1\nu_1\|x_2 - y_2\|] \\
&= (\sqrt{1 - 2\rho_1k_1 + \rho_1^2\mu_1^2} + \rho_1\nu_1)\|x_2 - y_2\| \\
&= (\sqrt{1 - 2\rho_1k_1 + \rho_1^2\mu_1^2} + \rho_1\nu_1)\|J_{\partial\phi_2}^{\rho_2}[x_3 - \rho_2(T_2x_3 + S_2x_3)] \\
&\quad - J_{\partial\phi_2}^{\rho_2}[y_3 - \rho_2(T_2y_3 + S_2y_3)]\| \\
&\leqslant (\sqrt{1 - 2\rho_1k_1 + \rho_1^2\mu_1^2} + \rho_1\nu_1)\|x_3 - y_3 - \rho_2[(S_2x_3 - S_2y_3) + (T_2x_3 - T_2y_3)]\| \\
&\leqslant (\sqrt{1 - 2\rho_1k_1 + \rho_1^2\mu_1^2} + \rho_1\nu_1)\|x_3 - y_3 - \rho_2(S_2x_3 - S_2y_3)\| + \rho_2\|T_2x_3 - T_2y_3\|
\end{aligned}$$

$$\leqslant (\sqrt{1-2\rho_1 k_1+\rho_1^2\mu_1^2}+\rho_1\nu_1)\times$$
$$\times [\sqrt{\|x_3-y_3\|^2-2\rho_2\langle S_2x_3-S_2y_3,x_3-y_3\rangle+\rho_2^2\|S_2x_3-S_2y_3\|^2}+\rho_2\nu_2\|x_3-y_3\|]$$
$$\leqslant (\sqrt{1-2\rho_1 k_1+\rho_1^2\mu_1^2}+\rho_1\nu_1)(\sqrt{1-2\rho_2 k_2+\rho_2^2\mu_2^2}+\rho_2\nu_2)\|x_3-y_3\|$$
$$\leqslant \cdots \leqslant \prod_{i=1}^{n-1}(\sqrt{1-2\rho_i k_i+\rho_i^2\mu_i^2}+\rho_i\nu_i)\|x_n-y_n\|$$
$$=\prod_{i=1}^{n-1}(\sqrt{1-2\rho_i k_i+\rho_i^2\mu_i^2}+\rho_i\nu_i)\|J_{\partial\phi_n}^{\rho_n}[x-\rho_n(T_nx+S_nx)]-J_{\partial\phi_n}^{\rho_n}[y-\rho_n(T_ny+S_ny)]\|$$
$$\leqslant \prod_{i=1}^{n-1}(\sqrt{1-2\rho_i k_i+\rho_i^2\mu_i^2}+\rho_i\nu_i)[\|x-y-\rho_n(S_nx)-S_ny)\|+\rho_n\|T_nx-T_ny\|]$$
$$\leqslant \prod_{i=1}^{n-1}(\sqrt{1-2\rho_i k_i+\rho_i^2\mu_i^2}+\rho_i\nu_i)[\sqrt{\|x-y\|^2-2\rho_n\langle S_nx-S_ny,x-y\rangle+\rho_n^2\|S_nx-S_ny\|^2}$$
$$+\rho_n\nu_n\|x-y\|]$$
$$\leqslant \prod_{i=1}^{n}(\sqrt{1-2\rho_i k_i+\rho_i^2\mu_i^2}+\rho_i\nu_i)\|x-y\|. \tag{4.3}$$

It follows from (4.1) that

$$0<\prod_{i=1}^{n}(\sqrt{1-2\rho_i k_i+\rho_i^2\mu_i^2}+\rho_i\nu_i)<1.$$

Thus, (4.3) implies that F is a contractive mapping and so there exists a point $x_1^*\in H$ such that

$$x_1^*=F(x_1^*)$$

Let

$$x_i^*=J_{\partial\phi_i}^{\rho_i}[x_{i+1}^*-\rho_i(T_i(x_{i+1}^*)+S_i(x_{i+1}^*))],\ i=1,2,\cdots,n-1$$
$$x_n^*=J_{\partial\phi_n}^{\rho_n}[x_1^*-\rho_n(T_n(x_1^*)+S_n(x_1^*))]$$

then by the definition of F, we have,

$$\begin{cases} x_1^*=J_{\partial\phi_1}^{\rho_1}[x_2^*-\rho_1(T_1x_2^*+S_1x_2^*)],\\ x_2^*=J_{\partial\phi_2}^{\rho_2}[x_3^*-\rho_2(T_2x_3^*+S_2x_3^*)],\\ \qquad\cdots,\\ x_{n-1}^*=J_{\partial\phi_{n-1}}^{\rho_{n-1}}[x_n^*-\rho_{n-1}(T_{n-1}x_n^*+S_{n-1}x_n^*)],\\ x_n^*=J_{\partial\phi_n}^{\rho_n}[x_1^*-\rho_n(T_nx_1{}^*+S_nx_1{}^*)]\end{cases}$$

i.e. $(x_1^*,x_2^*,\cdots,x_n^*)$ is a solution of problem (3.1).

Then, we show the uniqueness of the solution. Let $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ be another solution of problem (3.1). It follows from lemma 3.1 that

$$\begin{cases} \bar{x}_1 = J_{\partial\phi_1}^{\rho_1}[\bar{x}_2 - \rho_1(T_1\bar{x}_2 + S_1\bar{x}_2)], \\ \bar{x}_2 = J_{\partial\phi_2}^{\rho_2}[\bar{x}_3 - \rho_2(T_2\bar{x}_3 + S_2\bar{x}_3)], \\ \qquad \cdots, \\ \bar{x}_{n-1} = J_{\partial\phi_{n-1}}^{\rho_{n-1}}[\bar{x}_n - \rho_{n-1}(T_{n-1}\bar{x}_n + S_{n-1}\bar{x}_n)], \\ \bar{x}_n = J_{\partial\phi_n}^{\rho_n}[\bar{x}_1 - \rho_n(T_n\bar{x}_1 + S_n\bar{x}_1)] \end{cases}$$

As the proof of (4.3), we have

$$\|x_1^* - \bar{x}_1\| \leqslant \prod_{i=1}^{n}[\sqrt{1 - 2\rho_i k_i + \rho_i^2\mu_i^2} + \rho_i\nu_i]\|x_1^* - \bar{x}_1\|$$

It follows from (4.1) that

$$0 < \prod_{i=1}^{n}[\sqrt{1 - 2\rho_i k_i + \rho_i^2\mu_i^2} + \rho_i\nu_i] < 1.$$

Hence

$$x_1^* = \bar{x}_1$$

and so for $i = 2, 3, \cdots, n$, we have

$$x_i^* = \bar{x}_i.$$

This completes the proof. $\square$

REMARK 4.1. By Theorem 4.1, it is easy to get the existence and uniqueness of solutions for the special cases of problem (3.1), now we give some examples as follows.

For $i = 1, 2, ..., n$, let $T_i = 0$ and $\phi_i = \delta_{k_i}$, by Theorem 4.1, we have

COROLLARY 4.2. *For $i = 1, 2, \cdots, n$, let $S_i : H \longrightarrow H$ be strongly monotone and Lipschitz continuous with constants k_i and μ_i, respectively. If for each $i = 1, 2, \cdots, n$,*

$$0 < \rho_i < \frac{2k_i}{\mu_i^2}. \tag{4.4}$$

then problem (3.4) has a unique solution $(x_1^*, x_2^*, ..., x_n^*) \in H^n$.

Let $n = 2$, by Theorem 4.1 and Corollary 4.2, respectively, we have

COROLLARY 4.3. *For $i = 1, 2$, let $S_i : H \longrightarrow H$ be strongly monotone and Lipschitz continuous with constants k_i and μ_i, respectively, $T_i : H \longrightarrow H$ be Lipschitz continuous with constant ν_i. If for each $i = 1, 2$,*

$$0 < \rho_i < \min\left\{\frac{2(k_i - \nu_i)}{\mu_i^2 - \nu_i^2}, \frac{1}{\nu_i}\right\}, \nu_i < k_i. \tag{4.5}$$

then problem (3.2) has a unique solution $(x_1^*, x_2^*) \in H^2$.

COROLLARY 4.4. *For $i = 1,2$, let $S_i : H \longrightarrow H$ be strongly monotone and Lipschitz continuous with constants k_i and μ_i, respectively. If for each $i = 1,2$,*

$$0 < \rho_i < \frac{2k_i}{\mu_i^2}. \tag{4.6}$$

then problem (3.5) has a unique solution $(x_1^*, x_2^*) \in H^2$.

REMARK 4.2. If $\phi_1 = \phi_2 = \varphi$, then by Corollary 4.3, we recover Theorem 2.1 in [45]. And so Theorem 4.1 extends and improves the corresponding results in [43] and [45] in several aspects.

5. Algorithms and Convergence

This section deals with an introduction of some n-step iterative sequences with mixed errors for problem (3.1) and its special cases that can be applied to the convergence analysis of the iterative sequences generated by the algorithms.

ALGORITHM 5.1. For any given point $x_0 \in H$, define the generalized N-step iterative sequences $\{x_{1,k}\}, \{x_{2,k}\}, \cdots, \{x_{n,k}\}$ as follows:

$$\begin{cases} x_{1,k+1} = (1-\alpha_k)x_{1,k} + \alpha_k J_{\partial\phi_1}^{\rho_1}[x_{2,k} - \rho_1(T_1 x_{2,k} + S_1 x_{2,k})] + \alpha_k u_{1,k} + w_k, \\ x_{2,k} = J_{\partial\phi_2}^{\rho_2}[x_{3,k} - \rho_2(T_2 x_{3,k} + S_2 x_{3,k})] + u_{2,k}, \\ \qquad\qquad \cdots, \\ x_{n-1,k} = J_{\partial\phi_{n-1}}^{\rho_{n-1}}[x_{n,k} - \rho_{n-1}(T_{n-1} x_{n,k} + S_{n-1} x_{n,k})] + u_{n-1,k}, \\ x_{n,k} = J_{\partial\phi_n}^{\rho_n}[x_{1,k} - \rho_n(T_n x_{1,k} + S_n x_{1,k})] + u_{n,k}, \end{cases} \tag{5.1}$$

where $x_{1,1} = x_0$, $\{\alpha_k\}$ is a sequence in [0,1], and $\{u_{i,k}\} \subset H (i = 1,2,\cdots,n), \{w_k\} \subset H$ are the sequences satisfying the following conditions:

$$\sum_{k=1}^{\infty} \alpha_k = +\infty; \quad \sum_{k=1}^{\infty} \|w_k\| < +\infty; \quad \lim_{k\to\infty} \|u_{i,k}\| = 0,\ i = 1,2,\cdots,n. \tag{5.2}$$

We can also construct some new iterative algorithms for the special cases of problem (3.1). For examples, we give the following iterative algorithms for problem (3.2), (3.4) and (3.5).

ALGORITHM 5.2. For any given point $x_0 \in H$, define the generalized n-step iterative sequences $\{x_{1,k}\}, \{x_{2,k}\}, \cdots, \{x_{n,k}\}$ as follows:

$$\begin{cases} x_{1,k+1} = (1-\alpha_k)x_{1,k} + \alpha_k P_{K_1}[x_{2,k} - \rho_1 S_1 x_{2,k}] + \alpha_k u_{1,k} + w_k, \\ x_{2,k} = P_{K_2}[x_{3,k} - \rho_2 S_2 x_{3,k}] + u_{2,k}, \\ \qquad\qquad \cdots, \\ x_{n-1,k} = P_{K_{n-1}}[x_{n,k} - \rho_{n-1} S_{n-1} x_{n,k}] + u_{n-1,k}, \\ x_{n,k} = P_{K_n}[x_{1,k} - \rho_n S_n x_{1,k}] + u_{n,k}, \end{cases}$$

where $x_{1,1} = x_0$, P_{K_i} is the projection of H onto K_i, $\rho_i > 0$, is a constant, $\{\alpha_k\} \subset [0,1]$ and $\{u_{i,k}\} \subset H$ $(i = 1,2,\cdots,n)$, $\{w_k\} \subset H$ are the sequences satisfying the following conditions:

$$\sum_{k=1}^{\infty} \alpha_k = +\infty; \quad \sum_{k=1}^{\infty} \|w_k\| < +\infty; \quad \lim_{k\to\infty} \|u_{i,k}\| = 0, \ i = 1,2,\cdots,n.$$

ALGORITHM 5.3. For any given point $x_0 \in H$, define the generalized n-step iterative sequences $\{x_{1,k}\},\{x_{2,k}\},\cdots,\{x_{n,k}\}$ as follows:

$$\begin{cases} x_{1,k+1} = (1-\alpha_k)x_{1,k} + \alpha_k P_{K_1}[x_{2,k} - \rho_1 S_1 x_{2,k}], \\ x_{2,k} = P_{K_2}[x_{3,k} - \rho_2 S_2 x_{3,k}], \\ \qquad\qquad \cdots, \\ x_{n-1,k} = P_{K_{n-1}}[x_{n,k} - \rho_{n-1} S_{n-1} x_{n,k}], \\ x_{n,k} = P_{K_n}[x_{1,k} - \rho_n S_n x_{1,k}], \end{cases}$$

where $x_{1,1} = x_0$, P_{K_i} is the projection of H onto K_i and $\rho_i > 0$ is a constant for $i = 1,2,...,n$, $\{\alpha_k\} \subset [0,1]$ are the sequences satisfying the following conditions:

$$\sum_{k=1}^{\infty} \alpha_k = +\infty.$$

ALGORITHM 5.4. For any given point $x_0 \in H$, define the generalized N-step iterative sequences $\{x_{1,k}\},\{x_{2,k}\}\}$ as follows:

$$\begin{cases} x_{1,k+1} = (1-\alpha_k)x_{1,k} + \alpha_k J_{\partial\phi_1}^{\rho_1}[x_{2,k} - \rho_1(T_1 x_{2,k} + S_1 x_{2,k})] + \alpha_k u_{1,k} + w_k, \\ x_{2,k} = J_{\partial\phi_2}^{\rho_2}[x_{1,k} - \rho_2(T_2 x_{1,k} + S_2 x_{1,k})] + u_{2,k}, \end{cases}$$

where $x_{1,1} = x_0$, $\{\alpha_k\}$ is a sequence in [0,1], and $\{u_{i,k}\} \subset H(i = 1,2), \{w_k\} \subset H$ are the sequences satisfying the following conditions:

$$\sum_{k=1}^{\infty} \alpha_k = +\infty; \quad \sum_{k=1}^{\infty} \|w_k\| < +\infty; \quad \lim_{k\to\infty} \|u_{i,k}\| = 0, \ i = 1,2.$$

ALGORITHM 5.5. For any given point $x_0 \in H$, define the two-step iterative sequences $\{x_{1,k}\},\{x_{2,k}\}$ as follows:

$$\begin{cases} x_{1,k+1} = (1-\alpha_k)x_{1,k} + \alpha_k P_{K_1}[x_{2,k} - \rho_1 S_1 x_{2,k}], \\ x_{2,k} = P_{K_2}[x_{1,k} - \rho_2 S_2 x_{1,k}], \end{cases}$$

where $x_{1,1} = x_0$, P_{K_i} is the projection of H onto K_i, $\rho_i > 0$, is a constant, and $\{\alpha_k\} \subset [0,1]$ are the sequences satisfying the following conditions:

$$\sum_{k=1}^{\infty} \alpha_k = +\infty.$$

THEOREM 5.1. *Let* T_i *and* S_i *be the same as in Theorem 4.1, and suppose that the sequences* $\{x_{1,k}\}, \{x_{2,k}\}, \cdots, \{x_{n,k}\}$ *are generated by algorithm 4.1. If the condition (4.1) holds, then* $(x_{1,k}, x_{2,k}, \cdots, x_{n,k})$ *converges strongly to the unique solution* $(x_1^*, x_2^*, \cdots, x_n^*)$ *of the problem (3.1).*

Proof. By the Theorem 4.1, we know that problem (3.1) has a unique solution $(x_1^*, x_2^*, \cdots, x_n^*)$, It follows from Lemma 3.1 that

$$\begin{cases} x_1^* = J_{\partial\phi_1}^{\rho_1}[x_2^* - \rho_1(T_1x_2^* + S_1x_2^*)], \\ x_2^* = J_{\partial\phi_2}^{\rho_2}[x_3^* - \rho_2(T_2x_3^* + S_2x_3^*)], \\ \qquad \cdots, \\ x_{n-1}^* = J_{\partial\phi_{n-1}}^{\rho_{n-1}}[x_n^* - \rho_{n-1}(T_{n-1}x_n^* + S_{n-1}x_n^*)], \\ x_n^* = J_{\partial\phi_n}^{\rho_n}[x_1^* - \rho_n(T_nx_1^* + S_nx_1^*)], \end{cases} \tag{5.3}$$

By (5.1) and (5.3), we have

$$\begin{aligned} &\|x_{1,k+1} - x_1^*\| \\ &\quad = \|(1-\alpha_k)x_{1,k} + \alpha_k J_{\partial\phi_1}^{\rho_1}[x_{2,k} - \rho_1(T_1x_{2,k} + S_1x_{2,k})] + \alpha_k u_{1,k} + w_k - x_1^*\| \\ &\quad \leqslant (1-\alpha_k)\|x_{1,k} - x_1^*\| + \alpha_k\|J_{\partial\phi_1}^{\rho_1}[x_{2,k} - \rho_1(T_1x_{2,k} + S_1x_{2,k})] \\ &\qquad - J_{\partial\phi_1}^{\rho}[x_2^* - \rho_1(T_1x_2^* + S_1x_2^*)]\| + \alpha_k\|u_{1,k}\| + \|w_k\| \\ &\quad \leqslant (1-\alpha_k)\|x_{1,k} - x_1^*\| + \alpha_k\|(x_{2,k} - x_2^*) - \rho_1[(T_1x_{2,k} + S_1x_{2,k}) - (T_1x_2^* + S_1x_2^*)]\| \\ &\qquad + \alpha_k\|u_{1,k}\| + \|w_k\|. \end{aligned} \tag{5.4}$$

For $i = 1, 2, \cdots, n$, since S_i is strongly monotone and Lipschitz continuous with constants k_i and μ_i, respectively, and T_i is Lipschitz continuous with constant ν_i, we get for $i = 1, 2, ..., n-1$

$$\begin{aligned} &\|(x_{i+1,k} - x_{i+1}^*) - \rho_i[(T_ix_{i+1,k} + S_ix_{i+1,k}) - (T_ix_{i+1}^* + S_ix_{i+1}^*)]\| \\ &\quad \leqslant \|(x_{i+1,k} - x_{i+1}^*) - \rho_i(S_ix_{i+1,k} - S_ix_{i+1}^*)\| + \rho_i\|T_ix_{i+1,k} - T_ix_{i+1}^*\| \\ &\quad \leqslant \sqrt{\|x_{i+1,k} - x_{i+1}^*\|^2 - 2\rho_i\langle S_ix_{i+1,k} - S_ix_{i+1}^*, x_{i+1,k} - x_{i+1}^*\rangle + \rho_i^2\|S_ix_{i+1,k} - S_ix_{i+1}^*\|} \\ &\qquad\quad + \rho_i\nu_i\|x_{i+1,k} - x_{i+1}^*\| \\ &\quad \leqslant \xi_i\|x_{i+1,k} - x_{i+1}^*\|, \end{aligned} \tag{5.5}$$

where $\xi_i = \sqrt{1 - 2\rho_ik_i + \rho_i^2\mu_i^2} + \rho_i\nu_i$, $i = 1, 2, ..., n-1$.

It follows from (5.4) and (5.5) that

$$\|x_{1,k+1} - x_1^*\| \leqslant (1-\alpha_k)\|x_{1,k} - x_1^*\| + \alpha_k\xi_1\|x_{2,k} - x_2^*\| + \alpha_k\|u_{1,k}\| + \|w_k\|. \tag{5.6}$$

By (5.1), (5.3) and (5.5), we have

$$\begin{aligned}
&\|x_{2,k} - x_2^*\| \\
&\quad = \|J_{\partial\phi_2}^{\rho_2}[x_{3,k} - \rho_2(T_2x_{3,k} + S_2x_{3,k}] - J_{\partial\phi_2}^{\rho_2}[x_3^* - \rho_2(T_2x_3^* + S_2x_3^*)] + u_{2,k}\| \\
&\quad \leqslant \|(x_{3,k} - x_3^*) - \rho_2[(T_2x_{3,k} + S_2x_{3,k}) - (T_2x_3^* + S_2x_3^*)]\| + \|u_{2,k}\| \\
&\quad \leqslant \xi_2\|x_{3,k} - x_3^*\| + \|u_{2,k}\|, \qquad (5.7)
\end{aligned}$$

and

$$\begin{aligned}
&\|x_{3,k} - x_3^*\| \\
&\quad = \|J_{\partial\phi_3}^{\rho_3}[x_{4,k} - \rho_3(T_3x_{4,k} + S_3x_{4,k}] - J_{\partial\phi_3}^{\rho_3}[x_4^* - \rho_3(T_3x_4^* + S_3x_4^*)] + u_{3,k}\| \\
&\quad \leqslant \|(x_{4,k} - x_4^*) - \rho_3[(T_3x_{4,k} + S_3x_{4,k}) - (T_3x_4^* + S_3x_4^*)]\| + \|u_{3,k}\| \\
&\quad \leqslant \xi_3\|x_{4,k} - x_4^*\| + \|u_{3,k}\|, \qquad (5.8)
\end{aligned}$$

$$\cdots,$$

$$\begin{aligned}
&\|x_{n-1,k} - x_{n-1}^*\| \\
&\quad = \|J_{\partial\phi_{n-1}}^{\rho_{n-1}}[x_{n,k} - \rho_{n-1}(T_{n-1}x_{n,k} + S_{n-1}x_{n,k}] \\
&\qquad - J_{\partial\phi_{n-1}}^{\rho_{n-1}}[x_n^* - \rho_{n-1}(T_{n-1}x_n^* + S_{n-1}x_n^*)] + u_{n-1,k}\| \\
&\quad \leqslant \|(x_{n,k} - x_n^*) - \rho_{n-1}[(T_{n-1}x_{n,k} + S_{n-1}x_{n,k}) - (T_{n-1}x_n^* + S_{n-1}x_n^*)]\| + \|u_{n-1,k}\| \\
&\quad \leqslant \xi_{n-1}\|x_{n,k} - x_n^*\| + \|u_{n-1,k}\|, \qquad (5.9)
\end{aligned}$$

since S_n is strongly monotone and Lipschitz continuous with constants k_n and μ_n, respectively, and T_n is Lipschitz continuous with constant ν_n, we get

$$\begin{aligned}
&\|x_{n,k} - x_n^*\| \\
&\quad = \|J_{\partial\phi_n}^{\rho_n}[x_{1,k} - \rho_n(T_nx_{1,k} + S_nx_{1,k}] - J_{\partial\phi_n}^{\rho_n}[x_1^* - \rho_n(T_nx_1^* + S_nx_1^*)] + u_{n,k}\| \\
&\quad \leqslant \|(x_{1,k} - x_1^*) - \rho_n[(T_nx_{1,k} + S_nx_{1,k}) - (T_nx_1^* + S_nx_1^*)]\| + \|u_{n,k}\| \\
&\quad \leqslant \sqrt{\|(x_{1,k} - x_1^*\|^2 - 2\rho_n\langle S_nx_{1,k}) - S_nx_1^*, x_{1,k} - x_1^*\rangle + \rho_n^2\|S_nx_{1,k}) - S_nx_1^*\|^2} \\
&\qquad\qquad + \rho_n\|T_nx_{1,k} - T_nx_1^*\| \\
&\quad \leqslant \xi_n\|x_{1,k} - x_1^*\| + \|u_{n,k}\|, \qquad (5.10)
\end{aligned}$$

where $\xi_n = \sqrt{1 - 2\rho_nk_n + \rho_n^2\mu_n^2} + \rho_n\nu_n$.

It follows from (5.6)–(5.10) that

$$\begin{aligned}
&\|x_{1,k+1} - x_1^*\| \\
&\quad \leqslant (1-\alpha_k)\|x_{1,k} - x_1^*\| + \alpha_k\xi_1\|x_{2,k} - x_2^*\| + \alpha_k\|u_{1,k}\| + \|w_k\| \\
&\quad \leqslant (1-\alpha_k)\|x_{1,k} - x_1^*\| + \alpha_k\xi_1[\xi_2\|x_{3,k} - x_3^*\| + \|u_{2,k}\|] + \alpha_k\|u_{1,k}\| + \|w_k\| \\
&\quad \leqslant (1-\alpha_k)\|x_{1,k} - x_1^*\| + \alpha_k\xi_1\xi_2\|x_{3,k} - x_3^*\| + \alpha_k\xi_1\|u_{2,k}\| + \alpha_k\|u_{1,k}\| + \|w_k\|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant (1-\alpha_k)\|x_{1,k}-x_1^*\| + \alpha_k\xi_1\xi_2\xi_3\|x_{4,k}-x_4^*\| + \alpha_k\xi_1\xi_2\|u_{3,k}\| + \alpha_k\xi_1\|u_{2,k}\| \\
&\quad + \alpha_k\|u_{1,k}\| + \|w_k\| \\
&\leqslant \cdots \leqslant (1-\alpha_k)\|x_{1,k}-x_1^*\| + \alpha_k\xi_1\xi_2\cdots\xi_{n-1}\|x_{n,k}-x_n^*\| \\
&\quad + \alpha_k\xi_1\xi_2\cdots\xi_{n-2}\|u_{n-1,k}\| + \cdots + \alpha_k\xi_1\|u_{2,k}\| + \alpha_k\|u_{1,k}\| + \|w_k\| \\
&\leqslant (1-\alpha_k)\|x_{1,k}-x_1^*\| + \alpha_k\xi_1\xi_2\cdots\xi_n\|x_{1,k}-x_1^*\| + \alpha_k\xi_1\xi_2\cdots\xi_{n-1}\|u_{n,k}\| \\
&\quad + \alpha_k\xi_1\xi_2\cdots\xi_{n-2}\|u_{n-1,k}\| + \cdots + \alpha_k\xi_1\|u_{2,k}\| + \alpha_k\|u_{1,k}\| + \|w_k\| \\
&= [1-\alpha_k(1-\xi_1\xi_2\cdots\xi_n)]\|x_{1,k}-x_1^*\| + \alpha_k(1-\xi_1\xi_2\cdots\xi_n)\frac{1}{1-\xi_1\xi_2\cdots\xi_n}\times \\
&\quad \times(\|u_{1,k}\| + \xi_1\|u_{2,k}\| + \cdots + \xi_1\xi_2\cdots\xi_{n-1}\|u_{n,k}\|) + \|w_k\|. \qquad (5.11)
\end{aligned}
$$

Let

$$
\begin{aligned}
&a_k = \|x_{1,k}-x_1^*\|, \quad t_k = \alpha_k(1-\xi_1\xi_2\cdots\xi_n), \quad c_k = \|w_k\|, \\
&b_k = \frac{1}{1-\xi_1\xi_2\cdots\xi_n}(\|u_{1,k}\| + \xi_1\|u_{2,k}\| + \cdots + \xi_1\xi_2\cdots\xi_{n-1}\|u_{n,k}\|).
\end{aligned}
$$

Then (5.11) can be written as follows:

$$a_{k+1} \leqslant (1-t_k)a_k + b_kt_k + c_k.$$

From the assumption (5.2), we know that $\{a_k\}$, $\{b_k\}$, $\{t_k\}$, $\{c_k\}$ satisfy the conditions of Lemma 2.2.

Thus $a_k \to 0 (k\to\infty)$, that is, $\|x_{1,k}-x_1^*\| \to 0 (k\to\infty)$. It follows from (5.6)–(5.10) that $\|x_{n,k}-x_n^*\| \to 0 \ (k\to\infty)$, $\|x_{n-1,k}-x_{n-1}^*\| \to 0 \ (k\to\infty)$,..., $\|x_{2,k}-x_2^*\| \to 0 \ (k\to\infty)$.

And so $x_{i,k} \to x_i^*(k\to\infty)$ for $i = 1,2,...,n$. That is, $(x_{1,k},x_{2,k},\cdots,x_{n,k})$ converges strongly to the unique solution $(x_1^*,x_2^*,\cdots,x_n^*)$ of the problem (3.1).

By using similar argument with the proof of Theorem 5.1, we have

COROLLARY 5.2. *Let S_i be the same as in Corollary 4.2, and suppose that the sequences $\{x_{1,k}\},\{x_{2,k}\},\cdots,\{x_{n,k}\}$ are generated by algorithm 4.2. If the condition (4.4) holds, then $(x_{1,k},x_{2,k},\cdots,x_{n,k})$ converges strongly to the unique solution $(x_1^*,x_2^*,\cdots,x_n^*)$ of the problem (3.4).*

COROLLARY 5.3. *Let S_i be the same as in Corollary 4.2, and suppose that the sequences $\{x_{1,k}\},\{x_{2,k}\},\cdots,\{x_{n,k}\}$ are generated by algorithm 4.3. If the condition (4.4) holds, then $(x_{1,k},x_{2,k},\cdots,x_{n,k})$ converges strongly to the unique solution $(x_1^*,x_2^*,\cdots,x_n^*)$ of the problem (3.4).*

Let $n=2$, by Theorem 5.1 and Corollary 5.3, respectively, we have

COROLLARY 5.4. *Let T_i and S_i be the same as in Corollary 4.3, and suppose that the sequences $\{x_{1,k}\},\{x_{2,k}\}$ are generated by algorithm 4.4. If the condition (4.5) holds, then $(x_{1,k},x_{2,k},\cdots,x_{n,k})$ converges strongly to the unique solution (x_1^*,x_2^*) of the problem (3.2).*

COROLLARY 5.5. *Let S_i be the same as in Corollary 4.5, and suppose that the sequences $\{x_{1,k}\}, \{x_{2,k}\}$ are generated by algorithm 4.5. If the condition (4.6) holds, then $(x_{1,k}, x_{2,k})$ converges strongly to the unique solution (x_1^*, x_2^*) of the problem (3.5).*

REMARK 5.1. By Corollary 5.4 and Corollary 5.5, respectively, we can recover Theorem 3.5 in [25] and Theorem 2.1(b) in [41]. And so Theorem 5.1, Corollary 5.2 and Corollary 5.3 extend and improve the corresponding results in [41–44, 25].

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