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ON THE ELLIPTIC INEQUALITY $\mathbf{L}u \leqslant 0$

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Abstract. In this note we obtain a strictly positive function u in $\overline{\Omega}$ satisfying $\mathbf{L}u \leqslant 0$ in Ω, where $\mathbf{L}$ is a general elliptic operator of second order. Some immediate applications are also indicated.

1. Introduction

It is known (see for example [3], p.73) that it is not always possible to find a strictly positive function u in the bounded domain $\overline{\Omega} \subset \mathbb{R}^n, n \geqslant 2$ satisfying

$$\mathbf{L}u \equiv a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u \leqslant 0 \quad \text{in } \Omega, \tag{1}$$

where $a^{ij} = a^{ji}$ and c is of arbitrary sign in Ω. We note that if $c \leqslant 0$ then it is trivial to find positive functions satisfying (1). For example, we can use constant functions or existence theorems ([1], Theorem 6.13, p.106) to produce positive nonconstant functions.

The object of the this note is to indicate conditions where it is possible to produce such a function (Lemma). This result has various applications. For example, it can be used to present some results of interest for second or higher order elliptic problems such as uniqueness results (Theorem 1), comparison-type results (Theorem 2), Harnack inequalities for inhomogeneous equations ([3], Remark iv), p.117), Phragmèn-Lindelöf principles ([3], Theorem 19, p.97) etc.

2. Results

LEMMA 1. *Let*

$$a(x) \geqslant \frac{\operatorname{diam}\Omega(\operatorname{diam}\Omega + \delta)(\sqrt{n}+1)}{2(n-1)} c_0 \quad \textit{in } \Omega \tag{2}$$

where $c_0 = \sup_\Omega c$, δ *is any positive constant and* $\operatorname{diam}\Omega$ *is the diameter of the bounded domain* $\Omega \subset \mathbb{R}^n, n \geqslant 2$. *Let* $i_1, \ldots, i_n \in \{1, \ldots, n\}$ *be distinct numbers. If* $a^{ii} = a,\ i = 1, \ldots, n$ *in* Ω *and one of the following conditions holds*

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c_0). $a^{ij} \leqslant 0$ for all $i,j = 1,\dots,n,\ i \neq j$ and $b^k \geqslant 0$ for all $k = 1,\dots,n$ in Ω; or
c_1). there exist(s) $i_1,\dots,i_q$ $(1 \leqslant q \leqslant n-1)$ such that $a^{i_1 i_k},\dots,a^{i_q i_k} \geqslant 0,\ b^{i_1},\dots,b^{i_q} \leqslant 0$ in Ω for all $i_k \neq i_1,\dots,i_k \neq i_q$, the rest of the coefficients a^{ij}, $i \neq j$ are nonpositive and the rest of coefficients b^k are nonnegative in Ω; or
c_2). $a^{ij} \leqslant 0$ for all $i,j = 1,\dots,n, i \neq j$ and $b^k \leqslant 0$ for all $k = 1,\dots,n$ in Ω;
then there exists a strictly positive function u in $\overline{\Omega}$ that satisfies (1).

Proof. We select two points $x_0, y_0 \in \partial\Omega$ such that $|x_0 - y_0| = \operatorname{diam}\Omega$ and construct a ball B_1 of radius $\operatorname{diam}\Omega/2$ such that $B_1 \supseteq \Omega$ and $x_0, y_0 \in \partial B_1$.

The ball B_1 may be imbedded in a cube K, parallel to the coordinate axes and side length $\operatorname{diam}\Omega$. Rotating, we can assume without loss of generality that $x_0, y_0 \in$ diagK where diagK represents a diagonal of K.

Let y be a vertex of K such that $y \in$ diagK, i.e., x_0, y_0 and y need to be colinear.

Further, we construct a ball B with center y such that $x_0 \in \partial B$. Of course, $y \notin \overline{\Omega}$ and $\Omega \subset B$.

Now we can define the smooth function (in $\overline{\Omega}$)

$$u(x) = \operatorname{dist}(x, \partial B) + \delta, \quad x \in \overline{\Omega}, \tag{3}$$

where δ is an arbitrary positive constant.

A straightforward calculation gives

$$D_i u = -\frac{x_i - y_i}{|x-y|}, \quad i = 1,\dots,n, \tag{4}$$

$$D_{ij} u = -\frac{\delta^{ij}}{|x-y|} + \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^3}, \quad i,j = 1,\dots,n. \tag{5}$$

Since

$$0 < |x-y| \leqslant |x_0 - y| = \frac{\operatorname{diam}\Omega(\sqrt{n}+1)}{2} \quad \forall\, x \in \overline{\Omega}, \tag{6}$$

it follows that if the relation (2) and if one of the conditions $c_q).,q = 0,1,2$ is fulfilled (choose y corresponding to each individual case $c_q).,q = 0,1,2$ then,

$$\mathbf{L}u \leqslant 0 \quad \text{in } \Omega. \tag{7}$$

□

Now we indicate some immediate applications.

The following uniqueness result is stated ([3], Theorem 11, p.73):

If there exists a function $w > 0$ in $\overline{\Omega}$ that satisfies (1) in Ω and if Ω is bounded, then the problem

$$\begin{aligned} \mathbf{L}u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{8}$$

has at most one solution. Here $\mathbf{L}$ is an uniformly elliptic operator.

Using this result and the Lemma we can prove the following version of the above mentioned uniqueness result

THEOREM 1. *The problem (8) has at most one classical solution. Here the condition (2) is satisfied, one of the conditions $c_q).,q = 0,1,2$ is satisfied and $a^{ii} = a, i = 1,\dots,n$ in Ω.*

An interesting comparison principle for fourth-order operators was proved by V. B. Goyal and P. W. Schaefer ([2], Theorem 1).

Let c *be a strictly positive constant and assume that there exists a strictly positive function* u in $\overline{\Omega}$ *satisfying*

$$\Delta u + cu \leqslant 0 \text{ in } \Omega. \tag{9}$$

If $v, w \in C^4(\Omega) \cap C^2(\overline{\Omega})$ *satisfy*

$$\begin{aligned} \Delta^2 v - c^2 v \leqslant \Delta^2 w - c^2 w \quad & \text{in } \Omega, \\ v \leqslant w, \ \Delta w \leqslant \Delta v \quad & \text{on } \partial\Omega, \end{aligned} \tag{10}$$

then

$$v \leqslant w \quad \text{in } \Omega.$$

We note that in [2] it is not indicated when the comparison principle is valid, i.e., it wasn't indicate a function $u > 0$ in $\overline{\Omega}$ satisfying (9).

Using the above mentioned comparison principle and our Lemma we obtain the following improved result:

THEOREM 2. *Let* c *be a strictly positive constant such that*

$$c \leqslant \frac{2(n-1)}{\operatorname{diam}\Omega(\operatorname{diam}\Omega + \delta)(\sqrt{n}+1)}, \tag{11}$$

where δ *is any positive constant. Assume that* $v, w \in C^4(\Omega) \cap C^2(\overline{\Omega})$ *satisfy (10). Then*

$$v \leqslant w \quad \text{in } \Omega. \tag{12}$$

REMARKS.

1. Protter and Weinberger gave a method for determining a strictly positive function that satisfies (1) ([3], p.73-74). Here we offer alternate conditions for the existence of a strictly positive function u in $\overline{\Omega}$ satisfying (1). For example, the authors in [3] cannot handle the case $\mathbf{L}$ elliptic or/and all b^i are unbounded by below or/and c unbounded in Ω, while we can. Hence Theorem 1 works if all b^i are unbounded by below in Ω.

2. Suppose that $\Omega \subset \mathbb{R}^n, n \geqslant 2$ is a ball of radius R. We see that if we use our results instead of the results in [3], p.73-74 then, the comparison principle (Theorem 2) works under the less restrictive condition $c \leqslant n-1/R(2R+\delta)(\sqrt{n}+1)$ (i.e., condition (11)) than $c \leqslant 4/(2R)^2 e^2$ (i.e., the condition imposed in [3], relation (6), p.74).

3. The constant

$$C(\Omega, n) = \frac{\operatorname{diam}\Omega(\operatorname{diam}\Omega + \delta)(\sqrt{n}+1)}{2(n-1)} \tag{13}$$

(see relation (2)) is not sharp. For some domains with corners, e.g., a pyramid, a parallepiped, a cone etc., $C(\Omega)$ can be taken

$$C(\Omega, n) = \frac{(\operatorname{diam}\Omega + \delta)^2}{(n-1)}. \tag{14}$$

4. Improvements of (14) are also possible for particular domains (e.g., an annular domain) when all coefficients $a^{ij} \equiv 0,\ i \neq j$ and $b^i \equiv 0$ in Ω.

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