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ON AN ABSTRACT VERSION OF A FUNCTIONAL INEQUALITY

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Abstract. The purpose of the present paper is to provide a joint generalization of some previously obtained results connected with functional inequalities, stability of functional equations and stability of functional inequalities.

1. Introduction

Throughout the paper it is assumed that $\mathbb{N} = \{1,2,\dots\}$, $\mathbb{R}$ is the real line and $[c,+\infty) = \{t \in \mathbb{R} : t \geqslant c\}$ for $c \in \mathbb{R}$. We use additive notation in abelian groups; abelian group Y is uniquely 2-divisible if the mapping $Y \ni t \mapsto t+t \in Y$ is bijective. By a topological group Y we understand a group endowed with a topology such that Y is a Hausdorff space and both mappings

$$Y \ni t \mapsto -t \in Y$$

and

$$Y \times Y \ni (s,t) \mapsto s+t \in Y$$

are continuous. In particular, each real or complex linear-topological space is a uniquely 2-divisible abelian topological group. A systematic study of the theory of topological groups can be found in the monograph of E. Hewitt and K. Ross [5].

Let X and Y be abelian groups. A mappping $a: X \to Y$ is called *additive* if it satisfies the Cauchy functional equation:

$$a(x+y) = a(x) + a(y), \quad x,y \in X.$$

A map $A: X \to \mathbb{R}$ is *subadditive* if

$$A(x+y) \leqslant A(x) + A(y), \quad x,y \in X,$$

a mappping $A: X \to \mathbb{R}$ is *superadditive* if $-A$ is subadditive. Function $q: X \to Y$ is *quadratic* if it satisfies the following functional equation:

$$q(x+y) + q(x-y) = 2q(x) + 2q(y), \quad x,y \in X.$$

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Finally, a mapping $B: X \times X \to Y$ is *symmetric and biadditive* if

$$B(x,y) = B(y,x), \quad x,y \in X,$$

and for each $x \in X$ the function $B(x,\cdot): X \to Y$ is additive.

Relationships between quadratic mappings and biadditive and symmetric functionals are described by the following theorem (see J. Aczél & J. Dhombres [1, Chapter 11, Proposition 1]).

THEOREM A. *Let X be an abelian group and Y a uniquely 2-divisible abelian group. A mapping $q: X \to Y$ is quadratic if and only if there exists a biadditive and symmetric functional $B: X \times X \to Y$ such that*

$$q(x) = \frac{1}{2}B(x,x), \quad x \in X.$$

Moreover,

$$q(x+y) - q(x) - q(y) = B(x,y), \quad x,y \in X.$$

From this theorem it follows that if $K \subset Y$ is an arbitrary nonempty set, $q: X \to Y$ is a quadratic mapping, $B: X \times X \to Y$ is the corresponding biadditive and symmetric functional, $A: X \to Y$ satisfies

$$A(x+y) - A(x) - A(y) \in K, \quad x,y \in X, \tag{1}$$

and $f = q + A$, then the following relation holds true:

$$f(x+y) - f(x) - f(y) - B(x,y) \in K, \quad x,y \in X. \tag{2}$$

In the present paper we are interested in situations, where the converse implication is valid. At least two special cases of this problem have already been investigated.

In [2] and [3] we have dealt with the following functional inequality:

$$f(x+y) - f(x) - f(y) \geqslant \phi(x,y), \quad x,y \in X,$$

where $(X,+)$ is an abelian group and f and ϕ are real mappings defined on X. We have found conditions that are sufficient for the representation

$$f(x) = \frac{1}{2}\phi(x,x) + A(x), \quad x \in X,$$

where $A: X \to \mathbb{R}$ is a superadditive mapping and $\phi: X \times X \to \mathbb{R}$ is biadditive and symmetric (see [2, Theorem 1 and Corollary 1] and [3, Theorem 16]).

In [4] we have investigated some Hyers-Ulam stability problems connected with quadratic mappings. In particular, we have solved the following system:

$$\begin{aligned} \|f(x+y) - f(x) - f(y) - 2\phi(x,y)\| &\leqslant \varepsilon, \quad x,y \in X, \\ \|\phi(x,y) + \phi(x,-y)\| &\leqslant \eta, \quad x,y \in X, \end{aligned}$$

where $\varepsilon \geqslant 0$, $\eta \geqslant 0$ and f and ϕ map an abelian group into a Banach space ([4, Corollary 3]).

The purpose of the present paper is to generalize the above-mentioned results. We will be working in the following framework: functions considered will have their values in a uniquely 2-divisible topological group Y, a fixed set $K \subset Y$ will be playing the role of the closed halfline $[0,+\infty) \subset \mathbb{R}$ or of the closed ball $B(0,\varepsilon_1) \subset X$ with the center at 0 and of radius $\varepsilon_1 = \max\{\varepsilon,\eta\}$, respectively.

2. Main results

Let $(Y,+)$ be a uniquely 2-divisible abelian group and let $K \subset Y$; denote $K^- := K \cap (-K)$. Let us start with enumeration of assumptions we impose upon the set K.

$$\underbrace{K + \ldots + K}_{n \text{ times}} \subset nK, \quad n \in \mathbb{N}, \tag{K1}$$

$$\bigcap_{n\in\mathbb{N}} \frac{1}{4^n} K^- = \{0\}, \tag{K2}$$

if $(x_n)_{n\in\mathbb{N}}$ is a sequence from Y having the following two properties : (K3)

- $x_{n+1} - x_n \in \frac{1}{2^n}K$ for sufficiently large $n \in \mathbb{N}$,
- there exists $c \in Y$ such that $c - x_n \in \frac{1}{2^n}K$ for sufficiently large $n \in \mathbb{N}$,

then the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent.

Note that from (K1) and (K2) it follows in particular that $0 \in K$ and $nK \subset mK$ for $n, m \in \mathbb{N}$ such that $n \leqslant m$.

We start with a lemma, which is a generalization of [3, Lemma 3]. To prove this lemma we do not need (K3).

LEMMA 1. *Let X be an abelian group, Y a uniquely 2-divisible abelian group and let $K \subset Y$ fulfill* (K1) *and* (K2). *Assume that $f: X \to Y$ and $\phi: X\times X \to Y$ satisfy*

$$f(x+y) - f(x) - f(y) - \phi(x,y) \in K, \quad x,y \in X, \tag{3}$$

$$\phi(x,y) + \phi(x,-y) \in K, \quad x,y \in X, \tag{4}$$

and

$$f(2x) = 3f(x) + f(-x), \quad x \in X. \tag{5}$$

Then, there exist an additive mapping $a: X \to Y$ and a quadratic mapping $q: X \to Y$ such that $f = a + q$. Moreover,

$$f(x+y) - f(x) - f(y) - \phi(x,y) \in 2K^-, \quad x,y \in X. \tag{6}$$

Proof. Apply (3) with y replaced by $-y$ to get

$$f(x-y) - f(x) - f(-y) - \phi(x,-y) \in K, \quad x,y \in X.$$

Then, add side by side this relation, (3) and (4) and then use (K1) to obtain

$$f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) \in K + K + K = 3K, \quad x,y \in X. \tag{7}$$

Now, split f into its odd and even part: $f = f_e + f_o$, where

$$f_e(x) := \frac{f(x) + f(-x)}{2}, \quad f_o(x) := \frac{f(x) - f(-x)}{2}, \quad x \in X.$$

From (7) it follows that

$$\begin{aligned} f_e(x+y) &+ f_e(x-y)-2f_e(x)-2f_e(y) \\ &= \frac{1}{2}\left[f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\right] \\ &+ \frac{1}{2}\left[f(-x-y)+f(-x+y)-2f(-x)-f(-y)-f(y)\right] \\ &\in \frac{1}{2}\left[3K+3K\right]=3K, \quad x,y\in X. \end{aligned} \tag{8}$$

On the other hand, (5) gives us

$$f_e(2x)=4f_e(x), \quad x\in X. \tag{9}$$

In particular, $f(0)=f_e(0)=0$.

Now, let us apply (8) with x replaced by $x+y$ and y replaced by $x-y$. Taking (9) into account we get

$$\begin{aligned} 4f_e(x) &+ 4f_e(y)-2f_e(x+y)-2f_e(x-y) \\ &= f_e(2x)+f_e(2y)-2f_e(x+y)-2f_e(x-y) \\ &= f_e((x+y)+(x-y))+f_e((x+y)-(x-y))-2f_e(x+y)-2f_e(x-y) \\ &\in 3K, \quad x,y\in X, \end{aligned}$$

which means that

$$f_e(x+y)+f_e(x-y)-2f_e(x)-2f_e(y)\in -\frac{3}{2}K, \quad x,y\in X.$$

This, jointly with (8) and with (K1) implies that

$$f_e(x+y)+f_e(x-y)-2f_e(x)-2f_e(y)\in 3K\cap\left(-\frac{3}{2}K\right)\subset 3K^-, \quad x,y\in X.$$

From this and from (9) we derive that

$$\begin{aligned} 4^n[f_e(x+y) &+ f_e(x-y)-2f_e(x)-2f_e(y)] \\ &= f_e(2^nx+2^ny)+f_e(2^nx-2^ny)-2f_e(2^nx)-2f_e(2^ny) \\ &\in 3K^-, \quad x,y\in X,\ n\in\mathbb{N}, \end{aligned}$$

which together with (K2) leads to the equality

$$f_e(x+y)+f_e(x-y)-2f_e(x)-2f_e(y)=0, \quad x,y\in X.$$

Now, this compared with (7) gives us

$$\begin{aligned} f_o(x+y) &+ f_o(x-y)-2f_o(x) \\ &= f_o(x+y)+f_o(x-y)-2f_o(x)-f_o(y)-f_o(-y) \\ &= f(x+y)+f(x-y)-2f(x)-f(y)-f(-y) \\ &- [f_e(x+y)+f_e(x-y)-2f_e(x)-f_e(y)-f_e(-y)] \\ &\in 3K, \quad x,y\in X. \end{aligned} \tag{10}$$

After replacing x by $-x$ and y by $-y$ and using the oddness of f_o we arrive at

$$f_o(x+y)+f_o(x-y)-2f_o(x)\in 3K^-,\quad x,y\in X.$$

Observe also that from (5) and the definition of f_o it follows that

$$f_o(4x)=4f_o(x),\quad x\in X.$$

Using this we get the relation

$$\begin{aligned}4^n[f_o(x+y) &+ f_o(x-y)-2f_o(x)]\\ &= f_o(4^nx+4^ny)+f_o(4^nx-4^ny)-2f_o(4^nx)\\ &\in 3K^-,\quad x,y\in X,\, n\in\mathbb{N},\end{aligned}$$

which jointly with (K2) implies that

$$f_o(x+y)+f_o(x-y)-2f_o(x)=0,\quad x,y\in X.$$

Since f_o is odd, then f_o is additive.

Denote $q:=f_e$, $a:=f_o$ and define $B\colon X\times X\to Y$ by

$$B(x,y):=q(x+y)-q(x)-q(y),\quad x,y\in X.$$

Theorem A states that B is biadditive and symmetric. We have

$$B(x,y)-\phi(x,y)=f(x+y)-f(x)-f(y)-\phi(x,y)\in K,\quad x,y\in X.$$

On the other hand, this together with (4) gives us

$$\begin{aligned}-B(x,y)+\phi(x,y) &= B(x,-y)-\phi(x,-y)+\phi(x,y)+\phi(x,-y)\\ &\in K+K=2K,\quad x,y\in X.\end{aligned}$$

Therefore, we eventually arrive at

$$B(x,y)-\phi(x,y)\in K\cap(-2K)\subset 2K^-,\quad x,y\in X,$$

and (6) follows. □

Now, we are able to state and prove our main result. The following theorem is a generalization of [3, Theorem 16].

THEOREM 2. *Let X be an abelian group, Y a uniquely 2-divisible abelian topological group and let $K\subset Y$ be a closed set which fulfills* (K1), (K2) *and* (K3). *Assume that $f\colon X\to Y$ and $\phi\colon X\times X\to Y$ satisfy* (3), (4) *and*

$$\left.\begin{array}{c}\forall_{x\in X}\exists_{c\in Y}\left[c-\frac{1}{4^n}\phi(2^nx,2^nx)\in\frac{1}{4^n}K \text{ for sufficiently large } n\right],\\ \forall_{x,y\in X}\left[\lim_{n\to+\infty}\frac{1}{4^n}\phi(2^nx,2^ny)-\phi(x,y)\in K\right].\end{array}\right\}\tag{11}$$

If for each $x\in X$ the sequence

$$\left(\frac{f(2^nx)-f(-2^nx)}{2^{n+1}}\right)_{n\in\mathbb{N}}\tag{12}$$

is convergent and its pointwise limit satisfies (1)*, then there exist a quadratic mapping* $q\colon X \to Y$ *and a solution* $A\colon X \to Y$ *of*

$$A(x+y) - A(x) - A(y) \in 18K, \quad x, y \in X, \tag{13}$$

such that $f = q + A$*. Moreover,*

$$q(x+y) - q(x) - q(y) - \phi(x,y) \in 17K^-, \quad x, y \in X. \tag{14}$$

Proof. Apply (3) and (4) for $x = y = 0$ to get that $-f(0) - \phi(0,0) \in K$ and $\phi(0,0) \in \frac{1}{2}K$. From this we deduce that $-f(0) \in \frac{3}{2}K$. Next, put $y = -x$ in (3) and use (4) to obtain

$$\begin{aligned} \phi(x,x) &- f(x) - f(-x) \\ &= [f(0) - f(x) - f(-x) - \phi(x,-x)] + [\phi(x,x) + \phi(x,-x)] - f(0) \\ &\in K + K + \frac{3}{2}K = \frac{7}{2}K, \quad x \in X. \end{aligned} \tag{15}$$

On the other hand, on applying (3) for $y = x$ we derive that

$$f(2x) - 2f(x) - \phi(x,x) \in K, \quad x \in X,$$

which jointly with (15) implies that

$$f(2x) - 3f(x) - f(-x) \in K + \frac{7}{2}K = \frac{9}{2}K, \quad x \in X. \tag{16}$$

Now, for each $x \in X$ define two sequences of elements of Y in the following way:

$$\alpha_n(x) := \frac{f(2^n x) - f(-2^n x)}{2^{n+1}}, \quad \beta_n(x) := \frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n}, \quad n \in \mathbb{N}.$$

The first sequence is convergent by assumption; therefore a map $\alpha\colon X \to Y$ is well defined by

$$\alpha(x) := \lim_{n \to +\infty} \alpha_n(x), \quad x \in X,$$

and by assumption it satisfies

$$\alpha(x+y) - \alpha(x) - \alpha(y) \in K, \quad x, y \in X. \tag{17}$$

We will check that the second sequence is convergent. This follows from (15), (16) and from (K3). Indeed,

$$\begin{aligned} \beta_{n+1}(x) - \beta_n(x) &= \frac{f(2^{n+1}x) + f(-2^{n+1}x) - 4f(2^n x) - 4f(-2^n x)}{2 \cdot 4^{n+1}} \\ &= \frac{1}{2 \cdot 4^{n+1}} [f(2 \cdot 2^n x) - 3f(2^n x) - f(-2^n x)] \\ &+ \frac{1}{2 \cdot 4^{n+1}} [f(2 \cdot (-2^n x)) - 3f(-2^n x) - f(2^n x)] \\ &\in \frac{1}{2 \cdot 4^{n+1}} \left[\frac{9}{2}K + \frac{9}{2}K\right] \subset \frac{1}{4^{n-1}}K, \quad x \in X, \, n \in \mathbb{N}. \end{aligned}$$

Next, let $c \in Y$ (possibly depending upon x) be chosen so that the first part of (11) is fulfilled. We have

$$\begin{aligned} \frac{1}{2}c - \beta_n(x) &= \frac{1}{2}c - \frac{1}{2\cdot 4^n}\phi(2^n x, 2^n x) \\ &+ \frac{1}{2\cdot 4^n}\left[\phi(2^n x, 2^n x) - f(2^n x) - f(-2^n x)\right] \\ &\in \frac{1}{2\cdot 4^n}K + \frac{1}{2\cdot 4^n}\cdot\frac{7}{2}K \subset \frac{1}{4^{n-1}}K, \quad x \in X,\ n \in \mathbb{N}. \end{aligned}$$

Therefore, $(\beta_n(x))_{n\in\mathbb{N}}$ is convergent. Let $\beta\colon X \to Y$ be the limit function:

$$\beta(x) := \lim_{n\to+\infty} \beta_n(x), \quad x \in X.$$

Now, it is clear that

$$\alpha(2x) = 2\alpha(x), \quad \beta(2x) = 4\beta(x), \quad x \in X.$$

Moreover, α is odd, whereas β is even. Next, put $\varphi := \alpha + \beta$ and observe that

$$\varphi(2x) = 3\varphi(x) + \varphi(-x), \quad x \in X.$$

Define, $\phi_1\colon X \times X \to Y$ by

$$\phi_1(x,y) := \frac{1}{2}\left[\phi(x,y) + \phi(-x,-y)\right], \quad x,y \in X.$$

With the aid of (3) we get

$$\frac{1}{2\cdot 4^n}\left[f(2^n x + 2^n y) - f(2^n x) - f(2^n y) - \phi(2^n x, 2^n y)\right] \in \frac{1}{2\cdot 4^n}K, \quad x,y \in X.$$

This, the second part of (11) and the fact that K is closed lead to the following estimations:

$$\begin{aligned} \beta(x+y) &- \beta(x) - \beta(y) - \phi_1(x,y) \\ &= \lim_{n\to+\infty} \frac{1}{2\cdot 4^n}\left[f(2^n x + 2^n y) - f(2^n x) - f(2^n y) - \phi(2^n x, 2^n y)\right] \\ &+ \frac{1}{2\cdot 4^n}\left[f(-2^n x - 2^n y) - f(-2^n x) - f(-2^n y) - \phi(-2^n x, -2^n y)\right] \\ &+ \frac{1}{2}\left[\frac{1}{4^n}\phi(2^n x, 2^n y) - \phi(x,y) + \frac{1}{4^n}\phi(-2^n x, -2^n y) - \phi(-x,-y)\right] \\ &\in \frac{1}{2}\left[K + K\right] = K, \quad x,y \in X. \end{aligned} \tag{18}$$

Now, deduce that

$$\begin{aligned} \varphi(x+y) &- \varphi(x) - \varphi(y) - \phi_1(x,y) \\ &= \alpha(x+y) - \alpha(x) - \alpha(y) + \beta(x+y) - \beta(x) - \beta(y) - \phi_1(x,y) \\ &\in K + K = 2K, \quad x,y \in X, \end{aligned} \tag{19}$$

and

$$\begin{aligned}\phi_1(x,y)+\phi_1(x,-y) &= \frac{1}{2}\left[\phi_1(x,y)+\phi_1(x,-y)+\phi_1(-x,-y)+\phi_1(-x,y)\right]\\ &\in \frac{1}{2}\left[K+K\right]=K\subset 2K, \quad x,y\in X.\end{aligned}$$

Thus, the assumptions of Lemma 1 are satisfied by φ, ϕ_1 and $2K$. From this we get that α is additive, β is quadratic and

$$\varphi(x+y)-\varphi(x)-\varphi(y)-\phi_1(x,y)\in 4K^-, \quad x,y\in X,$$

which implies that

$$\beta(x+y)-\beta(x)-\beta(y)-\phi_1(x,y)\in 4K^-, \quad x,y\in X. \tag{20}$$

Next, let us compare this with some inclusions used to check (18). In particular, for each $x,y\in X$ the sequence

$$\left(\frac{1}{4^n}\left[f(2^nx+2^ny)-f(2^nx)-f(2^ny)\right]\right)_{n\in\mathbb{N}}$$

is convergent and

$$\lim_{n\to+\infty}\frac{1}{4^n}\left[f(2^nx+2^ny)-f(2^nx)-f(2^ny)\right]-\phi(x,y)\in K, \quad x,y\in X.$$

On the other hand, we have

$$\begin{aligned}&\lim_{n\to+\infty}\frac{1}{4^n}\left[f(2^nx+2^ny)-f(2^nx)-f(2^ny)\right]-\phi(x,y)\\ &= 2\left[\beta(x+y)-\beta(x)-\beta(y)-\phi_1(x,y)\right]\\ &- \frac{1}{4^n}\left[f(-2^nx-2^ny)-f(-2^nx)-f(-2^ny)\right]+\phi(-x,-y)\\ &\in 8K^- - K\subset -9K, \quad x,y\in X.\end{aligned}$$

Joining this two facts we obtain

$$\lim_{n\to+\infty}\frac{1}{4^n}\left[f(2^nx+2^ny)-f(2^nx)-f(2^ny)\right]-\phi(x,y)\in -9K\cap K\subset 9K^-, \tag{21}$$

for each $x,y\in X$.

Now, put $f_1:=f-\varphi$ and $\phi_2:=\phi-\phi_1$; clearly,

$$\begin{aligned}f_1(x+y) &- f_1(x)-f_1(y)-\phi_2(x,y)\\ &= f(x+y)-f(x)-f(y)-\phi(x,y)\\ &- \varphi(x+y)+\varphi(x)+\varphi(y)+\phi_1(x,y)\\ &\in K-4K^-\subset 5K, \quad x,y\in X.\end{aligned} \tag{22}$$

Moreover, from (20) and (21) we derive that

$$\begin{aligned}
&\lim_{n\to+\infty}\frac{1}{4^n}\left[f_1(2^nx+2^ny)-f_1(2^nx)-f_1(2^ny)\right]-\phi_2(x,y)\\
&= \lim_{n\to+\infty}\frac{1}{4^n}\left[f(2^nx+2^ny)-f(2^nx)-f(2^ny)\right]-\phi(x,y)\\
&- \lim_{n\to+\infty}\frac{1}{4^n}\left[\varphi(2^nx+2^ny)-\varphi(2^nx)-\varphi(2^ny)\right]+\phi_1(x,y)\\
&\in 9K^- -\beta(x+y)+\beta(x)+\beta(y)+\phi_1(x,y)\\
&\subset 9K^- -4K^- = 13K^-, \quad x,y\in X.
\end{aligned} \tag{23}$$

Let us introduce two new functions, $P\colon X\to Y$ and $g\colon X\to Y$ by

$$P(x) := \frac{f_1(x)+f_1(-x)}{2}, \quad g(x) := \frac{f_1(x)-f_1(-x)}{2}, \quad x\in X.$$

Observe also that

$$\phi_2(x,y) = -\phi_2(-x,-y), \quad x,y\in X,$$

and using this and (22) deduce that

$$\begin{aligned}
P(x+y) &- P(x)-P(y)\\
&= \frac{1}{2}\left[f_1(x+y)-f_1(x)-f_1(y)+f_1(-x-y)-f_1(-x)-f_1(-y)\right]\\
&= \frac{1}{2}\left[f_1(x+y)-f_1(x)-f_1(y)-\phi_2(x,y)\right]\\
&+ \frac{1}{2}\left[f_1(-x-y)-f_1(-x)-f_1(-y)-\phi_2(-x,-y)\right]\\
&\in \frac{1}{2}\left[5K+5K\right] = 5K, \quad x,y\in X.
\end{aligned} \tag{24}$$

Put $x=y=0$ in (24) to obtain that $-P(0)\in 5K$. Now, apply (24) with $x=y$ to get that $P(2x)-2P(x)\in 5K$ for each $x\in X$. Next, (24) used for $y=-x$ leads to

$$P(0)-2P(x)\in 5K, \quad x\in X.$$

Therefore, we have the following relations:

$$\begin{aligned}
\frac{1}{2^{n+1}}P(2^{n+1}x)-\frac{1}{2^{n+1}}P(2^nx) &\in \frac{5}{2^{n+1}}K, \quad x\in X,\ n\in\mathbb{N},\\
\frac{1}{2^{n+1}}P(0)-\frac{1}{2^n}P(2^nx) &\in \frac{5}{2^{n+1}}K, \quad x\in X,\ n\in\mathbb{N}.
\end{aligned}$$

(K3) implies that the sequence $(\frac{1}{2^n}P(2^nx))_{n\in\mathbb{N}}$ is pointwise convergent. In particular, $\lim_{n\to+\infty}\frac{1}{4^n}P(2^nx)=0$ for each $x\in X$. On joining this observation with (23) we arrive at

$$\lim_{n\to+\infty}\frac{1}{4^n}\left[g(2^nx+2^ny)-g(2^nx)-g(2^ny)\right]-\phi_2(x,y)\in 13K^-, \quad x,y\in X.$$

On the other hand,

$$\begin{aligned}\lim_{n\to+\infty}\frac{g(2^nx)}{2^n} &= \lim_{n\to+\infty}\frac{f_1(2^nx)-f_1(-2^nx)}{2^{n+1}}\\ &= \lim_{n\to+\infty}\frac{f(2^nx)-f(-2^nx)}{2^{n+1}} - \lim_{n\to+\infty}\frac{\varphi(2^nx)-\varphi(-2^nx)}{2^{n+1}}\\ &= \alpha(x)-\alpha(x)=0, \quad x\in X.\end{aligned}$$

Therefore, we have proved that $\phi_2(x,y)\in 13K^-$ for each $x,y\in X$.

Put $A:=f_1+\alpha$ and $q:=\beta$. To finish the proof one needs to join the last observation with (20) to obtain (14):

$$\begin{aligned}q(x+y) &- q(x)-q(y)-\phi(x,y)\\ &= \beta(x+y)-\beta(x)-\beta(y)-\phi_1(x,y)-\phi_2(x,y)\in 4K^-+13K^-\\ &= 17K^-, \quad x,y\in X.\end{aligned}$$

And joining this with (3) one may prove (13):

$$\begin{aligned}A(x+y) &- A(x)-A(y)\\ &= f(x+y)-f(x)-f(y)-\phi(x,y)\\ &- [q(x+y)-q(x)-q(y)-\phi(x,y)]\\ &\in K+17K^-\subset 18K, \quad x,y\in X. \qquad \square\end{aligned}$$

Now, let us derive two corollaries from Theorem 2.

COROLLARY 3. *Let X be an abelian group, Y a uniquely 2-divisible abelian topological group and let $K\subset Y$ be a closed set which fulfills* (K1), (K2) *and* (K3). *Assume that $f:X\to Y$ and $\phi:X\times X\to Y$ satisfy* (3), (4) *and* (11). *If the map f is even, then there exist a quadratic mapping $q:X\to Y$ and an even solution $A:X\to Y$ of* (13) *such that $f=q+A$. Moreover, relation* (14) *is satisfied.*

Proof. Since f is even, then the sequence (12) from Theorem 2 is constant and equal to zero and thus the assumptions of this theorem are satisfied. The evenness of A follows from the evenness of f and q. $\square$

COROLLARY 4. *Let X be an abelian group, Y a uniquely 2-divisible abelian topological group and let $K\subset Y$ be a closed set which fulfills* (K1), (K2) *and* (K3). *Assume that $f:X\to Y$ and $\phi:X\times X\to Y$ satisfy* (3), (4), (11) *and*

$$\phi(x,y)=\phi(-x,-y), \quad x,y\in X. \tag{25}$$

Then there exist a quadratic mapping $q:X\to Y$ and a solution $A:X\to Y$ of (13) *such that $f=q+A$. Moreover, relation* (14) *is satisfied.*

Proof. Split f into its even and odd parts, say f_e and f_o, and check that the even part f_e satisfies assumptions of Corollary 3. In particular, relation (14) holds true with a quadratic mapping q. Now, it sufficies to use this and repeat the last calculation from the proof of Theorem 2 to check that the map $A:=f-q$ fulfills (13). $\square$

3. Concluding remarks

We will terminate this paper with some additional remarks.

REMARK 5. In Lemma 1 it is not stated that functional ϕ is biadditive and symmetric. This was proved in [3, Lemma 3] in the special case $K = [0,+\infty)$. Clearly, it holds true if $K^- = \{0\}$; one may find an example which shows that this is not true in general. Take $X = Y = \mathbb{R}$, $c > 0$ and put $f = 0$, $\phi = c$ and $K = [-2c,+2c]$ to see that the assumptions of Lemma 1 are satisfied by these mappings. Therefore, Lemma 1 cannot be strengthen in that way.

REMARK 6. Constants 17 and 18 appearing in Corollary 3 and in Corollary 4 can be sharpened by a thorough inspection of the proof of Theorem 2 and by applying there the evenness of f or the assumption (25), respectively. Moreover, some additional informations about the set K may lead to further improvements of these estimations. In particular, if the set K is not symmetric with respect to 0, then inclusion $K^- \subset K$, which was frequently used in the proof of Theorem 2, may be replaced by a sharper one.

REMARK 7. If Y is a Banach space and the set K is bounded, then the celebrated Hyers Theorem (see D.H. Hyers [6]) can be applied to relation (13). Therefore, on joining the presented results with the Hyers Theorem one may obtain that in this case f is uniformly close to the sum $a+q$ of an additive mapping a and a quadratic one q.

REMARK 8. We will provide one more example of sets K satisfying assumptions (K1), (K2) and (K3). Let Y be a real locally convex linear-topological space. Denote by Y^* the family of all linear and continuous functionals on Y. For any subfamily $E^* \subset Y^*$ let $\mathscr{T}(E^*)$ stand for the weakest topology on Y such that all elements of E^* remain continuous under this topology. Therefore, $\mathscr{T}(Y^*)$ is the weak topology on Y.

If Y is considered with the weak topology, then in particular Y is a uniquely 2-divisible abelian group and each nonempty set $K \subset Y$ having the following property:

$$\mathscr{T}\left(\{y^* \in Y^* : \forall_{x \in K}\, [y^*(x) \geqslant -c]\}\right) = \mathscr{T}(Y^*)$$

for a certain fixed $c \geqslant 0$ is closed and satisfies (K1), (K2) and (K3). In particular, taking $Y = \mathbb{R}^2$ and $c = 0$ we can see that each closed quarter of the real plane $\mathbb{R}^2$ fulfills this condition.

Similarly, each nonempty set $K \subset Y$ satisfying

$$\mathscr{T}\left(\{y^* \in Y^* : \forall_{x \in K}\, [-c \leqslant y^*(x) \leqslant c]\}\right) = \mathscr{T}(Y^*)$$

fulfills all this conditions and additionally is compact.

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