Mathematical
Inequalities
& Applications
Volume 11, Number 3 (2008), 549–557

STABILITY OF LINEAR MAPPINGS IN QUASI–BANACH MODULES

MOHAMMAD SAL MOSLEHIAN AND GHADIR SADEGHI

(*communicated by H. Srivastava*)

Abstract. A quasi norm is a non-negative function $\|.\|$ on a linear space $\mathscr{X}$ satisfying the same axioms as a norm except for the triangle inequality, which is replaced by the weaker condition that "there is a constant $K \geqslant 1$ such that $\|x+y\| \leqslant K(\|x\|+\|y\|)$ for all $x,y \in \mathscr{X}$". In this paper, we prove the Hyers–Ulam–Rassias stability of linear mappings in quasi-Banach modules associated to the Cauchy functional equation and a generalized Jensen functional equation.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of S.M. Ulam [18] concerning the stability of group homomorphisms: Given a group $\mathscr{G}_1$, a metric group $(\mathscr{G}_2, d)$ and a positive number ϵ, does there exist a number $\delta > 0$ such that if a function $f : \mathscr{G}_1 \to \mathscr{G}_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in \mathscr{G}_1$ then there exists a homomorphism $T : \mathscr{G}_1 \to \mathscr{G}_2$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in \mathscr{G}_1$?

If the answer is affirmative, we say that the equation of homomorphism $T(xy) = T(x)T(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

D. H. Hyers [6] gave a first affirmative answer to the question of Ulam for Banach spaces: Let E_1 be a normed space and E_2 be a Banach space. Suppose that for some $\varepsilon \geqslant 0$, the mapping $f : E_1 \to E_2$ satisfies $\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon$ for all $x, y \in E_1$. Then there exists a unique additive mapping $T : E_1 \to E_2$ such that $\|f(x) - T(x)\| \leqslant \varepsilon$ for all $x \in E_1$.

Let $f : E_1 \to E_2$ be a mapping from a normed space E_1 into a Banach space E_2 such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Th.M. Rassias [13] introduced the following inequality: Assume that there exist constants $\varepsilon \geqslant 0$ and $p \in [0,1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p) \qquad (x, y \in E_1).$$

Mathematics subject classification (2000): 39B82, 39B52, 46L05, 47Jxx.

Key words and phrases: Cauchy equation, generalized Jensen equation, quasi-Banach module, C^*-algebra, Hyers–Ulam–Rassias stability, p-Banach module.

The first author was in part supported by a grant from IPM (No. 85390031).

Then there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leqslant \frac{2\varepsilon}{|2-2^p|}\|x\|^p \qquad (x \in E_1).$$

(of course, his proof still works in the case where $p < 1$).

The above inequality has provided a lot of influence in the development of what is now known as *Hyers–Ulam–Rassias stability* of functional equations. In 1991, Z. Gajda [4] following the same approach as in [13] proved that Th.M. Rassias' result is also valid for the case $p > 1$. It is shown that there is no analogue of Th.M. Rassias result for $p = 1$ (see [4, 14]). The topic of approximate mappings or the stability of functional equations was studied and extended by a number of mathematicians; see [3, 5, 7, 8, 10, 15] and references therein.

For the sake of convenience, we mention some basic facts about quasi-Banach spaces.

DEFINITION 1.1. ([2, 16]) Let $\mathscr{X}$ be a linear space. A *quasi-norm* is a real-valued function on $\mathscr{X}$ satisfying the following:

(1) $\|x\| > 0$ for all $x \neq 0$ in $\mathscr{X}$;
(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all scalars λ and all $x \in \mathscr{X}$.
(3) There is a constant $K \geqslant 1$ such that $\|x+y\| \leqslant K(\|x\|+\|y\|)$ for all $x, y \in \mathscr{X}$.

Then the pair $(\mathscr{X}, \|\cdot\|)$ is said to be a *quasi-normed space*. The smallest possible K is called the *quasi-norm constant* of $\|\cdot\|$. It is easy to see that the balls with respect to $\|\cdot\|$ define a linear topology on $\mathscr{X}$. In this way, $\mathscr{X}$ becomes a locally bounded space (i.e. it has a bounded neighborhood of 0), and conversely, every locally bounded topology on a vector space comes from a quasi-norm. A *quasi-Banach space* is a complete quasi-normed space, i.e. a quasi-normed space in which each $\|\cdot\|$-Cauchy sequence is convergent. This class includes Banach spaces. The most significant class of quasi-Banach spaces which are not Banach spaces are the L_p spaces for $0 < p < 1$ with the L_p-norm $\|\cdot\|_p$.

A quasi-norm $\|\cdot\|$ is said to be a p*-norm* $(0 < p \leqslant 1)$ if

$$\|x+y\|^p \leqslant \|x\|^p + \|y\|^p \qquad (x, y \in \mathscr{X}).$$

In this case, a quasi-Banach space is called a p*-Banach space*.

DEFINITION 1.2. Let $(\mathscr{A}, \|\cdot\|)$ be a normed algebra. A (left) *quasi-normed module* over $\mathscr{A}$ is a quasi-normed space $(\mathscr{X}, \|\cdot\|)$ which is an algebraic (left) $\mathscr{A}$-module and $\|ax\| \leqslant \|a\| \cdot \|x\|$ for all $a \in \mathscr{A}$ and all $x \in \mathscr{X}$.

The notions of *quasi-Banach module* and p*-Banach module* can be defined in a similar fashion.

Given a p-norm, the formula $d(x,y) := \|x-y\|^p$ gives us a translation invariant metric on $\mathscr{X}$. By the Aoki–Rolewicz theorem [16] (see also [2]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

In this paper, we prove the Hyers–Ulam–Rassias stability of linear mappings in quasi-Banach modules associated to the Cauchy functional equation and the generalized Jensen functional equation by using some ideas of [17].

Throughout the paper, let $\mathscr{X}$ be a quasi-normed $\mathscr{A}$-module with quasi-norm $\|\cdot\|_{\mathscr{X}}$ and $\mathscr{Y}$ be a p-Banach $\mathscr{A}$-module with p-norm $\|\cdot\|_{\mathscr{Y}}$ and the quasi-norm constant K. These modules are assumed to be unit-linked, in the sense of $1.x = x$ for x in the modules.

2. Stability of linear mappings associated to Cauchy equation

In this section, we prove the Hyers–Ulam–Rassias stability of linear mappings in quasi-Banach modules over a unital C^*-algebra $\mathscr{A}$ associated to the Cauchy functional equation $f(x+y) = f(x)+f(y)$ (see also [11]). We denote the unitary group and the positive part of the unit ball $\mathscr{A}_1$ of $\mathscr{A}$ by $U(\mathscr{A})$ and $\mathscr{A}_1^+$, respectively.

THEOREM 2.1. *Let $r > 1$ and ε be a positive real number. Let $f : \mathscr{X} \to \mathscr{Y}$ be a mapping such that*

$$\|f(ux+y)-uf(x)-f(y)\|_{\mathscr{Y}} \leqslant \varepsilon(\|x\|_{\mathscr{X}}^r + \|y\|_{\mathscr{X}}^r) \tag{2.1}$$

for all $x, y \in \mathscr{X}$ and all $u \in U(\mathscr{A})$. Then there exists a unique $\mathscr{A}$-linear mapping $L : \mathscr{X} \to \mathscr{Y}$ such that

$$\|f(x)-L(x)\|_{\mathscr{Y}} \leqslant \frac{2\varepsilon}{(2^{pr}-2^p)^{\frac{1}{p}}}\|x\|_{\mathscr{X}}^r \qquad (x \in \mathscr{X}). \tag{2.2}$$

Proof. Fix $x \in \mathscr{X}$. Replace both y and x by $x/2$ and put $u = 1 \in U(\mathscr{A})$ in (2.1) to get

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\|_{\mathscr{Y}} \leqslant \frac{2\varepsilon}{2^r}\|x\|_{\mathscr{X}}^r, \tag{2.3}$$

whence

$$\left\|2^jf\left(\frac{x}{2^j}\right)-2^{j+1}f\left(\frac{x}{2^{j+1}}\right)\right\|_{\mathscr{Y}} \leqslant \frac{\varepsilon}{2^{j(r-1)+r-1}}\|x\|_{\mathscr{X}}^r$$

for all non-negative integers j. Since $\mathscr{Y}$ is a p-Banach $\mathscr{A}$-module,

$$\begin{aligned}\left\|2^lf\left(\frac{x}{2^l}\right)-2^mf\left(\frac{x}{2^m}\right)\right\|_{\mathscr{Y}}^p &\leqslant \sum_{j=l}^{m-1}\left\|2^jf\left(\frac{x}{2^j}\right)-2^{j+1}f\left(\frac{x}{2^{j+1}}\right)\right\|_{\mathscr{Y}}^p \\ &\leqslant \frac{\varepsilon^p}{2^{p(r-1)}}\|x\|_{\mathscr{X}}^{pr}\sum_{j=l}^{m-1}\left(\frac{1}{2^{(r-1)p}}\right)^j \end{aligned} \tag{2.4}$$

for all non-negative integers m and l with $m > l$. It follows from (2.4) that the sequence $\{2^nf(\frac{x}{2^n})\}$ is a Cauchy sequence. Since $\mathscr{Y}$ is complete, the sequence $\{2^nf(\frac{x}{2^n})\}$ is convergent. Hence one can define the mapping $L : \mathscr{X} \to \mathscr{Y}$ by

$$L(x) := \lim_{n\to\infty} 2^nf\left(\frac{x}{2^n}\right) \qquad (x \in \mathscr{X})$$

Set $u = 1$ in (2.1) to obtain

$$\begin{aligned}\|L(x+y) - L(x) - L(y)\|_{\mathscr{Y}} &= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right\|_{\mathscr{Y}} \\ &\leqslant \lim_{n\to\infty} \frac{\varepsilon}{2^{n(r-1)}}(\|x\|^r_{\mathscr{X}} + \|y\|^r_{\mathscr{X}}) = 0 \qquad (x, y \in \mathscr{X})\end{aligned}$$

Therefore

$$L(x+y) = L(x) + L(y) \qquad (x, y \in \mathscr{X}).$$

Moreover, letting $l = 0$ and $m \to \infty$ in (2.4), we get (2.2).

Put $y = 0$ in (2.1) to get

$$\begin{aligned}\|L(ux) - uL(x)\|_{\mathscr{Y}} &= \lim_{n\to\infty} 2^n \left\| f\left(\frac{ux}{2^n}\right) - uf\left(\frac{x}{2^n}\right)\right\|_{\mathscr{Y}} \\ &\leqslant \lim_{n\to\infty} \frac{\varepsilon}{2^{n(r-1)}} \|x\|^r_{\mathscr{X}} = 0 \qquad (x \in \mathscr{X}, u \in \mathscr{A}).\end{aligned}$$

So $L(ux) = uL(x)$ for all $x \in \mathscr{X}$ and all $u \in U(\mathscr{A})$.

For the proof of $\mathscr{A}$-linearity of L we use a standard strategy. Let $a \in \mathscr{A}$ $(a \neq 0)$. Assume that M is a natural number greater than $4\|a\|$. Then $\|\frac{a}{M}\| < \frac{1}{4} < \frac{1}{3} = 1 - \frac{2}{3}$. By Theorem 1 of [9], there exist three unitaries $u_1, u_2, u_3 \in U(\mathscr{A})$ such that $3\frac{a}{M} = u_1 + u_2 + u_3$. By the additivity of L we get $L(rx) = rL(x)$ for all $x \in \mathscr{X}$ and all rational number r. Hence

$$\begin{aligned}L(ax) &= L\left(\frac{M}{3}\cdot 3 \cdot \frac{a}{M}x\right) = \frac{M}{3}L\left(3\cdot\frac{a}{M}x\right) \\ &= \frac{M}{3}L(u_1x + u_2x + u_3x) = \frac{M}{3}(L(u_1x) + L(u_2x) + L(u_3x)) \\ &= \frac{M}{3}(u_1 + u_2 + u_3)L(x) = \frac{M}{3}\cdot 3 \cdot \frac{a}{M}L(x) = aL(x) \qquad (x \in \mathscr{X}).\end{aligned}$$

Since $\mathscr{A}$ is unital we conclude that L is $\mathscr{A}$-linear.

Now, let $T : \mathscr{X} \to \mathscr{Y}$ be another additive mapping satisfying (2.2). Then we have

$$\begin{aligned}\|L(x) - T(x)\|_{\mathscr{Y}} &= 2^n \left\| L\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\|_{\mathscr{Y}} \\ &\leqslant 2^n K\left(\left\| L\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_{\mathscr{Y}} + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_{\mathscr{Y}}\right) \\ &\leqslant \frac{4K\varepsilon}{(2^{pr} - 2^p)^{\frac{1}{p}} 2^{n(r-1)}} \|x\|^r_{\mathscr{X}},\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in \mathscr{X}$. So we can conclude that $L(x) = T(x)$ for all $x \in \mathscr{X}$. This proves the uniqueness of L. $\square$

THEOREM 2.2. *Let* $r < 1$ *and* ε *be a positive real number. Let* $f : \mathscr{X} \to \mathscr{Y}$ *be a mapping satisfying*

$$\|f(ax+y) - af(x) - f(y)\|_{\mathscr{Y}} \leqslant \varepsilon(\|x\|_{\mathscr{X}}^r + \|y\|_{\mathscr{X}}^r) \tag{2.5}$$

for all $x, y \in \mathscr{X}$ *and all* $a \in \mathscr{A}_1^+ \cup \{\mathrm{i}\}$. *Then there exists a unique* $\mathscr{A}$*-linear mapping* $L : \mathscr{X} \to \mathscr{Y}$ *such that*

$$\|f(x) - L(x)\|_{\mathscr{Y}} \leqslant \frac{2\varepsilon}{(2^p - 2^{pr})^{\frac{1}{p}}} \|x\|_{\mathscr{X}}^r \qquad (x \in \mathscr{X}). \tag{2.6}$$

Proof. Fix $x \in \mathscr{X}$. Letting $y = x$ and $a = 1$ in (2.5), we obtain

$$\left\| f(x) - \frac{1}{2} f(2x) \right\|_{\mathscr{Y}} \leqslant \varepsilon \|x\|_{\mathscr{X}}^r$$

Since $\mathscr{Y}$ is a p-Banach $\mathscr{A}$-module,

$$\begin{aligned} \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\|_{\mathscr{Y}}^p &\leqslant \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\|_{\mathscr{Y}}^p \\ &\leqslant \varepsilon^p \|x\|_{\mathscr{X}}^{pr} \sum_{j=l}^{m-1} \left(\frac{1}{2^{p(1-r)}} \right)^j \end{aligned} \tag{2.7}$$

for all non-negative integers m and l with $m > l$. It follows from (2.7) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence. Since $\mathscr{Y}$ is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is convergent. Hence one can define the mapping $L : \mathscr{X} \to \mathscr{Y}$ by

$$L(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x) \qquad (x \in \mathscr{X}).$$

Using the same reasoning as in the proof of Theorem 2.1, L is the unique additive mapping satisfying (2.6) and

$$L(ax) = aL(x) \qquad (a \in \mathscr{A}_1^+ \cup \{\mathrm{i}\}, x \in \mathscr{X}).$$

Next let $a \in \mathscr{A}^+$ and $x \in \mathscr{X}$. Then there exists a positive integer M such that $\frac{\|a\|}{M} 1 \in \mathscr{A}_1^+$. So

$$\begin{aligned} L(ax) &= L\left(\frac{a}{\|a\|} . \|a\| x \right) = \frac{a}{\|a\|} L\left(\frac{\|a\|}{M} . Mx \right) \\ &= \frac{a}{\|a\|} . \frac{\|a\|}{M} L(Mx) = \frac{a}{M} M L(x) = aL(x) \end{aligned}$$

We also have $L(-x) = L(\mathrm{i}^2x) = \mathrm{i}L(\mathrm{i}x) = -L(x)$. Since each element a of $\mathscr{A}$ has a decomposition of positive elements as $a = (a_1 - a_2) + \mathrm{i}(a_3 - a_4)$, we conclude that L is $\mathscr{A}$-linear mapping. $\square$

3. Stability of linear mappings associated to Jensen equation

In this section, we prove the Hyers–Ulam–Rassias stability of linear mappings in quasi-Banach modules over a unital normed algebra $\mathscr{A}$, associated to the generalized Jensen functional equation $Nf\left(\frac{x+y}{N}\right) = f(x) + f(y)$ (see also [1, 12]). Let $S(\mathscr{A})$ denotes the unite sphere $\{a \in \mathscr{A} : \|a\| = 1\}$ of $\mathscr{A}$.

THEOREM 3.1. *Let $r < 1$, $\varepsilon > 0$ and $N > 1$ be an integer. Let $f : \mathscr{X} \to \mathscr{Y}$ be a mapping with $f(0) = 0$ such that*

$$\left\| Nf\left(\frac{ax+y}{N}\right) - af(x) - f(y)\right\|_{\mathscr{Y}} \leqslant \varepsilon(\|x\|^r_{\mathscr{X}} + \|y\|^r_{\mathscr{X}}) \qquad (x, y \in \mathscr{X}, a \in S(\mathscr{A})). \tag{3.1}$$

Then there exists a unique $\mathscr{A}$-linear mapping $L : \mathscr{X} \to \mathscr{Y}$ such that

$$\|f(x) - L(x)\|_{\mathscr{Y}} \leqslant \frac{N^r\varepsilon}{(N^p - N^{pr})^{\frac{1}{p}}}\|x\|^r_{\mathscr{X}} \qquad (x \in \mathscr{X}). \tag{3.2}$$

Proof. Fix $x \in \mathscr{X}$. Setting $y = 0$ and $a = 1$ and replacing x by Nx in (3.1) to get

$$\left\| f(x) - \frac{1}{N}f(Nx)\right\|_{\mathscr{Y}} \leqslant \varepsilon N^{r-1}\|x\|^r_{\mathscr{X}},$$

whence

$$\left\| \frac{1}{N^j}f(N^jx) - \frac{1}{N^{j+1}}f(N^{j+1}x)\right\|_{\mathscr{Y}} \leqslant \frac{\varepsilon}{N^{j(1-r)+1-r}}\|x\|^r_{\mathscr{X}}$$

for all non-negative integers j. Since $\mathscr{Y}$ is a p-Banach $\mathscr{A}$-module,

$$\begin{aligned}\left\| \frac{1}{N^l}f(N^lx) - \frac{1}{N^m}f(N^mx)\right\|^p_{\mathscr{Y}} &\leqslant \sum_{j=l}^{m-1}\left\| \frac{1}{N^j}f(N^jx) - \frac{1}{N^{j+1}}f(N^{j+1}x)\right\|^p_{\mathscr{Y}} \\ &\leqslant \frac{\varepsilon^p}{N^{p(1-r)}}\|x\|^{pr}_{\mathscr{X}}\sum_{j=l}^{m-1}\left(\frac{1}{N^{p(1-r)}}\right)^j \end{aligned} \tag{3.3}$$

for all non-negative integers m and l with $m > l$. It follows from (3.3) that the sequence $\left\{\frac{1}{N^n}f(N^nx)\right\}$ is a Cauchy sequence. Since $\mathscr{Y}$ is complete, the sequence $\left\{\frac{1}{N^n}f(N^nx)\right\}$ is convergent. Hence one can define the mapping $L : \mathscr{X} \to \mathscr{Y}$ by

$$L(x) := \lim_{n\to\infty}\frac{1}{N^n}f(N^nx) \qquad (x \in \mathscr{X}).$$

By (3.1),

$$\begin{aligned}\left\|NL\left(\frac{x+y}{N}\right)-L(x)-L(y)\right\|_{\mathscr{Y}} &= \lim_{n\to\infty}\frac{1}{N^n}\left\|Nf\left(N^n\frac{x+y}{N}\right)-f(N^nx)-f(N^ny)\right\|_{\mathscr{Y}}\\ &\leqslant \lim_{n\to\infty}\frac{N^{rn}}{N^n}\varepsilon(\|x\|^r_{\mathscr{X}}+\|y\|^r_{\mathscr{X}})=0 \qquad (x,y\in\mathscr{X}).\end{aligned}$$

So

$$NL\left(\frac{x+y}{N}\right)=L(x)+L(y) \qquad (x,y\in\mathscr{X}).$$

Thus $NL(\frac{x}{N})=L(x)$ for each $x\in\mathscr{X}$, and so $L(x+y)=NL(\frac{x+y}{N})=L(x)+L(y)$ for all $x,y\in\mathscr{X}$. Using the same strategy as in the proof of Theorem 2.1, one can show that $L(ax)=aL(x)$ for all $x\in\mathscr{X}$ and all $a\in S(\mathscr{A})$. Now let μ be a scalar. There is a positive integer M such that $|\frac{\mu}{M}|<1$. Using a geometric argument, one can easily show that there exist complex numbers λ_1,λ_2 such that $|\lambda_1|=|\lambda_2|=1$ and $\frac{\mu}{M}=\frac{\lambda_1+\lambda_2}{2}$. Then for each $x\in\mathscr{X}$, we have

$$L(\mu x)=\frac{M}{2}L((\lambda_1+\lambda_2)x)=\frac{M}{2}(L(\lambda_1x)+L(\lambda_2x))=\frac{M}{2}(\lambda_1+\lambda_2)L(x)=\mu L(x).$$

Thus

$$L(ax)=L\left(\frac{a}{\|a\|}.\|a\|x\right)=\|a\|L\left(\frac{a}{\|a\|}x\right)=\|a\|.\frac{a}{\|a\|}L(x)=aL(x)$$

for all $x\in\mathscr{X}$ and all $a\in\mathscr{A}$. It follows that L is an $\mathscr{A}$-linear mapping. Moreover, letting $l=0$ and $m\to\infty$ in (3.3), we get (3.2).

Now, let $T:\mathscr{X}\to\mathscr{Y}$ be another additive mapping satisfying (3.2). Then we have

$$\begin{aligned}\|L(x)-T(x)\|^p_{\mathscr{Y}} &= \frac{1}{N^{pn}}\|L(N^nx)-T(N^nx)\|^p_{\mathscr{Y}}\\ &\leqslant \frac{1}{N^{pn}}(\|L(N^nx)-f(N^nx)\|^p_{\mathscr{Y}}+\|T(N^nx)-f(N^nx)\|^p_{\mathscr{Y}})\\ &\leqslant 2\cdot\frac{N^{prn}}{N^{pn}}\cdot\frac{\varepsilon^pN^{rp}}{N^p-N^{pr}}\|x\|^{pr}_{\mathscr{X}},\end{aligned}$$

which tends to zero as $n\to\infty$ for all $x\in\mathscr{X}$. So we can conclude that $L(x)=T(x)$ for all $x\in\mathscr{X}$. This proves the uniqueness of L. □

THEOREM 3.2. *Let $r>1$, $\varepsilon>0$ and $N>1$ be an integer. Let $f:\mathscr{X}\to\mathscr{Y}$ be a mapping with $f(0)=0$ satisfying. Then there exists a unique $\mathscr{A}$-linear mapping $L:\mathscr{X}\to\mathscr{Y}$ such that*

$$\|f(x)-L(x)\|_{\mathscr{Y}}\leqslant\frac{N^r\varepsilon}{(N^{pr}-N^p)^{\frac{1}{p}}}\|x\|^r_{\mathscr{X}} \qquad (x\in\mathscr{X}).$$

Proof. Fix $x \in \mathscr{X}$. Putting $y = 0$ and $a = 1$ in (3.1) we get

$$\left\|f(x) - Nf\left(\frac{x}{N}\right)\right\|_{\mathscr{Y}} \leqslant \varepsilon\|x\|_{\mathscr{X}}^r.$$

Since $\mathscr{Y}$ is a p-Banach $\mathscr{A}$-module,

$$\begin{aligned}\left\|N^l f\left(\frac{x}{N^l}\right) - N^m f\left(\frac{x}{N^m}\right)\right\|_{\mathscr{Y}}^p &\leqslant \sum_{j=l}^{m-1}\left\|N^j f\left(\frac{x}{N^j}\right) - N^{j+1} f\left(\frac{x}{N^{j+1}}\right)\right\|_{\mathscr{Y}}^p \\ &\leqslant \varepsilon^p\|x\|_{\mathscr{X}}^{pr}\sum_{j=l}^{m-1}\left(\frac{1}{N^{p(r-1)}}\right)^j \qquad (3.4)\end{aligned}$$

for all non-negative integers m and l with $m > l$. It follows from (3.4) that the sequence $\left\{N^n f\left(\frac{x}{N^n}\right)\right\}$ is a Cauchy sequence. Since $\mathscr{Y}$ is complete, the sequence $\left\{N^n f\left(\frac{x}{N^n}\right)\right\}$ converges. So one can define the mapping $L : \mathscr{X} \to \mathscr{Y}$ by

$$L(x) := \lim_{n\to\infty} N^n f\left(\frac{x}{N^n}\right) \qquad (x \in \mathscr{X}).$$

The rest of the proof is similar to the proof of Theorems 3.1. □

REMARK 3.3. Let $r = 0$ in all of our results. Adding some suitable conditions, e.g. the continuity of f at a point (cf. [6]), we can deduce the continuity of the obtained $\mathscr{A}$-linear mapping L.

REFERENCES

[1] M. AMYARI, F. RAHBARNIA AND GH. SADEGHI, *Some results on stability of extended derivations*, J. Math. Anal. Appl. **329** (2007), no. 2, 753–758.

[2] Y. BENYAMINI AND J. LINDENSTRAUSS, *Geometric Nonlinear Functional Analysis, Vol. 1*, Colloq. Publ. **48**, Amer. Math. Soc., Providence, 2000.

[3] S. CZERWIK, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, 2002.

[4] Z. GAJDA, *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14** (1991), 431–434.

[5] P. GĂVRUTA, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[6] D. H. HYERS, *On the stability of the linear functional equation*, Pro. Nat'l. Acad. Sci. U.S.A. **27** (1941), 222–224.

[7] D. H. HYERS, G. ISAC AND TH. M. RASSIAS, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[8] S.-M. JUNG, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press lnc. Palm Harbor, Florida, 2001.

[9] R. V. KADISON AND G. K. PEDERSEN, *Means and convex combinations of unitary operators*, Math. Scan. **57** (1985), 249–266.

[10] M. S. MOSLEHIAN, *Approximately vanishing of topological cohomology groups*, J. Math. Anal. Appl. **318** (2006), no. 2, 758–771.

[11] C. PARK, *On the stability of the linear mapping in Banach modules*, J. Math. Anal. Appl. **275** (2002), 711–720.

[12] C. PARK AND W. PARK, *On the Jensen's equation in Banach modules*, Taiwanese J. Math. **6** (2002), 523–531.

[13] TH. M. RASSIAS, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[14] TH. M. RASSIAS AND P. ŠEMRL, *On the behavior of mappings which do not satisfy Hyers–Ulam stability*, Proc. Amer. Math. Soc **114** (1992), 989–993.
[15] TH. M. RASSIAS, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2003.
[16] S. ROLEWICZ, *Metric Linear Spaces*, PWN-Polish Sci. Publ., Reidel and Dordrecht, 1984.
[17] J. TABOR, *Stability of the Cauchy functional equation in quasi-Banach spaces*, Ann. Polon. Math. **83** (2004), no. 3, 243–255.
[18] S. M. ULAM, *Problems in Modern Mathematics*, Wiley, New York, 1960.

(Received October 10, 2006)

Mohammad Sal Moslehian
Department of Mathematics
Ferdowsi University of Mashhad
P. O. Box 1159
Mashhad 91775
Iran
and
Institute for Studies in Theoretical
Physics and Mathematics (IPM)
Iran
e-mail: moslehian@ferdowsi.um.ac.ir
moslehian@ams.org

Ghadir Sadeghi
Department of Mathematics
Ferdowsi University of Mashhad
P. O. Box 1159
Mashhad 91775
Iran
and
Banach Mathematical Research Group (BMRG)
Mashhad
Iran
e-mail: ghadir54@yahoo.com