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MAJORIZATION INEQUALITIES RELATED TO INCREASING CONVEX FUNCTIONS IN A SEMIFINITE VON NEUMANN ALGEBRA

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Abstract. Let $\mu_s(x)$ denote the generalized s-number of an operator x. We show a mojorization inequality $\int_0^t \mu_s(f(a+b))\,ds \geqslant \int_0^t \mu_s(f(a)+f(b))\,ds$ for every increasing convex function with $f(0)=0$ and positive τ-measurable operators a, b affiliated with a semi-finite von Neumann algebra $\mathfrak{A}$.

1. Introduction

In 1999, Ando and Zhan proved [1]

$$\sum_{j=1}^{k} \lambda_j(f(a+b)) \geqslant \sum_{j=1}^{k} \lambda_j(f(a)+f(b)) \quad (1 \leqslant j \leqslant n) \tag{1.1}$$

for every operator convex function with $f(0) = 0$ and $n \times n$ positive semidefinite matrices a, b. Here, $\lambda(a) = (\lambda_1(a), \cdots, \lambda_n(a))$ and $\lambda(b) = (\lambda_1(b), \cdots, \lambda_n(b))$ denote the eigenvectors of a and b in decreasing order, respectively. Recently, an extended result of the above inequality was shown by Kosem [5]. He proved the inequality (1.1) when f is an increasing convex function with $f(0) = 0$ by using the above Ando-Zhan result and an approximation method. In this paper, we prove the inequality (1.1) in a semi-finite von Neumann algebra. More precisely, let $\mathfrak{A}$ be a semi-finite von Neumann algebra with faithful normal semi-finite trace τ and f be an increasing convex function with $f(0) = 0$. We show

$$\int_0^t \mu_s(f(a+b))\,ds \geqslant \int_0^t \mu_s(f(a)+f(b))\,ds \quad (0 \leqslant t \leqslant \tau(1)) \tag{1.2}$$

for positive operators a, b in $\mathfrak{A}$. Here, we denote by $\mu_s(a)$ the generalized s-number of an element a in $\mathfrak{A}$ (see [4]). We first prove the inequality (1.2) for every operator convex function with $f(0) = 0$. Since Ando and Zhan used the $n \times m$ matrix trick to prove the inequality (1.1), we think that we can not apply the same method. So we

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modify the proof related to this trick (Lemma 3.1). Finally, we show the inequality (1.2) for an increasing convex function f with $f(0) = 0$ and positive τ-measurable operators a, b.

2. Notation

Throughout this paper, $\mathfrak{A}$ denotes a semi-finite von Neumann algebra on a Hilbert space $\mathfrak{H}$ with a faithful normal semi-finite trace τ.

A (continuous) non-negative function $f(t)$ on $[0,\infty)$ is said to be operator monotone if

$$0 \leqslant a \leqslant b \;\Rightarrow f(a) \leqslant f(b)$$

for any such matrices a, b of all orders n. This is equivalent to saying that the above implication is valid in the space of bounded linear operators on an (infinite dimensional) Hilbert space (see [2]). The following integral representation theorem is espacislly famous and important; for each non-nagatiove operator monotone $f(t)$ there are uniquely constants $\alpha, \beta \geqslant 0$ and a non-negative measure $\mu(\cdot)$ on $[0,\infty)$ such that

$$f(t) = \alpha + \beta t + \int_0^\infty \frac{st}{s+t}\, d\mu(s) \quad (t \in [0,\infty\rangle\,).$$

(See [2], V. 4.) A non-negative function $g(t)$ on $[0,\infty\rangle$ is said to be operator convex (resp. operator concave) if

$$g[\lambda a + (1-\lambda)b] \leqslant \lambda g(a) + (1-\lambda)g(b) \quad (a, b \geqslant 0;\; 0 \leqslant \lambda \leqslant 1)$$

(resp. reversed inequality holds in the above). It is known (see [2], V. 2) that operator concavity concides with operator monotonity on $[0,\infty\rangle$, and that a non-negative function $g(t)$ with $g(0) = 0$ is operator convex if and only if $\frac{g(t)}{t}$ is operator monotone.

A densely-defined closed operator a affiliated with $\mathfrak{A}$ is said to be τ-measurable if, for each $\delta > 0$, there exists a projection e in $\mathfrak{A}$ such that $e\mathfrak{H} \subset D(a)$ and $\tau(1-e) < \delta$ where $D(a)$ denotes the domain of a. Let $\overline{\mathfrak{A}}$ denote the set of all τ-measurable operators affiliated with $\mathfrak{A}$, which becomes a complete Hausdorff topological $*$-algebra equipped with the measure topology (see [7], [8]). Let $|a| = \int_0^\infty \lambda\, de_\lambda(|a|)$ be the spectral decomposition. Then it is easy to check that a is τ-measurable if and only if $\tau(1 - e_\lambda(|a|)) < \infty$ for λ large enough (cf. [4], [7]). The generalized s-number $\mu_t(a)$, $t > 0$, of $a \in \overline{\mathfrak{A}}$ is defined by

$$\mu_t(a) = \inf\{s : \tau(e_{(s,\infty)}(|a|)) \leqslant t\} \quad (0 < t < \infty).$$

The above definition corresponds to the decreasing rearrangement of the eigenvalues of $|a|$ (see [4]).

3. Main theorem

For the proof of the main theorem the following Lemma is crucial. The analogous inequality can be found in [3].

LEMMA 3.1. *For a positive operator c in $\mathfrak{A}$ let q_1 and q_2 be projections commuting with c such that $e_{[0,s\rangle}(c) \leqslant q_1 \leqslant e_{[0,s]}(c)$ and $e_{\langle s,\infty\rangle}(c) \leqslant q_2 \leqslant e_{[s,\infty\rangle}$ where $e_B(c)$ denotes the spectral projection for c corresponding to a Borel set B in $[0,\infty\rangle$. Then*

$$||hcq_1||_2 \leqslant ||chq_1||_2, \tag{3.1}$$

$$||hcq_2||_2 \geqslant ||chq_2||_2 \tag{3.2}$$

for every self-adjoint operator h in $\mathfrak{A}$. Here, $||x||_2$ denotes the L^2-norm of an element x. (i.e. $||x||_2 = \tau(x^*x)^{\frac{1}{2}}$.)

Proof. Notice that $q_1c^2q_1 \leqslant s^2q_1$ and $q_1^{\perp}c^2q_1^{\perp} \geqslant s^2q_1^{\perp}$, since $e_{[0,s]}(c)c^2e_{[0,s]}(c) \leqslant s^2e_{[0,s]}(c)$ and $e_{[s,\infty]}(c)c^2e_{[s,\infty]}(c) \geqslant s^2e_{[s,\infty]}(c)$. Here, $q_1^{\perp} = 1 - q_1$. We compute

$$\begin{aligned}
||hcq_1||_2^2 &= ||(q_1 + q_1^{\perp})hcq_1||_2^2 \\
&= ||q_1hcq_1 + q_1^{\perp}hcq_1||_2^2 \\
&= \tau((q_1chq_1 + q_1chq_1^{\perp})(q_1hcq_1 + q_1^{\perp}hcq_1)) \\
&= \tau(|q_1hcq_1|^2 + |q_1^{\perp}hcq_1|^2) \\
&= ||q_1hcq_1||_2^2 + ||q_1^{\perp}hcq_1||_2^2 \\
&\leqslant ||q_1hcq_1||_2^2 + s^2||q_1^{\perp}hq_1||_2^2
\end{aligned}$$

because

$$\begin{aligned}
||q_1^{\perp}hcq_1||_2^2 &= ||q_1chq_1^{\perp}||_2^2 \\
&= \tau(q_1^{\perp}hcq_1 \cdot q_1chq_1^{\perp}) \\
&= \tau(q_1^{\perp}h \cdot cq_1c \cdot hq_1^{\perp}) \\
&\leqslant s^2\tau(q_1^{\perp}hq_1 \cdot q_1hq_1^{\perp}) \\
&\text{(since } cq_1c = q_1c^2q_1 \leqslant s^2q_1) \\
&= s^2||q_1^{\perp}hq_1||_2^2.
\end{aligned}$$

In a similar way we have

$$\begin{aligned}
||chq_1||_2^2 &= ||q_1hc||_2^2 \\
&= ||q_1hc(q_1 + q_1^{\perp})||_2^2 \\
&= ||q_1hcq_1 + q_1hcq_1^{\perp}||_2^2 \\
&= \tau((q_1chq_1 + q_1^{\perp}chq_1)(q_1hcq_1 + q_1hcq_1^{\perp})) \\
&= \tau(|q_1hcq_1|^2 + |q_1hcq_1^{\perp}|^2) \\
&= ||q_1hcq_1||_2^2 + ||q_1hcq_1^{\perp}||_2^2 \\
&\geqslant ||q_1hcq_1||_2^2 + s^2||q_1^{\perp}hq_1||_2^2.
\end{aligned}$$

These two inequalities yield the first inequality of the assertion. Similary for the second inequality, we get the conclusion. □

LEMMA 3.2. *For positive operators a, b in $\mathfrak{A}$ and a projection q commuting with $a+b$ such that $e_{\langle s,\infty\rangle}(a+b) \leqslant q \leqslant e_{[s,\infty\rangle}(a+b)$*

$$\tau\left(q(a(a+1)^{-1}+b(b+1)^{-1})q\right) \geqslant \tau\left(q(a+b)(a+b+1)^{-1}q\right).$$

Proof. We set $c=(a+b+1)^{-\frac{1}{2}}$. Since

$$(a+b)(a+b+1)^{-1} = c(a+b)c = cac+cbc,$$

it suffices to prove that

$$\tau(qa(a+1)^{-1}q) \geqslant \tau(qcacq).$$

Notice here that $e_{[s,\infty\rangle}(a+b) = e_{[0,\frac{1}{\sqrt{s+1}}]}(c)$ and $e_{\langle s,\infty\rangle}(a+b) = e_{[0,\frac{1}{\sqrt{s+1}}\rangle}(c)$.

Applying Lemma 3.1 to $a^{\frac{1}{2}}$ in place of h we have

$$\begin{aligned}
\tau(qcacq) &= ||a^{\frac{1}{2}}cq||_2^2 \\
&\leqslant ||ca^{\frac{1}{2}}q||_2^2 \\
&= \tau(qa^{\frac{1}{2}}c^2a^{\frac{1}{2}}q) \\
&\leqslant \tau(qa^{\frac{1}{2}}(a+1)^{-1}a^{\frac{1}{2}}q) \\
&\quad (\text{since } c^2=(a+b+1)^{-1} \leqslant (a+1)^{-1}) \\
&= \tau(qa(a+1)^{-1}q).
\end{aligned}$$

□

In next proposition, we prove the main result for operator monotone and operator convex functions.

PROPOSITION 3.1. *Let a, b be positive operators in $\mathfrak{A}$. Then:*

(1) *For a non-negative operator monotone function $f(t)$ on $[0,\infty\rangle$*

$$\int_0^t \mu_s(f(a)+f(b))\,ds \geqslant \int_0^t \mu_s(f(a+b))\,ds.$$

(2) *For every non-negative increasing function $g(t)$ on $[0,\infty\rangle$ with $g(0)=0$ and $g(\infty)=\infty$, whose inverse function is operator monotone*

$$\int_0^t \mu_s(g(a)+g(b))\,ds \leqslant \int_0^t \mu_s(g(a+b))\,ds.$$

Proof. There exists a positive measure $\mu(s)$ on $[0,\infty\rangle$ such that

$$f(t) = \alpha+\beta t+\int_0^\infty \frac{st}{s+t}\,d\mu(s) \quad (t\in[0,\infty\rangle).$$

So, for any positive operator $a \in \mathfrak{A}$ and any projection $e \in \mathfrak{A}$ we have

$$\tau(f(a)e) = \tau(\alpha e) + \tau(\beta ae) + \int_0^\infty \tau(sa(a+s)^{-1}e)\, d\mu(s).$$

Applying Lemma 3.2 for a projection p commuting with $a+b$ such that $e_{\langle \gamma,\infty\rangle}(a+b) \leqslant p \leqslant e_{[\gamma,\infty\rangle}(a+b)$ $(0 \leqslant \gamma \leqslant \infty)$ we have

$$\tau(p(sa(a+s)^{-1} + sb(b+s)^{-1})p) \geqslant \tau(p(s(a+b)(a+b+s)^{-1})p).$$

Therefore we get

$$\tau((f(a)+f(b))p) \geqslant \tau(f(a+b)p).$$

If $\mathfrak{A}$ has no minimal projection, then there is a projection $p_t \in \mathfrak{A}$ commuting with $a+b$ such that

$$e_{\langle \mu_t(a+b),\infty\rangle}(a+b) \leqslant p_t \leqslant e_{[\mu_t(a+b),\infty\rangle}(a+b), \quad \text{and} \quad \tau(p_t) = t.$$

Since f is increasing, we have

$$e_{\langle \mu_t(f(a+b)),\infty\rangle}(f(a+b)) \leqslant p_t \leqslant e_{[\mu_t(f(a+b)),\infty\rangle}(f(a+b)).$$

So, we have

$$\tau(f(a+b)p_t) = \int_0^t \mu_s(f(a+b))\, ds$$

(see p202, [9]). Especially we set $p = p_t$. We get

$$\begin{aligned}\int_0^t \mu_s(f(a)+f(b))\, ds &\geqslant \tau((f(a)+f(b))p_t) \quad \text{(Lemma 4.1, [4])}\\ &\geqslant \tau(f(a+b)p_t)\\ &= \int_0^t \mu_s(f(a+b))\, ds.\end{aligned}$$

Next we prove the second assertion. From the above inequality we get

$$\begin{aligned}\int_0^t \mu_s(a+b)ds &= \int_0^t \mu_s(g^{-1}(g(a)) + g^{-1}(g(b)))\, ds\\ &\geqslant \int_0^t \mu_s(g^{-1}(g(a)+g(b)))\, ds.\end{aligned}$$

Since $g(t)$ is a non-decreasing convex function, by the majorization principle (see [6]) this implies

$$\int_0^t \mu_s(g(a+b))ds \geqslant \int_0^t \mu_s(g(a)+g(b))\, ds.$$

□

REMARK 3.1. In the above proof, we assume that $\mathfrak{A}$ has no minimal projection. But this assumption is not restrictive, since we can always embed $\mathfrak{A}$ into $\mathfrak{A} \otimes L^\infty_{([0,1];dt)}$ without changing the generalized s-number. (See p. 286, [4].)

Finally, we show the main result for every increasing convex function with $f(0) = 0$ and positive τ-measurable operators.

THEOREM 3.1. *Let f be a non-negative convex function on $[0,\infty\rangle$ with $f(0) = 0$. Then*

$$\int_0^t \mu_s(f(a+b))\,ds \geqslant \int_0^t \mu_s(f(a)+f(b))\,ds$$

for positive elements a, b in $\overline{\mathfrak{A}}$.

Proof. By Proposition 3.1 we can prove

$$\int_0^t \mu_s(f(a+b))\,ds \geqslant \int_0^t \mu_s(f(a)+f(b))\,ds \tag{3.3}$$

for positive operators a, b in $\mathfrak{A}$. Here exactly the same argument as in [5] works so that details are left to the reader.

We then extend (3.3) to (not necessarily bounded) τ-measurable operators a, b. We set $a_n = \int_0^n \lambda de_\lambda$ and $b_n = \int_0^n \lambda de_\lambda$. Now, $a_n \leqslant a$, $b_n \leqslant b$ and $a_n \nearrow a$, $b_n \nearrow b$ in measure. By a simple argument we see $f(a_n) \nearrow f(a)$, $f(b_n) \nearrow f(b)$ in measure. By Theorem 3.5, [4] we get

$$\begin{aligned}
\int_0^t \mu_s(f(a)+f(b))\,ds &\leqslant \int_0^t \liminf_{n\to\infty} \mu_s(f(a_n)+f(b_n))\,ds \\
&\leqslant \liminf_{n\to\infty} \int_0^t \mu_s(f(a_n)+f(b_n))\,ds \\
&\leqslant \liminf_{n\to\infty} \int_0^t \mu_s(f(a_n+b_n))\,ds.
\end{aligned}$$

Since $a_n + b_n \leqslant a+b$ and f is non-decreasing, we have

$$\int_0^t \mu_s(f(a_n+b_n))\,ds \leqslant \int_0^t \mu_s(f(a+b))\,ds.$$

Combining the above estimates, we obtain (3.3) for positive τ-measurable operators. □

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