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ON SOME CONVOLUTION NORM INEQUALITIES IN WEIGHTED $L_P(\mathbb{R}^n,\rho)$ SPACES AND THEIR APPLICATIONS

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Abstract. In this paper, we give the inequalities in convolutions in weighted $L_p(\mathbb{R}^n,\rho)$ spaces and their important applications to partial differential equations and integral transforms.

1. Introduction

It is well known, the Young's inequality

$$\|f*g\|_r \leqslant \|f\|_p\|g\|_q, \quad f\in L_p(\mathbb{R}^n),\ g\in L_q(\mathbb{R}^n);$$

$$r^{-1}=p^{-1}+q^{-1}-1\ (p,q,r>0),$$

is very fundamental. ([10])

In a series of papers, S. Saitoh ([7], [8], [9]) derived new type norm inequalities in convolutions in some several weighted L_2 spaces using the theory of reproducing kernels. Specially S.Saitoh ([6]) obtained convolution norm inequalities in the form

$$\|f*g\|_p \leqslant \|f\|_p\|g\|_p \qquad (p>1)$$

by considering the L_p - norms in more naturally determined weighted spaces.

Recently, we ([5]) gave new type of convolution inequality in weighted $L_p(\mathbb{R}^2,\rho)$ $(p>1)$ spaces

$$\left\|((F_1\rho_1)*(F_2\rho_2))(\rho_1*\rho_2)^{\frac{1}{p}-1}\right\|_p \leqslant \|F_1\|_{L_p(\mathbb{R}^2,|\rho_1|)}\,\|F_2\|_{L_p(\mathbb{R}^2,|\rho_2|)}. \tag{11}$$

holds for $F_j(\xi,\tau)\in L_p(\mathbb{R}^2,|\rho_j(\xi,\tau)|d\xi d\tau)$ $(j=1,2)$.

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In this paper, we will introduce the general inequalities in convolutions in weighted $L_p(\mathbb{R}^n,\rho)(p>1)$ spaces and their applications.

2. The Main Results

For brevity of presentation we shall use the following notation.

2.1. Notation

By $\mathbb{R}^n$ we denote the $n-$*dimensional Euclidean space*, $n\in\mathbb{N}$. This is the set of all $n-$ tupels of real numbers, $\mathbf{x}=(x_1,...,x_n)$, $x_j\in\mathbb{R}$, $j=1,2,...,n$ with the linear operations

$$\begin{aligned}\mathbf{x}+\mathbf{y} &= (x_1+y_1,...,x_n+y_n),\ \ \mathbf{x},\mathbf{y}\in\mathbb{R}^n, &(22)\\ \lambda\mathbf{x} &= (\lambda x_1,...,\lambda x_n),\ \ \lambda\in\mathbb{R},\ \ \mathbf{x}\in\mathbb{R}^n, &(23)\end{aligned}$$

the scalar product

$$\mathbf{x}\mathbf{y}=x_1y_1+\cdots+x_ny_n,\ \ \mathbf{x},\mathbf{y}\in\mathbb{R}^n, \tag{24}$$

and the norm

$$\|\mathbf{x}\|=(\mathbf{x}\mathbf{x})^{\frac{1}{2}}=(x_1^2+\cdots+x_n^2)^{\frac{1}{2}},\ \ \mathbf{x}\in\mathbb{R}^n. \tag{25}$$

We shall write $\mathbf{x}>\mathbf{y}$ instead of $x_j>y_j$, $j=1,2,...,n$. Anologously one has to understand $\mathbf{x}\geqslant\mathbf{y}$, $\mathbf{x}<\mathbf{y}$, $\mathbf{x}\leqslant\mathbf{y}$. In particular let

$$\mathbf{1}=(1,1,...,1),\quad \mathbf{2}=(2,2,...,2),... \tag{26}$$

We shall denote some subsets of $\mathbb{R}^n$

$$\begin{aligned}\mathbb{R}^n_+ &= \{\mathbf{x}:\mathbf{x}\in\mathbb{R}^n,\mathbf{x}>\mathbf{0}\} &(27)\\ \mathbb{R}^n_+(\mathbf{t}) &= \{\mathbf{x}:\mathbf{x}\in\mathbb{R}^n,\mathbf{0}<\mathbf{x}<\mathbf{t}\}. &(28)\end{aligned}$$

Now let $\mathbf{z},\alpha\in\mathbb{R}^n$. Then we set

$$\mathbf{z}^\alpha=\prod_{j=1}^n z_j^{\alpha_j}. \tag{29}$$

Finally, we shall denote some integrals

$$\begin{aligned}\int_{\mathbb{R}^n} f(\mathbf{x})d\mathbf{x} &= \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} f(x_1,...,x_n)dx_1\cdots dx_n, &(210)\\ \int_{\mathbb{R}^n_+(\mathbf{t})} f(\mathbf{x})d\mathbf{x} &= \int_0^{t_1}\cdots\int_0^{t_n} f(x_1,...,x_n)dx_1\cdots dx_n, &(211)\\ \int_{\mathbb{R}^{n-1}} f(\mathbf{x})d\mathbf{x}^i &= \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} f(x_1,...,x_n)dx_1\cdots dx_{i-1}dx_{i+1}\cdots dx_n. &(212)\end{aligned}$$

2.2. The Inequalities

In order to show our new type inequalities and their essentials simply, we will state the inequality as follows :

THEOREM 1. *For two non-vanishing functions* $\rho_j(\mathbf{x})(j = 1,2)$ *belonging to* $L_1(\mathbb{R}^n, d\mathbf{x})$ *and for* $p > 1$ *we have* L_p *weighted convolution inequality*

$$\int_{\mathbb{R}^n} \frac{\left|\int_{\mathbb{R}^n} F_1(\xi)\rho_1(\xi)F_2(\mathbf{x}-\xi)\rho_2(\mathbf{x}-\xi)\,d\xi\right|^p}{\left(\int_{\mathbb{R}^n}|\rho_1(\xi)|\,|\rho_2(\mathbf{x}-\xi)|\,d\xi\right)^{p-1}}d\mathbf{x} \leqslant \int_{\mathbb{R}^n}|F_1(\xi)|^p|\rho_1(\xi)|d\xi\int_{\mathbb{R}^n}|F_2(\xi)|^p|\rho_2(\xi)|d\xi \tag{213}$$

holds for $F_j(\mathbf{x}) \in L_p(\mathbb{R}^n, |\rho_j(\mathbf{x})|d\mathbf{x})(j = 1,2)$. *Equality holds for* F_j *if and only if* F_j *are represented in the form*

$$F_j(\mathbf{x}) = C_j e^{\alpha\mathbf{x}};\ C_j:\ \textit{constant} \tag{214}$$

where $\alpha = (\alpha_1,...,\alpha_n) \in \mathbb{R}^n$ *is a constant such that* $F_j \in L_p(\mathbb{R}^n, |\rho_j|d\mathbf{x})(j = 1,2)$.

Proof. By Hölder's inequality and Fubini's theorem, we obtain directly

$$\begin{aligned}\int_{\mathbb{R}^n} &\frac{\left|\int_{\mathbb{R}^n} F_1(\xi)\rho_1(\xi)F_2(\mathbf{x}-\xi)\rho_2(\mathbf{x}-\xi)\,d\xi\right|^p}{\left(\int_{\mathbb{R}^n}|\rho_1(\xi)|\,|\rho_2(\mathbf{x}-\xi)|\,d\xi\right)^{p-1}}d\mathbf{x}\\ &\leqslant \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|F_1(\xi)|^p|\rho_1(\xi)||F_2(\mathbf{x}-\xi)|^p|\rho_2(\mathbf{x}-\xi)|d\xi d\mathbf{x}\\ &= \int_{\mathbb{R}^n}|F_1(\xi)|^p|\rho_1(\xi)|d\xi\int_{\mathbb{R}^n}|F_2(\xi)|^p|\rho_2(\xi)|d\xi.\end{aligned}$$

Equality holds if and only if for a function $k(\mathbf{x})$ in $\mathbf{x} \in \mathbb{R}^n$

$$F_1(\xi)F_2(\mathbf{x}-\xi) = k(\mathbf{x}) \text{ a.e. on } \mathbb{R}^n,$$

that is

$$F_1(\mathbf{x})F_2(\mathbf{y}) = k(\mathbf{x}+\mathbf{y}) \text{ a.e. on } \mathbb{R}^n \times \mathbb{R}^n.$$

From this function equation, we have the desired result for the equality problem in (214). □

In the inequality (213), we can write the following norm

$$\left\|((F_1\rho_1)*(F_2\rho_2))(\rho_1*\rho_2)^{\frac{1}{p}-1}\right\|_p \leqslant \|F_1\|_{L_p(\mathbb{R}^n,|\rho_1|)}\,\|F_2\|_{L_p(\mathbb{R}^n,|\rho_2|)}. \tag{215}$$

REMARK 1. 1. In the inequality (213), in many cases the convolution will be given in the form

$$\rho_2(\xi) \equiv 1, \text{ and } F_2(\mathbf{x}-\xi) = G(\mathbf{x}-\xi)$$

for some Green's functions $G(\mathbf{x}-\xi)$. Then, we have the inequality

$$\begin{aligned} &\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} F(\xi)\, \rho(\xi)\, G(\mathbf{x}-\xi)\, d\xi \right|^p d\mathbf{x} \\ &\qquad \leqslant \left(\int_{\mathbb{R}^n} |\rho(\xi)|\, d\xi \right)^{p-1} \int_{\mathbb{R}^n} |F(\xi)|^p |\rho(\xi)| d\xi \int_{\mathbb{R}^n} |G(\xi)|^p d\xi. \end{aligned} \tag{216}$$

REMARK 1. 2. In general, in (216) we have a generalization

$$\begin{aligned} &\int_{\mathbf{c}}^{\mathbf{d}} \left| \int_{\mathbb{R}^n} F(\xi)\, \rho(\xi)\, G(\mathbf{x}-\xi)\, d\xi \right|^p d\mathbf{x} \\ &\qquad \leqslant \left(\int_{\mathbb{R}^n} |\rho(\xi)|\, d\xi \right)^{p-1} \int_{\mathbb{R}^n} |F(\xi)|^p |\rho(\xi)| d\xi \int_{\mathbf{c}-\xi}^{\mathbf{d}-\xi} |G(\mathbf{x})|^p d\mathbf{x}. \end{aligned} \tag{217}$$

Morever, by the Hölder's inequality and Fubini's theorem and by changing the variables in integral we obtain the following inequalities

THEOREM 2. *For two non-vanishing functions* $\rho_j(\mathbf{x})(j=1,2)$ *belonging to* $L_1(\mathbb{R}^n, d\mathbf{x})$, *for* $p>1, q>1, p^{-1}+q^{-1}=1$ *and for* $(\mathbf{x},t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, *we have the inequality*

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} \frac{\left| \int_{\mathbb{R}^n} F_1(\xi,t)\, \rho_1(\xi,t)\, F_2(\mathbf{x}-\xi,t)\, \rho_2(\mathbf{x}-\xi,t)\, d\xi dt \right|^p}{\left(\int_{\mathbb{R}^n} |\rho_1(\xi,t)|\, |\rho_2(\mathbf{x}-\xi,t)|\, d\xi dt \right)^{p-1}} d\mathbf{x} \\ &\qquad \leqslant \left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} |F_1(\xi,t)|^p |\rho_1(\xi,t)| d\xi \right]^p dt \right)^{\frac{1}{p}} \\ &\qquad \times \left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} |F_2(\xi,t)|^p |\rho_2(\xi,t)| d\xi \right]^q dt \right)^{\frac{1}{q}}, \end{aligned} \tag{218}$$

where $F_j(j=1,2)$ *are such that the right hand side of* (218) *is finite.*

Proof. Put

$$N = \left| \int_{\mathbb{R}^n} F_1(\xi,t)\, \rho_1(\xi,t)\, F_2(\mathbf{x}-\xi,t)\, \rho_2(\mathbf{x}-\xi,t)\, d\xi dt \right|^p.$$

Applying Hölder inequality, we obtain

$$\begin{aligned} N \leqslant &\left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} |F_1(\xi,t)|^p |\rho_1(\xi,t)| F_2(\mathbf{x}-\xi,t)|^p |\rho_2(\mathbf{x}-\xi,t)| d\xi \right]^{\frac{1}{p}} \right. \\ &\qquad \left. \cdot \left[\int_{\mathbb{R}^{n-1}} |\rho_1(\xi,t)| \rho_2(\mathbf{x}-\xi,t)| d\xi \right]^{\frac{1}{q}} dt \right)^p \end{aligned}$$

$$\leqslant \int_{\mathbb{R}^n} |F_1(\xi,t)|^p |\rho_1(\xi,t)| F_2(\mathbf{x}-\xi,t)|^p |\rho_2(\mathbf{x}-\xi,t)| d\xi dt$$

$$\cdot \left(\int_{\mathbb{R}^n} |\rho_1(\xi,t)| \rho_2(\mathbf{x}-\xi,t)| d\xi dt \right)^{p-1}.$$

Therefore, by changing the variables in integrals and Fubini's theorem we get

$$M = \int_{\mathbb{R}^{n-1}} \frac{\left| \int_{\mathbb{R}^n} F_1(\xi,t)\, \rho_1(\xi,t)\, F_2(\mathbf{x}-\xi,t)\, \rho_2(\mathbf{x}-\xi,t)\, d\xi dt \right|^p}{\left(\int_{\mathbb{R}^n} |\rho_1(\xi,t)|\, |\rho_2(\mathbf{x}-\xi,t)|\, d\xi dt \right)^{p-1}} d\mathbf{x}$$

$$\leqslant \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} |F_1(\xi,t)|^p |\rho_1(\xi,t)| F_2(\mathbf{x}-\xi,t)|^p |\rho_2(\mathbf{x}-\xi,t)| d\xi dt d\mathbf{x}$$

$$= \int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} |F_1(\xi,t)|^p |\rho_1(\xi,t)| d\xi \int_{\mathbb{R}^{n-1}} |F_2(\xi,t)|^p |\rho_2(\xi,t)| d\xi \right] dt.$$

Then, by Hölder inequality, we have directly

$$M \leqslant \left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} |F_1(\xi,t)|^p |\rho_1(\xi,t)| d\xi \right]^p dt \right)^{\frac{1}{p}}$$

$$\times \left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} |F_2(\xi,t)|^p |\rho_2(\xi,t)| d\xi \right]^q dt \right)^{\frac{1}{q}}.$$

The proof is complete. □

REMARK 2. 1. In the inequality (218), in many cases we consider

$$\rho_2(\xi,t) \equiv 1, \text{ and } F_2(\mathbf{x}-\xi,t) = G(\mathbf{x}-\xi,t)$$

for some Green's functions $G(\mathbf{x}-\xi,t)$. Then, we have the inequality

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} \left| \int_{\mathbb{R}^n} F(\xi,t)\, \rho(\xi,t)\, G(\mathbf{x}-\xi,t)\, d\xi dt \right|^p d\mathbf{x} \\ &\quad \leqslant \left(\int_{\mathbb{R}^n} |\rho(\xi,t)|\, d\xi dt \right)^{p-1} \left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} |F(\xi,t)|^p |\rho(\xi,t)| d\xi \right]^p dt \right)^{\frac{1}{p}} \\ &\quad\quad \cdot \left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} |G(\xi,t)|^p | d\xi \right]^q dt \right)^{\frac{1}{q}} \end{aligned} \tag{219}$$

for an $L_1(\mathbb{R}^n, d\mathbf{x})$ function ρ, and for functions F and G with finite integrals in the right hand side in (219).

REMARK 2. 2. In general, in (219) we have a generalization

$$\int_{\mathbb{R}^{n-1}}\left|\int_a^b dt\int_{\mathbb{R}^{n-1}} F(\xi,t)\,\rho(\xi,t)\,G(\mathbf{x}-\xi,t)\,d\xi\right|^p d\mathbf{x}$$
$$\leqslant\left(\int_a^b dt\int_{\mathbb{R}^{n-1}}|\rho(\xi,t)|\,d\xi\right)^{p-1}\left(\int_a^b\left[\int_{\mathbb{R}^{n-1}}|F(\xi,t)|^p|\rho(\xi,t)|d\xi\right]^p dt\right)^{\frac{1}{p}} \quad (220)$$
$$\cdot\left(\int_a^b\left[\int_{\mathbb{R}^{n-1}}|G(\xi,t)|^p d\xi\right]^q dt\right)^{\frac{1}{q}}.$$

In the sequel, we shall show several typical examples. We get not only L_p integral estimates for the several integral transform but also the solutions of homogeneous linear differential equations in the space $\mathbb{R}^n$ or $\mathbb{R}^{n+1}$ ([1], [2], [3], [4], [10]).

3. Applications

3.1. Laplace Transformation

In the Laplace transform

$$u(\mathbf{x}) = \mathrm{L}[\mathrm{F}\rho](\mathbf{x}) = \int_{\mathbb{R}^{\mathrm{n}}_{+}} \mathrm{e}^{-\mathbf{x}\xi}\mathrm{F}(\xi)\rho(\xi)\mathrm{d}\xi, \quad (321)$$

we have the inequality

$$\int_{\mathbb{R}^n_+}|u(\mathbf{x})|^p d\mathbf{x} \leqslant \frac{1}{p^n}\left(\int_{\mathbb{R}^n_+}|\rho(\xi)|d\xi\right)^{p-1}\int_{\mathbb{R}^n_+}|F(\xi)|^p|\rho(\xi)|\frac{d\xi}{\xi}, \quad (322)$$

where an $L_1(\mathbb{R}^n_+, d\xi)$ function ρ and for function $F(\xi)\in L_p(\mathbb{R}^n_+, |\rho(\xi)|\frac{\mathbf{d}\xi}{\xi})$.

3.2. Abel's Integral Transform

In the Abel's integral transform

$$f(\mathbf{x}) = \int_{\mathbb{R}^n_+(\mathbf{x})}\frac{F(\xi)\rho(\xi)}{(\mathbf{x}-\xi)^{\alpha}}d\xi,\ \mathbf{0}<\alpha<\mathbf{1}, \quad (323)$$

we have the inequality

$$\int_{\mathbb{R}^n_+(\mathbf{d})}\frac{|f(\mathbf{x})|^p}{\left(\int_{\mathbb{R}^n_+(\mathbf{x})}|\rho(\xi)|d\xi\right)^{p-1}}d\mathbf{x}$$
$$\leqslant\frac{\mathbf{1}}{\mathbf{1}-\alpha\mathbf{p}}\int_{\mathbb{R}^n_+(\mathbf{d})}|F(\xi)|^p|\rho(\xi)|(\mathbf{d}-\xi)^{\mathbf{1}-\alpha p}d\xi \ \ (\alpha p<\mathbf{1}), \quad (324)$$

for a function ρ such that

$$\int_{\mathbb{R}^n_+(\mathbf{x})} |\rho(\xi)| d\xi > 0 \text{ on } \mathbb{R}^n_+(\mathbf{d})$$

and for functions F with finite integrals in the right hand side in (324).

3.3. Heat Equation

In the integral transform

$$u(\mathbf{x},t) = \frac{1}{(2c\sqrt{\pi t})^n} \int_{\mathbb{R}^n} F(\xi)\rho(\xi) \exp\left\{-\frac{|\mathbf{x}-\xi|^2}{4c^2 t}\right\} d\xi, \tag{325}$$

which gives the solution $u(\mathbf{x},t)$ of the heat equation

$$u_t = c^2 \Delta_n u(\mathbf{x},t) \quad (\mathbf{x},t) \in \mathbb{R}^n \times \mathbb{R}_+ \tag{326}$$

satisfying the condition

$$u(\mathbf{x},0) = F(\mathbf{x})\rho(\mathbf{x}). \tag{327}$$

Then, we not only have the inequality

$$\int_{\mathbb{R}^n} |u(\mathbf{x},t)|^p d\mathbf{x} \leqslant \frac{1}{p^{\frac{n}{2}} (2c\sqrt{\pi t})^{n(p-1)}} I_n, \tag{328}$$

but also obtain

$$\int_{\mathbb{R}^{n-1}} |u(\mathbf{x},t)|^p d\mathbf{x}^i \leqslant \frac{1}{p^{\frac{n-1}{2}} (pq)^{\frac{1}{2q}} (2c\sqrt{\pi t})^{n(p-1)+\frac{1}{p}}} K_{n-1}, \tag{329}$$

where

$$I_n = \left(\int_{\mathbb{R}^n} |\rho(\xi)|\, d\xi\right)^{p-1} \int_{\mathbb{R}^n} |F(\xi)|^p |\rho(\xi)| d\xi,$$

and

$$K_{n-1} = \left(\int_{\mathbb{R}^n} |\rho(\xi,t)|\, d\xi dt\right)^{p-1} \left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} |F(\xi,t)|^p |\rho(\xi,t)| d\xi\right]^p dt\right)^{\frac{1}{p}},$$

for $\rho \in L_1(\mathbb{R}^n)$, $F \in L_p(\mathbb{R}^n;\rho)$.

3.4. Laplace Equation(Poisson Integrals)

We consider the Dirichlet problem for the Laplace Equation in a half-space of $\mathbb{R}^{n+1}$, i.e. the determination of the bounded solution of

$$\Delta_{n+1} u(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \mathbb{R}^n \times \mathbb{R}_+ \tag{330}$$

with the boundary condition

$$u(\mathbf{x},0) = F(\mathbf{x})\rho(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \tag{331}$$

we have the solution of the Dirichlet problem (330), (331) in the form

$$u(\mathbf{x},t)=\frac{2t}{\omega_{n+1}}\int_{\mathbb{R}^n}\frac{F(\mathbf{y})\rho(\mathbf{y})}{(t^2+|\mathbf{x}-\mathbf{y}|^2)^{\frac{n+1}{2}}}d\mathbf{y}, \tag{332}$$

where

$$\omega_n=\frac{1}{2\pi^{\frac{n+1}{2}}}\Gamma\left(\frac{n+1}{2}\right).$$

Then, we not only have the inequality

$$\int_{\mathbb{R}^n}|u(\mathbf{x},t)|^p d\mathbf{x}\leqslant\left(\frac{2}{\omega_{n+1}}\right)^p\frac{1}{t^{n(p-1)}}\prod_{i=1}^{n}B\left(\frac{1}{2},p\frac{n+1}{2}-\frac{i}{2}\right)I_n \tag{333}$$

but also obtain

$$\begin{aligned}\int_{\mathbb{R}^{n-1}}|u(\mathbf{x},t)|^p d\mathbf{x}^i\leqslant\left(\frac{2}{\omega_{n+1}}\right)^p\frac{1}{t^{n(p-1)+\frac{n}{p}}}\prod_{i=1}^{n-1}B\left(\frac{1}{2},p\frac{n+1}{2}-\frac{i}{2}\right)\\ \cdot B^{\frac{1}{q}}\left(\frac{1}{2},\frac{np+q(p+1)-1}{2}\right)K_{n-1},\end{aligned} \tag{334}$$

where

$$I_n=\left(\int_{\mathbb{R}^n}|\rho(\xi)|\,d\xi\right)^{p-1}\int_{\mathbb{R}^n}|F(\xi)|^p|\rho(\xi)|d\xi,$$

and

$$K_{n-1}=\left(\int_{\mathbb{R}^n}|\rho(\xi,t)|\,d\xi dt\right)^{p-1}\left(\int_{\mathbb{R}}\left[\int_{\mathbb{R}^{n-1}}|F(\xi,t)|^p|\rho(\xi,t)|d\xi\right]^p dt\right)^{\frac{1}{p}},$$

for $\rho\in L_1(\mathbb{R}^n)$, $F\in L_p(\mathbb{R}^n;\rho)$.

In the conjugate Poisson integral transform

$$v_i(\mathbf{x},t)=\frac{2}{\omega_{n+1}}\int_{\mathbb{R}^n}F(\mathbf{y})\rho(\mathbf{y})\frac{x_i-y_i}{(t^2+|\mathbf{x}-\mathbf{y}|^2)^{\frac{n+1}{2}}}d\mathbf{y}, \tag{335}$$

then we have

$$\begin{aligned}\int_{\mathbb{R}^n}|v_i(\mathbf{x},t)|^p d\mathbf{x}\leqslant\left(\frac{2}{\omega_{n+1}}\right)^p\frac{1}{t^{n(p-1)}}\prod_{i=1}^{n-1}B\left(\frac{1}{2},p\frac{n+1}{2}-\frac{i}{2}\right)\\ \times B\left(\frac{p+1}{2},\frac{n}{2}(p-1)\right)I_n,\end{aligned} \tag{336}$$

where

$$I_n=\left(\int_{\mathbb{R}^n}|\rho(\xi)|\,d\xi\right)^{p-1}\int_{\mathbb{R}^n}|F(\xi)|^p|\rho(\xi)|d\xi.$$

Moreover we also get

$$\int_{\mathbb{R}^{n-1}} |v_i(\mathbf{x},t)|^p d\mathbf{x}^i \leqslant \left(\frac{2}{\omega_{n+1}}\right)^p \frac{1}{t^{n(p-1)+\frac{1}{p}}} \prod_{i=1}^{n-1} B\left(\frac{1}{2}, p\frac{n+1}{2} - \frac{i}{2}\right) \times B^{\frac{1}{q}}\left(\frac{pq+1}{2}, \frac{np}{2}\right) K_{n-1}, \tag{337}$$

where

$$K_{n-1} = \left(\int_{\mathbb{R}^n} |\rho(\xi,t)|\, d\xi dt\right)^{p-1} \left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} |F(\xi,t)|^p |\rho(\xi,t)| d\xi\right]^p dt\right)^{\frac{1}{p}}.$$

Here $\rho \in L_1(\mathbb{R}^n)$, $F \in L_p(\mathbb{R}^n;\rho)$.

3.5. Biharmonic equation

3.5.1. Example 1

The solution of the biharmonic equation

$$\Delta_{n+1}^2 u(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \mathbb{R}^n \times \mathbb{R}_+, \quad \Delta_{n+1}^2 = \Delta_{n+1}(\Delta_{n+1}) \tag{338}$$

with the boundary conditions

$$u(\mathbf{x},0) = F(\mathbf{x})\rho(\mathbf{x}), \quad u_t(\mathbf{x},0) = 0, \tag{339}$$

is given by

$$u(\mathbf{x},t) = \frac{2(n+1)t^3}{\omega_{n+1}} \int_{\mathbb{R}^n} \frac{F(\mathbf{y})\rho(\mathbf{y})}{(t^2 + |\mathbf{x}-\mathbf{y}|^2)^{\frac{n+3}{2}}} d\mathbf{y}. \tag{340}$$

Then, we have the following inequalities:

$$\int_{\mathbb{R}^n} |u(\mathbf{x},t)|^p d\mathbf{x} \leqslant \left(\frac{2(n+1)}{\omega_{n+1}}\right)^p \frac{1}{t^{n(p-1)}} \prod_{i=1}^{n} B\left(\frac{1}{2}, p\frac{n+3}{2} - \frac{i}{2}\right) I_n, \tag{341}$$

$$\int_{\mathbb{R}^{n-1}} |u(\mathbf{x},t)|^p d\mathbf{x}^i \leqslant \left(\frac{2(n+1)}{\omega_{n+1}}\right)^p \frac{1}{t^{n(p-1)+\frac{n}{p}}} \prod_{i=1}^{n-1} B\left(\frac{1}{2}, p\frac{n+3}{2} - \frac{i}{2}\right) \times B^{\frac{1}{q}}\left(\frac{1}{2}, \frac{pn+q(3p+1)-1}{2}\right) K_{n-1}, \tag{342}$$

where

$$I_n = \left(\int_{\mathbb{R}^n} |\rho(\xi)|\, d\xi\right)^{p-1} \int_{\mathbb{R}^n} |F(\xi)|^p |\rho(\xi)| d\xi,$$

and

$$K_{n-1} = \left(\int_{\mathbb{R}^n} |\rho(\xi,t)|\, d\xi dt\right)^{p-1} \left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} |F(\xi,t)|^p |\rho(\xi,t)| d\xi\right]^p dt\right)^{\frac{1}{p}},$$

for $\rho \in L_1(\mathbb{R}^n)$, $F \in L_p(\mathbb{R}^n;\rho)$.

3.5.2. Example 2

The solution of the biharmonic equation

$$\Delta_{n+1}^2 u(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \mathbb{R}^n \times \mathbb{R}_+, \quad \Delta_{n+1}^2 = \Delta_{n+1}(\Delta_{n+1}) \tag{343}$$

with the boundary conditions

$$u(\mathbf{x},0) = 0, \quad u_t(\mathbf{x},0) = F(\mathbf{x})\rho(\mathbf{x}), \tag{344}$$

is given by

$$u(\mathbf{x},t) = \frac{2t^2}{\omega_{n+1}} \int_{\mathbb{R}^n} \frac{F(\mathbf{y})\rho(\mathbf{y})}{(t^2+|\mathbf{x}-\mathbf{y}|^2)^{\frac{n+1}{2}}} d\mathbf{y}. \tag{345}$$

Then, we not only have the inequality

$$\int_{\mathbb{R}^n} |u(\mathbf{x},t)|^p d\mathbf{x} \leqslant \left(\frac{2}{\omega_{n+1}}\right)^p \frac{1}{t^{n(p-1)-p}} \prod_{i=1}^{n} B\left(\frac{1}{2}, p\frac{n+1}{2} - \frac{i}{2}\right) I_n \tag{346}$$

but also obtain

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} |u(\mathbf{x},t)|^p d\mathbf{x}^i \leqslant \left(\frac{2}{\omega_{n+1}}\right)^p \frac{1}{t^{n(p-1)+\frac{n}{p}-p}} \prod_{i=1}^{n-1} B\left(\frac{1}{2}, p\frac{n+1}{2} - \frac{i}{2}\right) \\ \times B^{\frac{1}{q}}\left(\frac{1}{2}, \frac{pn+q(p+1)-1}{2}\right) K_{n-1},\end{aligned} \tag{347}$$

where

$$I_n = \left(\int_{\mathbb{R}^n} |\rho(\xi)|\, d\xi\right)^{p-1} \int_{\mathbb{R}^n} |F(\xi)|^p |\rho(\xi)| d\xi,$$

and

$$K_{n-1} = \left(\int_{\mathbb{R}^n} |\rho(\xi,t)|\, d\xi dt\right)^{p-1} \left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} |F(\xi,t)|^p |\rho(\xi,t)| d\xi\right]^p dt\right)^{\frac{1}{p}},$$

for $\rho \in L_1(\mathbb{R}^n)$, $F \in L_p(\mathbb{R}^n;\rho)$.

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