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THE STABILITY OF A FUNCTIONAL EQUATION OF MULTIPLICATIVE DERIVATION TYPE

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Abstract. In this paper, we investigate the stability in the sense of Ger of the following functional equation of multiplicative derivation type which was introduced by Gy. Maksa and Zs. Páles [5]:

$$\delta(xy) = M(x)\delta(y) + M(y)\delta(x)$$

for all $x, y \in (0,\infty)$, where $M : (0,\infty) \to (0,\infty)$ is a function satisfying $M(xy) = M(x)M(y)$ for all $x, y \in (0,\infty)$.

1. Introduction

"For what metric groups G is it true that an approximate homomorphism of G is necessarily near to a homomorphism?" The stability problem of this kind was raised by S. M. Ulam [7] and was first studied by D. H. Hyers [2]: If $\varepsilon > 0$ and $f : X \to Y$ is a mapping with X, Y Banach spaces, such that

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon$$

for all $x,\ y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leqslant \varepsilon$$

for all $x,\ y \in X$.

Recently, Gy. Maksa and Zs. Páles [5] considered the stability problem of the following functional equation

$$\delta(xy) = x^{\alpha}\delta(y) + y^{\alpha}\delta(x), \quad x, y \in (0,1], \tag{1.1}$$

where $\alpha \in \mathbb{R}$ is a fixed power and $f : (0,1] \to \mathbb{R}$. It is easy to see that the general solution of (1.1) is of the form

$$\delta(x) = x^{\alpha}\ell(x), \quad x \in (0,1],$$

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where $\ell : (0,1] \to \mathbb{R}$ satisfies the Cauchy equation

$$\ell(xy) = \ell(x) + \ell(y), \quad x, y \in (0,1].$$

In the case $\alpha = 1$, which defines the so-called multiplicative derivation, the stability problem is proposed by Gy. Maksa [4] at the 34th ISFE and an affirmative solution to the problem was found by J. Tabor [6]. Furthermore, by replacing the power function $t \mapsto t^{\alpha}$ in (1.1) by a function $M : (0,1] \to \mathbb{R}$ satisfying $M(xy) = M(x)M(y)$ for all $x, y \in (0,1]$, they [5] dealt with the hyperstability problem of the functional equation of multiplicative derivation type

$$\delta(xy) = M(x)\delta(y) + M(y)\delta(x), \quad x, y \in (0,1], \tag{1.2}$$

where $f : (0,1] \to \mathbb{R}$.

On the other hand, since the group structure in the range space of the exponential functional equation $f(x+y) = f(x)f(y)$ is the multiplication, R. Ger [1] pointed out that the superstability phenomenon of the functional inequality $|f(x+y) - f(x)f(y)| \leqslant \delta$ is caused by the fact that the natural group structure in the range space is disregarded. So, he posed the stability problem in the following form

$$\left| \frac{f(x+y)}{f(x)f(y)} - 1 \right| \leqslant \delta$$

and with this as a start, this stability problem is called *the stability in the sense of Ger*.

In this paper we are going to offer the stability in the sense of Ger of the functional equation (1.2).

2. Stability of the functional equation (1.2) in the sense of Ger

In this paper we assume that $\varphi : \mathbb{R} \times \mathbb{R} \to [0,1)$ is a function such that

$$\sum_{n=0}^{\infty} \varphi(x^{2^n}, y^{2^n}) \tag{2.1}$$

converges for all $x, y \in \mathbb{R}$. Moreover, we set

$$\phi(x,y) = \prod_{n=0}^{\infty} [1 - \varphi(x^{2^n}, y^{2^n})],$$

$$\psi(x,y) = \prod_{n=0}^{\infty} [1 + \varphi(x^{2^n}, y^{2^n})]$$

for all $x, y \in \mathbb{R}$.

THEOREM 1. *Let $M : \mathbb{R} \to (0,\infty)$ be a function satisfying $M(xy) = M(x)M(y)$ for all $x, y \in \mathbb{R}$. Suppose that the function $f : \mathbb{R} \to (0,\infty)$ satisfies the stability inequality*

$$\left| \frac{f(xy)}{M(x)f(y) + M(y)f(x)} - 1 \right| \leqslant \varphi(x,y) \tag{2.2}$$

for all $x, y \in \mathbb{R}$. *Then there exists a unique solution* $\delta : \mathbb{R} \to (0, \infty)$ *of the functional equation* (1.2) *such that the inequality*

$$\phi(x,x) \leqslant \frac{\delta(x)}{f(x)} \leqslant \psi(x,x)$$

holds for all $x \in \mathbb{R}$.

Proof. Let $y = x$ in (2.2). Then we get

$$\left| \frac{f(x^2)}{2M(x)f(x)} - 1 \right| \leqslant \varphi(x,x)$$

which implies

$$1 - \varphi(x,x) \leqslant \frac{f(x^2)}{2M(x)f(x)} \leqslant 1 + \varphi(x,x) \tag{2.3}$$

for all $x \in \mathbb{R}$.

If we replace x by x^{2^n} in (2.3), then we obtain

$$1 - \varphi(x^{2^n}, x^{2^n}) \leqslant \frac{f(x^{2^{n+1}})}{2M(x^{2^n})f(x^{2^n})} \leqslant 1 + \varphi(x^{2^n}, x^{2^n}) \tag{2.4}$$

for all $x \in \mathbb{R}$.

Let us define a function $g_n : \mathbb{R} \to (0, \infty)$ by

$$g_n(x) = \frac{f(x^{2^n})}{2^n M(x)^{2^n - 1}}$$

for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$. Then it follows from (2.4) that

$$1 - \varphi(x^{2^n}, x^{2^n}) \leqslant \frac{g_{n+1}(x)}{g_n(x)} \leqslant 1 + \varphi(x^{2^n}, x^{2^n})$$

for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$ which yields that

$$\prod_{k=m}^{n-1} [1 - \varphi(x^{2^k}, x^{2^k})] \leqslant \frac{g_n(x)}{g_m(x)} \leqslant \prod_{k=m}^{n-1} [1 + \varphi(x^{2^k}, x^{2^k})] \tag{2.5}$$

for all $x \in \mathbb{R}$ and all $n, m \in \mathbb{N}$ with $n > m$. From (2.5), it follows that

$$\sum_{k=m}^{n-1} \log[1 - \varphi(x^{2^k}, x^{2^k})] \leqslant \log g_n(x) - \log g_m(x) \leqslant \sum_{k=m}^{n-1} \log[1 + \varphi(x^{2^k}, x^{2^k})] \tag{2.6}$$

for all $x \in \mathbb{R}$ and for all $n, m \in \mathbb{N}$ with $n > m$, and from the assumption (2.1) that the series

$$\sum_{n=0}^{\infty} \log[1 - \varphi(x^{2^k}, x^{2^k})] \text{ and } \sum_{n=0}^{\infty} \log[1 + \varphi(x^{2^k}, x^{2^k})]$$

converge for all $x \in \mathbb{R}$. Hence (2.6) tells us that $\{\log g_n(x)\}$ is a Cauchy sequence for all $x \in \mathbb{R}$.

Now we define $\delta : \mathbb{R} \to (0,\infty)$ by $\delta(x) = e^{\lim_{n\to\infty} \log g_n(x)}$, i.e.,

$$\delta(x) = \lim_{n\to\infty} \frac{f(x^{2^n})}{2^n M(x)^{2^n-1}}$$

for all $x \in \mathbb{R}$.

We assert that the function δ satisfies the equation (1.2). Substituting $x = x^{2^n}$ and $y = y^{2^n}$ in (2.2) gives

$$\left| \frac{f(x^{2^n} y^{2^n})}{M(x)^{2^n} f(y^{2^n}) + M(y)^{2^n} f(x^{2^n})} - 1 \right| \leqslant \varphi(x^{2^n}, y^{2^n})$$

which can be rewritten as

$$\left| \frac{\frac{f(x^{2^n} y^{2^n})}{2^n M(x)^{2^n-1} M(y)^{2^n-1}}}{M(x)\frac{f(y^{2^n})}{2^n M(y)^{2^n-1}} + M(y)\frac{f(x^{2^n})}{2^n M(x)^{2^n-1}}} - 1 \right| \leqslant \varphi(x^{2^n}, y^{2^n}) \tag{2.7}$$

for all $x, y \in \mathbb{R}$.

If we pass the limit as $n \to \infty$ in (2.7), then, by using the definition of δ and the assumption (2.1), we obtain

$$\frac{\delta(xy)}{M(x)\delta(y) + M(y)\delta(x)} = 1$$

for all $x, y \in \mathbb{R}$, that is, δ satisfies the equation (1.2).

Taking to the limit as $n \to \infty$ in (2.5), we have

$$\prod_{k=m}^{\infty}[1 - \varphi(x^{2^k}, x^{2^k})] \leqslant \frac{\delta(x)}{g_m(x)} \leqslant \prod_{k=m}^{\infty}[1 + \varphi(x^{2^k}, x^{2^k})]$$

for all $x \in \mathbb{R}$ and in this inequality, setting $m = 0$ gives the inequality

$$\phi(x,x) \leqslant \frac{\delta(x)}{f(x)} \leqslant \psi(x,x)$$

for all $x \in \mathbb{R}$.

It remains to show that δ is uniquely defined. Let d be another solution of (1.2) with

$$\phi(x,x) \leqslant \frac{d(x)}{f(x)} \leqslant \psi(x,x) \tag{2.8}$$

for all $x \in \mathbb{R}$.

Putting $x = x^{2^n}$ in (2.8) yields

$$\phi(x^{2^n}, x^{2^n}) \leqslant \frac{d(x^{2^n})}{f(x^{2^n})} \leqslant \psi(x^{2^n}, x^{2^n}) \tag{2.9}$$

for all $x \in \mathbb{R}$.

Since d is also solution of (1.2), we see that it follows from induction that

$$d(x^{2^n}) = 2^n M(x)^{2^n-1} d(x) \tag{2.10}$$

holds for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$.

By (2.8) and (2.9), we deduce that

$$\phi(x^{2^n}, x^{2^n}) \leqslant \frac{d(x)}{\frac{f(x^{2^n})}{2^n M(x)^{2^n-1}}} \leqslant \psi(x^{2^n}, x^{2^n})$$

for all $x \in \mathbb{R}$.

Passing the limit as $n \to \infty$ in the above inequality and then using the the definition of ϕ and ψ, we conclude that $\delta(x) = d(x)$ is true for all $x \in \mathbb{R}$ and the proof of the theorem is complete. $\square$

If we define the function $M : (0,\infty) \to (0,\infty)$ by $M(x) = x^\alpha$ for all $x \in (0,\infty)$ and any fixed $\alpha \in \mathbb{R}$, then we obtain the following corollary:

COROLLARY 2. *Suppose that the function $f : (0,\infty) \to (0,\infty)$ satisfies the stability inequality*

$$\left| \frac{f(xy)}{x^\alpha f(y) + y^\alpha f(x)} - 1 \right| \leqslant \varphi(x,y) \tag{2.11}$$

for all $x, y \in (0,\infty)$. Then there exists a unique solution $\delta : (0,\infty) \to (0,\infty)$ of the functional equation (1.1) such that the inequality

$$\phi(x,x) \leqslant \frac{\delta(x)}{f(x)} \leqslant \psi(x,x)$$

holds for all $x \in (0,\infty)$.

From now on, let $\tilde{\varphi} : (-\infty,1) \times (-\infty,1) \to (0,1)$ be a function such that

$$\sum_{n=0}^{\infty} \tilde{\varphi}(1-(1-x)^{2^n},\, 1-(1-y)^{2^n})$$

converges for all $x, y \in (-\infty,1)$ and let

$$\tilde{\phi}(x,y) = \prod_{n=0}^{\infty} [\tilde{\varphi}(1-(1-x)^{2^n},\, 1-(1-y)^{2^n})],$$

$$\tilde{\psi}(x,y) = \prod_{n=0}^{\infty} [\tilde{\varphi}(1+(1-x)^{2^n},\, 1-(1-y)^{2^n})]$$

for all $x, y \in (-\infty,1)$.

Applying Corollary 2, we also obtain the stability result *in the sense of Ger* of the following functional equation from [3]:

$$f(x+y-xy) + xf(y) + yf(x) = f(x) + f(y) \tag{2.12}$$

for all $x, y \in (-\infty,1)$.

COROLLARY 3. *Suppose that the function* $f : (-\infty, 1) \to (-\infty, 1)$ *satisfies the stability inequality*

$$\left| \frac{f(x+y-xy)}{f(x)+f(y)-xf(y)-yf(x)} - 1 \right| \leqslant \tilde{\varphi}(x,y) \tag{2.13}$$

for all $x, y \in (-\infty, 1)$. *Then there exists a unique solution* $\varrho : (-\infty, 1) \to (-\infty, 1)$ *of the functional equation* (2.12) *such that the inequality*

$$\tilde{\phi}(x,x) \leqslant \frac{\varrho(x)}{f(x)} \leqslant \tilde{\psi}(x,x)$$

holds for all $x \in (-\infty, 1)$.

Proof. Replacing x by $1-x$ and y by $1-y$ in (2.13), we have

$$\left| \frac{f((1-x)+(1-y)-(1-x)(1-y))}{xf(1-y)+yf(1-x)} - 1 \right| \leqslant \tilde{\varphi}(1-x, 1-y) \tag{2.14}$$

for all $x, y \in (0, \infty)$.

By introducing a function $h : (0,\infty) \to (0,\infty)$ defined by $h(x) = f(1-x)$ for all $x \in (0,\infty)$, we see that the inequality (2.14) is equivalent to the inequality

$$\left| \frac{h(xy)}{xh(y)+yh(x)} - 1 \right| \leqslant \varphi(x,y)$$

for all $x, y \in (0,\infty)$, where $\varphi(x,y) = \tilde{\varphi}(1-x, 1-y)$ for all $x, y \in (0,\infty)$.

Therefore, by Corollary 3, there exists a unique solution $\chi : (0,\infty) \to (0,\infty)$ of the equation (1.1) with the case $\alpha = 1$ such that the inequality

$$\phi(x,x) \leqslant \frac{\chi(x)}{h(x)} \leqslant \psi(x,x) \tag{2.15}$$

holds for all $x \in (0,\infty)$, where $\phi(x,y) = \tilde{\phi}(1-x, 1-y)$ and $\psi(x,y) = \tilde{\psi}(1-x, 1-y)$ for all $x, y \in (0,\infty)$.

Let $\varrho : (-\infty, 1) \to (-\infty, 1)$ be a function defined by $\varrho(x) = \chi(1-x)$ for all $x \in (-\infty, 1)$. Then ϱ is a unique solution of the equation (2.12) and from (2.15) it follows that the inequality

$$\tilde{\phi}(x,x) \leqslant \frac{\varrho(x)}{f(x)} \leqslant \tilde{\psi}(x,x)$$

is satisfied for all $x \in (-\infty, 1)$ which completes the proof. $\square$

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