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ON CERTAIN MEAN VALUE THEOREMS

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Abstract. In this paper we have obtained a symmetric integral mean $M(a,b;p(r_{n,k}),q)$ involving functions which a generalization of the arithmetic-geometric mean of Gauss. We have also proved some characterization of the symmetric mean values for the twice continuously differentiable function p.

1. Introduction

The mean values are related to a continuously differentiable real valued function and a functional equation. The properties of the mean provide a bridge between the local and global properties of the functions. Some of the mean values are important in themselves, and some of them are required for applications in the theory of analytic inequalities(see references below and some of the references sited therein). We shall begin with the classical mean values. By classical mean value(or mean) we understand a special function $M:\langle 0,\infty\rangle^{2}\to\langle 0,\infty\rangle$, which satisfies the condition

$$\min\{a,b\}\leqslant M(a,b)\leqslant \max\{a,b\},\quad a,b>0.$$

Of course, a mean has the reflexivity property $M(a,a)=a$, for $a>0$. It is called symmetric of $M(a,b)=M(b,a)$, for $a,b>0$. The simplest and classical means are defined as follows: the arithmetic mean or average, $A(a,b)=(a+b)/2$; the geometric or mean proportional, $G(a,b)=\sqrt{ab}$, and the harmonic mean, $H(a,b)=G^2(a,b)/A(a,b)$. These means have been generalized, refined and extended in several directions. The root mean square is defined ad $N=(G+A)/2$, and the power mean as $M_r(a,b)=\left(\left(a^r+b^r\right)/2\right)^{1/r}$ for $r\neq 0$ with $M_0(a,b)=G(a,b)$.

Further developments led to definitions of other types of means, including; multi-variable means with $(a_1,a_1,\dots,a_n)$ replacing (a,b); abstracted mean

$$M_\varphi=\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right),$$

which reduce to M_r when $\varphi(a)=a^r$; weighted means given by $(1-\alpha)a+\alpha b$ and $a^{1-\alpha}b^{\alpha}, 0\leqslant\alpha\leqslant 1$; and Lehmer means $L_p(a,b)=(a^p+b^p)/\left(a^{p-1}+b^{p-1}\right)$ for

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$p > 0,$ which reduce to the anti-harmonic mean $L_2(a,b) = \left(a^2+b^2\right)/(a+b)$. Along with the means M_r there are more extended means of particular interest. Pólya and Szegö[13] defined the logarithmic mean L by

$$L = L(a,b) = \frac{a-b}{\ln a - \ln b} \tag{1.1}$$

for $a > 0, b > 0, a \neq b$ and $L(a,a) = a$. Galvani [5] consider the extended logarithmic means

$$S_p(a,b) = \left(\frac{b^p - a^p}{p(b-a)}\right)^{\frac{1}{p-1}}, \quad a \neq b,\ p \neq 0,1; \tag{1.2}$$

and $S_p(a,a) = a$; which is reduced to $S_0(a,b) = L(a,b)$, and the identical mean or the exponential mean $I(a,b)$;

$$S_1(a,b) = I(a,b) = e^{-1}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}, \quad a \neq b; \tag{1.3}$$

and $S_1(a,a) = I(a,a) = a$. The symmetric mean $Q_p(a,b)$ is also defined by

$$Q_p(a,b) = \frac{a^r b^s + a^s b^r}{2}, \tag{1.4}$$

where $r = (1+\sqrt{p})/2, s = (1-\sqrt{2})/2, p \geqslant 0$. The study of these means has a rich literature; for details one may see Alzer[2], Carlson[4], Mitrinović *at al.* [6], Kahlig and Matkowski [10], Kim and Rassias [12], Qi [14], Stolarsky [16] and Toader [17].

In what follows, we shall refer to another very well known example of mean. Given two positive numbers a and b, let us define successively the terms

$$a_{n+1} = A(a_n, b_n), \quad b_{n+1} = G(a_n, b_n), n \geqslant 0,$$

where $a_0 = a$ and $b_0 = b$. It is known (cf. [2]) that the sequences $\{a_n\}$ and $\{b_n\}$ are convergent to a common limit which is denoted by $A \cdot G(a,b)$. This function $A \cdot G(a,b)$ was investigated for the first time by Gauss. Thus, it is called *the arithmetic-geometric mean* of Gauss. It is also known (cf. [1]) the following representation formula

$$A \cdot G(a,b) = [I(a,b)]^{-1},$$

where

$$I(a,b) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}.$$

The proof of this formula is based on the fact that the function f verifies the functional relation

$$f\left(A(a,b), G(a,b)\right) = f(a,b),$$

which can be called *Gauss' functional equation.*

In [12], these results where generalized as follows. Let us denote by

$$r_{n,k}(\theta) = \left(a^{kn}\cos^2\theta + b^{kn}\sin^2\theta\right)^{\frac{1}{n}}, \quad (n,k \neq 0),$$

and

$$r_{0,k}(\theta) = \lim_{n\to 0} r_{n,k}(\theta) = a^{k\cos^2\theta} b^{k\sin^2\theta}, \quad (k \neq 0).$$

If $p : (0,\infty) \to R,\ p(x) = p$ is a strictly monotonic function and k, n are real numbers, then

$$M(a,b;p,r_{n,k}) = (ab)^{\frac{1-k}{2}} p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} p(r_{n,k}(\theta))\, d\theta\right) \tag{1.5}$$

defines a symmetric mean value on some bounded interval, that is such mean satisfies the property $0 < a \leqslant M(a,b;p,r_{n,k}) \leqslant b$ for $1 \leqslant b/a \leqslant T$, where T is a finite real number(see [1]). It is well-known that the arithmetic-geometric mean of Gauss is obtained for $k = 1, n = 2$ by $p(x) = x^{-1}$. The essential step was done in [8] by the consideration of the definition (1.5) for $k = 1, n = 2$ with an arbitrary $p(x)$. The case $n = 1$ was studied in [9] with $k = 1$. The general case (for arbitrary n) with $k = 1$ was studied in [17] and continued in [19].

In this paper, we obtain some new characterizations of the symmetric mean value form the right-hand side of (1.5) is replaced by the generalized mean $Q_p(a,b)$.

2. Characterizations of a symmetric mean

Let us consider the strictly monotonic function p. Using the function p we define a function

$$f(a,b;p,r_{n,k}) = \frac{1}{2\pi}\int_0^{2\pi} p(r_{n,k}(\theta))\, d\theta, \tag{2.1}$$

where $r_{n,k}(\theta)$ is defined by

$$r_{n,k}(\theta) = \left(a^{kn}\cos^2\theta + b^{kn}\sin^2\theta\right)^{\frac{1}{n}}, \quad (n,k \neq 0),$$

and

$$r_{0,k}(\theta) = \lim_{n\to 0} r_{n,k}(\theta) = a^{k\cos^2\theta} b^{k\sin^2\theta}, \quad (k \neq 0).$$

It is easy to prove that

$$M(a,b;p(r_{n,k}),q) = [Q_q(a,b)]^{(1-k)} p^{-1}[f(a,b;p(r_{n,k}))] \tag{2.2}$$

defines a mean on some bounded interval, where $Q_q(a,b)$ is defined by (1.4) as follows

$$Q_q(a,b) = \frac{a^r b^s + a^s b^r}{2}$$

for $r = (1+\sqrt{q})/2, s = (1-\sqrt{q})/2, g \geqslant 0$.

Certain characterizations of the mean $M(a,b;p(r_{n,k}),0)$ were studied in [12]. In this section, we obtain some new properties as special cases of the mean value defined by (2.2). If we set $p(s) = s, p(s) = 1/s, p(s) = 1/\log s, p(s) = 1/s^2$ in (2.2), by applying the three definite integral (see the paper [7], [8], [11] or [12])

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{a^k\cos^2\theta + b^k\sin^2\theta}\,d\theta = \frac{1}{(ab)^{\frac{k}{2}}},$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{(a^k\cos^2\theta + b^k\sin^2\theta)^2}\,d\theta = \frac{1}{2(ab)^{\frac{k}{2}}}\frac{a^k+b^k}{(ab)^k},$$

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left(a^k\cos^2\theta + b^k\sin^2\theta\right)\,d\theta = \log\left(\frac{a^{\frac{k}{2}}+b^{\frac{k}{2}}}{2}\right)^2,$$

we obtain the following results:

THEOREM 2.1. *Let $c_1(\neq 0), c_2$ and $k(\neq 0)$ be arbitrary real constants and let the function p be two times differentiable in $(0,\infty)$. Then*

(*i*) *$M(a,b;p(r_{1,k}),q) = Q_q^{1-k}(a,b)[(a^k+b^k)/2]$ holds for all positive real numbers a,b iff $p(s) = c_1 s + c_2$.*

(*ii*) *$M(a,b;p(r_{1,k}),q) = Q_q^{1-k}(a,b)(ab)^{k/2}$ holds for all positive real numbers a,b iff $p(s) = c_1(1/s) + c_2$.*

(*iii*) *$M(a,b;p(r_{1,k}),q) = Q_q^{1-k}(a,b)[(a^{k/2}+b^{k/2})/2]^2$ holds for all positive real numbers a,b iff $p(s) = c_1\log s + c_2$.*

(*iv*) *$M(a,b;p(r_{1,k}),q) = Q_q^{1-k}(a,b)[2(ab)^{3k/2}/(a^k+b^k)]^{1/2}$ holds for all positive real numbers a,b iff $p(s) = c_1(1/s^2) + c_2$.*

(*v*) *$M(a,b;p(r_{1,k}),q) = Q_q^{1-k}(a,b)[3(a^{2k}+b^{2k}) + 2(ab)^k/8]^{1/2}$ holds for all positive real numbers a,b iff $p(s) = c_1 s^2 + c_2$.*

Proof. We will show that the technique used in [7, 8, 9] for specific means works as well in the more general case stated here. We will prove (i). First, suppose that

$$M(a,b;p(r_{1,k}),q) = Q_q^{1-k}(a,b)\left(\frac{a^k+b^k}{2}\right)$$

for all positive numbers a and b. By the definition of $M(a,b;p(r_{1,k}),q)$, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}p(r_{1,k}(\theta))\,d\theta = p\left(\frac{a^k+b^k}{2}\right) \tag{2.3}$$

for all positive real numbers a and b. Setting $f(a,b;p(r_{1,k})) = (1/2\pi)\int_0^{2\pi}p(r_{1,k}(\theta))\,d\theta$ and applying (2.3) yields

$$f(a,b;p(r_{1,k})) = p\left(\frac{a^k+b^k}{2}\right) \tag{2.4}$$

for all positive real numbers a and b. Let c be an arbitrarily fixed positive real number. Operating on both sides (2.4) with $\partial^2/\partial a^2$ and setting $a = b = c$, in the resulting equality yields

$$f_{aa}(c,c;p(r_{1,k})) = \frac{1}{4}k^2c^{2k-2}p''\left(c^k\right) + \frac{1}{2}k(k-1)c^{k-2}p'\left(c^k\right). \tag{2.5}$$

The function f defined by (2.1) has the following partial derivative:

$$f_{aa}(c,c;p(r_{n,k})) = \frac{3}{8}k^2c^{2k-2}p''\left(c^k\right) + \frac{1}{8}\left(k^n + 3k^2 - 4k\right)c^{k-2}p'\left(c^k\right). \tag{2.6}$$

Combining (2.5) and (2.6), we obtain

$$p''(c^k) = 0. \tag{2.7}$$

Since c was and arbitrarily real number, we can replace c^k by a positive real variable s in the equality (2.7). Hence we have

$$p''(s) = 0. \tag{2.8}$$

Solving the differential equation (2.8) yields $p(s) = c_1 s + c_2$ in R^+, where c_1, c_2 are real constants with $c_1 \neq 0$.

Second we shall prove the "if" part. By using $r_{1,k}(\theta) = \left(a^k\cos^2\theta + b^k\sin^2\theta\right)$, $p^{-1} = (s - c_2)/c_1$ and $\int_0^{2\pi}\cos^2\theta\,d\theta = \int_0^{2\pi}\sin^2\theta\,d\theta = \pi$, we obtain

$$\begin{aligned} p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} p(r_{1,k}(\theta))\,d\theta\right) &= p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi}(c_1 r_{1,k}(\theta) + c_2)\,d\theta\right) \\ &= p^{-1}\left(\frac{1}{2\pi}c_1\left(a^k\pi + b^k\pi\right) + c_2\right) \\ &= \frac{a^k + b^k}{2} \end{aligned}$$

for all positive real numbers a and b. Hence, we get

$$M(a,b;p(r_{1,k}),q) = Q_q^{1-k}(a,b)\left(\frac{a^k + b^k}{2}\right).$$

The proofs of (ii), (iii), (iv) and (v) follow by an argument very similar to the one described in the proof of (i) with certain minor changes. □

By a reasoning similar to the Theorem 2.1 we can also prove the following assertions.

THEOREM 2.2. *Let $c_1(\neq 0), c_2$ and $k(\neq 0)$ be arbitrary real constants and let the function p be two times differentiable in $(0,\infty)$. Then*

(i) $M(a,b;p(r_{-1,k}),q) = Q_q^{1-k}(a,b)(ab)^{k/2}$ *holds for all positive real numbers a,b iff $p(s) = c_1 s + c_2$.*

(ii) $M(a,b;p(r_{-1,k}),q) = Q_q^{1-k}(a,b)[2a^kb^k/(a^k+b^k)]$ *holds for all positive real numbers a,b iff $p(s) = c_1(1/s) + c_2$.*

(iii) $M(a,b;p(r_{-1,k}),q) = Q_q^{1-k}(a,b)[4(ab)^k/(a^{k/2}+b^{k/2})^2]$ *holds for all positive real numbers a,b iff $p(s) = c_1\log s + c_2$.*

(iv) $M(a,b;p(r_{-1,k}),q) = Q_q^{1-k}(a,b)[8(ab)^{2k}/(3(a^{2k}+b^{2k}) + 2(ab)^k]^{1/2}$ *holds for all positive real numbers a,b iff $p(s) = c_1(1/s^2) + c_2$.*

(v) *$M(a,b;p(r_{-1,k}),q) = Q_q^{1-k}(a,b)[(ab)^{k/2}(a^k+b^k)/2]^{1/2}$ holds for all positive real numbers a,b iff $p(s) = c_1 s^2 + c_2$.*

Proof. We will prove (iii). First, suppose that

$$M(a,b;p(r_{-1,k}),q) = Q_q^{1-k}(a,b)\left[\frac{4(ab)^k}{\left(a^{\frac{k}{2}}+b^{\frac{k}{2}}\right)^2}\right]$$

for all positive numbers a and b. By the definition of $M(a,b;p(r_{1,k}),q)$, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} p(r_{-1,k}(\theta))\,d\theta = p\left(\frac{4a^k b^k}{\left(a^{\frac{k}{2}}+b^{\frac{k}{2}}\right)^2}\right) \tag{2.9}$$

for all positive real numbers a and b. Setting $f(a,b;p(r_{-1,k})) = (1/2\pi)\int_0^{2\pi} p(r_{-1,k}(\theta))\,d\theta$ and applying (2.9) yields

$$f(a,b;p(r_{-1,k})) = p\left(\frac{4a^k b^k}{\left(a^{\frac{k}{2}}+b^{\frac{k}{2}}\right)^2}\right) \tag{2.10}$$

for all positive real numbers a and b. Let c be an arbitrarily fixed positive real number. Operating on both sides (2.10) with $\partial^2/\partial a^2$ and setting $a=b=c$, in the resulting equality yields

$$f_{aa}(c,c;p(r_{-1,k})) = \frac{1}{4}k^2 c^{2k-2}p''\left(c^k\right) + \frac{1}{8}k(k-4)c^{k-2}p'\left(c^k\right). \tag{2.11}$$

The function f defined by (2.1) has the following partial derivative (2.6). Combining (2.6) and (2.11), we obtain

$$c^k p''\left(c^k\right) + p'\left(c^k\right) = 0.$$

Since c was and arbitrarily real number, we can replace c^k by a positive real variable s in the above equality. Hence we have

$$sp''(s) + p'(s) = 0.$$

with solution $p(s) = c_1 \log s + c_2$ in R^+, where c_1, c_2 are real constants with $c_1 \neq 0$.

To establish the other direction, by using $r_{-1,k}(\theta) = \left(a^k\cos^2\theta + b^k\sin^2\theta\right)^{-1}$,

$p^{-1}(s) = \exp((s-c_2)/c_1)$, after some standard calculations, one obtain

$$\begin{aligned} p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} p(r_{-1,k}(\theta))\,d\theta\right) &= p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi}(c_1\log(r_{1,k}(\theta))+c_2)\,d\theta\right) \\ &= p^{-1}\left(c_1\log\left(\frac{4a^kb^k}{\left(a^{\frac{k}{2}}+b^{\frac{k}{2}}\right)^2}\right)+c_2\right) \\ &= \frac{4a^kb^k}{\left(a^{\frac{k}{2}}+b^{\frac{k}{2}}\right)^2} \end{aligned}$$

for all positive real numbers a and b. Hence, we get

$$M(a,b;p(r_{1,k}),q) = Q_q^{1-k}(a,b)\left[\frac{4(ab)^k}{\left(a^{\frac{k}{2}}+b^{\frac{k}{2}}\right)^2}\right].$$

The proofs of (i), (ii), (iv) and (v) follow by an argument very similar to the one described in the proof of (iii) with some suitable changes. □

As it is shown in Theorem 2.1 and Theorem 2.2, the means $M(a,b;p(r_{n,k}),q)$ can represent some known means for a special choice of p and n. In the following, we shall determine the functions p for which $M(a,b;p(r_{r,k}),q)$ is the symmetric mean $Q_q(a,b)$, where $n(\neq 0), k(\neq 0)$ and $q(\geqslant 0)$ are arbitrary real numbers. We shall prove the following theorems.

THEOREM 2.3. *If for some twice continuously differentiable function p the mean $M(a,b;p(r_{n,k}),q)$ reduces to the symmetric mean $Q_q(a,b)$, then*

$$p(s) = \alpha s^{2q/k-n} + \beta, \tag{2.12}$$

where α and β are arbitrary real numbers.

Proof. Using function (2.1) and mean (2.2), we have

$$M(a,b;p(r_{n,k}),q) = Q_q(a,b)$$

if and only if

$$f(a,b;p(r_{n,k})) = p\left\{\left(\frac{a^rb^s+a^sb^r}{2}\right)^k\right\}$$

for $r=(1+\sqrt{q})/2, s=(1-\sqrt{q})/2, q\geqslant 0$. Applying $\partial^2/\partial a^2$ to both sides of the above equality, we get

$$f(a,b;p(r_{n,k})) = p''(C_1)\frac{\partial}{\partial a}C_1\frac{\partial}{\partial b}C_1 + p'(C_1)\frac{\partial^2}{\partial a\partial b}C_1,$$

where $C_1 = [a^r b^s + a^s b^r)/2]^k$. setting $a = b = c$, in the resulting equality yields

$$f_{ab}(c,c;p(r_{n,k})) = \frac{1}{4}k^2 c^{2k-2} p''\left(c^k\right) + \frac{1}{4}\left(k^2 - k - 4krs\right) c^{k-2} p'\left(c^k\right). \tag{2.13}$$

The function f defined by (2.1) has the following partial derivative:

$$f_{ab}(c,c;p(r_{n,k})) = \frac{1}{8}k^2 c^{2k-2} p''\left(c^k\right) + \frac{1}{8}(1-n)k^2 c^{k-2} p'\left(c^k\right). \tag{2.14}$$

Combining (2.13) and (2.14), we have

$$c^k p''\left(c^k\right) + \left(\frac{-2q}{k+n+1}\right) p'\left(c^k\right) = 0. \tag{2.15}$$

Since c^k was and arbitrarily real number, we can replace c^k by a positive real variable s in the equality (2.15). Hence we get

$$sp''(s) + \left(\frac{-2q}{k+n+1}\right) p'(s) = 0.$$

with solution (2.12). $\square$

In some case the condition given in Theorem 2.3 is also sufficient. We can formulate the following result for arbitrary n.

THEOREM 2.4. *The mean $M(a,b;p(r_{n,k}),q)$ reduces to the symmetric mean $Q_q(a,b)$, for some arbitrary n if*

$$p(s) = \alpha s^{2q/k-n} + \beta, \alpha, \beta \in R$$

and when takes the value $q = 0$.

Proof. It is enough to set $p(s) = s^{2q/k-n}$. Let $q = 0$, as $p(s) = s^{-n}$, we have

$$\begin{aligned} M(a,b;(r_{n,k})^{-n},0) &= (ab)^{\frac{1-k}{2}} \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{a^{kn}\cos^2\theta + b^{kn}\sin^2\theta}\right)^{-\frac{1}{n}} \\ &= (ab)^{\frac{1-k}{2}} \left(\frac{1}{\sqrt{a^{kn}b^{kn}}}\right)^{-\frac{1}{n}} \\ &= \sqrt{ab} = Q_0(a,b). \end{aligned}$$

$\square$

3. Some further characterizations

In this section, using the function (2.1) we can consider some further characterizations of the mean $M(a,b;p(r_{n,k}),q)$ for $p(s) = s^q$ and $p(s) = \log s$.

THEOREM 3.1. *Let $p(s) = s^q$, and let $q, q-1$ and $q-2$ be different from* 0, *then*

$$q\left[Q_q^{k-1}(a,b)M\left(a,b;(r_{1,k})^q,q\right)\right]^q = \left(q-\frac{1}{2}\right)\left(a^k+b^k\right)\left[Q_q^{k-1}(a,b)M(a,b;(r_{1,k})^{q-1},q)\right]^{q-1} - (q-1)a^kb^k\left[Q_q^{k-1}(a,b)M\left(a,b;(r_{1,k})^{q-2},q\right)\right]^{q-2}. \tag{3.1}$$

Proof. Applying the definition of the function $f\left(a,b;(r_{1,k})^q\right)$, we get

$$\begin{aligned} f\left(a,b;(r_{1,k})^q\right) &= \frac{2}{\pi}\int_0^{\pi/2}\left(a^k\cos^2\theta + b^k\sin^2\theta\right)^q d\theta \\ &= \frac{2}{\pi}\left(\frac{a^k+b^k}{2}\right)^q\int_0^{\pi/2}\left(1+\frac{a^k-b^k}{a^k+b^k}\cos 2\theta\right)^q d\theta. \\ &= \frac{2}{\pi}\left(\frac{a^k+b^k}{2}\right)^q J_q. \end{aligned}$$

If we denote $J_q = \int_0^{2/\pi}(1+h\cos 2\theta)^q d\theta$ and $h = \left(a^k-b^k\right)/\left(a^k+b^k\right)$., We look for a recurrence for J_q. We have

$$\begin{aligned} J_q &= \int_0^{2/\pi}(1+h\cos 2\theta)(1+h\cos 2\theta)^{q-1}d\theta \\ &= J_{q-1} + \int_0^{2/\pi}(h\cos 2\theta)(1+h\cos 2\theta)^{q-1}d\theta \\ &= J_{q-1} + h^2(q-1)\int_0^{2/\pi}\left(\sin^2 2\theta\right)(1+h\cos 2\theta)^{q-2}d\theta \\ &= J_{q-1} + h^2(q-1)J_{q-2} - (q-1)\int_0^{2/\pi}\left(h^2\cos^2 2\theta\right)(1+h\cos 2\theta)^{q-2}d\theta \\ &= J_{q-1} + h^2(q-1)J_{q-2} - (q-1)(J_q - 2J_{q-1} + J_{q-2}). \end{aligned}$$

Thus

$$qJ_q = (2q-1)J_{q-1} + (q-1)(h^2-1)J_{q-2}, \tag{3.3}$$

where $q, q-1$, and $q-2$ take values different from 0. From the equalities (3.2) and (3.3) one derives

$$\begin{aligned} qf\left(a,b;(r_{1,k})^q\right) &= \frac{2}{\pi}\left(\frac{a^k+b^k}{2}\right)^q\left[(2q-1)J_{q-1} + (q-1)\left(h^2-1\right)J_{q-2}\right] \qquad (3.4) \\ &= (2q-1)\left(\frac{a^k+b^k}{2}\right)f\left(a,b;(r_{1,k})^{q-1}\right) - (q-1)(ab)^k f\left(a,b;(r_{1,k})^{q-2}\right). \end{aligned}$$

The relations (3.4) gives the recurrence relation (3.1). □

COROLLARY 3.2. *Let* $p(s)=s^q$, *and let* $n,q,q-1$ *and* $q-2$ *take values different from* 0, *then*

$$q\left[Q_q^{k-1}(a,b)M(a,b;(r_{n,k})^{nq},q)\right]^{nq}$$
$$=\left(q-\frac{1}{2}\right)\left(a^{nk}+b^{nk}\right)\left[Q_q^{k-1}(a,b)M\left(a,b;(r_{n,k})^{n(q-1)},q\right)\right]^{n(q-1)}$$
$$-(q-1)a^{nk}b^{nk}\left[Q_q^{k-1}(a,b)M\left(a,b;(r_{n,k})^{n(q-2)},q\right)\right]^{n(q-2)}. \qquad (3.5)$$

Proof. Using the definition of the function $f\ (a,b;(r_{n,k})^q)$, one obtains

$$f\ (a,b;(r_{n,k})^q)=\frac{2}{\pi}\int_0^{\pi/2}\left(a^{nk}\cos^2\theta+b^{nk}\sin^2\theta\right)^{\frac{q}{n}}d\theta$$
$$=f\left(a,b;(r_{n,k})^{\frac{q}{n}}\right). \qquad (3.6)$$

From the equality (3.6), we derive the functional relation

$$f\ (a^n,b^n;(r_{1,k})^q)=f\ (a,b;(r_{n,k})^{qn}). \qquad (3.7)$$

In relation (3.4), replacing a and b by a^n and b^n, respectively. one gets the recurrence relation

$$qf\ (a^n,b^n;(r_{1,k})^q)=\left(q-\frac{1}{2}\right)\left(a^{nk}+b^{nk}\right)f\ \left(a^n,b^n;(r_{1,k})^{q-1}\right) \qquad (3.8)$$
$$-(q-1)(ab)^{nk}f\ \left(a^n,b^n;(r_{1,k})^{n(q-2)}\right).$$

From (3.7) and (3.8), we obtain the following recurrence relation

$$qf\ (a^n,b^n;(r_{n,k})^{nq})=\left(q-\frac{1}{2}\right)\left(a^{nk}+b^{nk}\right)f\ \left(a^n,b^n;(r_{n,k})^{n(q-1)}\right)$$
$$-(q-1)(ab)^{nk}f\ \left(a^n,b^n;(r_{n,k})^{n(q-2)}\right),$$

which is the same recurrence relation as (3.5). □

THEOREM 3.3. *Let* $n=0$ *and* $s\neq 0$, *then the following equalities hold true:*

(i) $M(a,b;\log(r_{0,k}),q)=(ab)^{\frac{k}{2}}\left[Q_q(a,b)\right]^{1-k}$.

(ii) $M_q\left(a,b;(r_{0,k})^s,q\right)=(ab)^{\frac{k}{2}}\left[Q_q(a,b)\right]^{1-k}\left[\sum_{i=0}^{\infty}\left(\frac{ks}{4}\log\frac{a}{b}\right)^{2i}\frac{1}{(i!)^2}\right]^{1/s}$.

Proof. By the definition of $r_{0,k}(\theta)$, one has

$$M(a,b;\log(r_{0,k}),q)=[Q_q(a,b)]^{1-k}\exp\left(\frac{2}{\pi}\int_0^{2\pi}\log\left(a^{k\cos^2\theta}b^{k\sin^2\theta}\right)d\theta\right)$$
$$=[Q_q(a,b)]^{1-k}\exp\left(\frac{2}{\pi}\int_0^{2\pi}\left(k\log a\cos^2\theta+k\log b\sin^2\theta\right)d\theta\right)$$
$$=(ab)^{\frac{k}{2}}[Q_q(a,b)]^{1-k}$$

The proof of (ii) follows by an argument similar to the one given in Section 5 in [19] with some suitable changes. □

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