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A NEW REFINEMENT OF YOUNG'S INEQUALITY

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(*communicated by S. Varošanec*)

Abstract. In this short note, the well-known Young's inequality is refined by a double inequality.

1. Introduction

The original Young's inequality reads as follows.

THEOREM A. [9] *Let* $f(x)$ *be a continuous and strictly increasing function on* $[0,A]$ *for* $A>0$. *If* $f(0)=0$, $a\in[0,A]$ *and* $b\in[0,f(A)]$, *then*

$$\int_0^a f(x)\mathrm{d}x+\int_0^b f^{-1}(x)\mathrm{d}x \geqslant ab, \tag{1}$$

where f^{-1} *is the inverse function of* f. *Equality in* (1) *is valid if and only if* $b=f(a)$.

The following theorem is a converse of Theorem A.

THEOREM B. [7] *If the functions* $f(x)$ *and* $g(x)$ *for* $x\geqslant 0$ *are continuous and strictly increasing with* $f(0)=g(0)=0$, $g^{-1}(x)\geqslant f(x)$ *and*

$$\int_0^a f(x)\mathrm{d}x+\int_0^b g(x)\mathrm{d}x \geqslant ab \tag{2}$$

for all $a>0$ *and* $b>0$, *then* $f=g^{-1}$.

The following reversed version of Young's inequality (1) was obtained in [8].

THEOREM C. [8] *Under the assumptions of Theorem A, the inequality*

$$\min\left\{1,\frac{b}{f(a)}\right\}\int_0^a f(t)\mathrm{d}t+\min\left\{1,\frac{a}{f^{-1}(b)}\right\}\int_0^b f^{-1}(t)\mathrm{d}t \leqslant ab \tag{3}$$

holds. Equality in (3) *is valid if and only if* $b=f(a)$.

For more information on Young's inequality, please refer to [2, pp. 651–653], [3, pp. 48–50], [4, Chapter XIV, pp. 379–389], [1, 6, 10] and the references therein.

In this short note, we would like to refine Young's inequality (1) by a double inequality below.

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THEOREM 1. *Let $f(x)$ be a continuous, differentiable and strictly increasing function on $[0,A]$ for $A>0$. If $f(0)=0$, $a\in[0,A]$, $b\in[0,f(A)]$ and $f'(x)$ is strictly monotonic on $[0,A]$, then*

$$\frac{m}{2}\left[a-f^{-1}(b)\right]^2 \leqslant \int_0^a f(x)\mathrm{d}x+\int_0^b f^{-1}(x)\mathrm{d}x-ab \leqslant \frac{M}{2}\left[a-f^{-1}(b)\right]^2, \tag{4}$$

where

$$m=\min\left\{f'(a),f'\big(f^{-1}(b)\big)\right\} \tag{5}$$

and

$$M=\max\left\{f'(a),f'\big(f^{-1}(b)\big)\right\}. \tag{6}$$

Equalities in (4) *are valid if and only if* $b=f(a)$.

2. Proof of Theorem 1

Changing variable of integration by $x=f(y)$ and integrating by part of the second integral in (4) yields

$$\begin{aligned}
\int_0^a f(x)\mathrm{d}x+\int_0^b f^{-1}(x)\mathrm{d}x &= \int_0^a f(x)\mathrm{d}x+\int_0^{f^{-1}(b)} yf'(y)\mathrm{d}y\\
&= \int_0^a f(x)\mathrm{d}x+bf^{-1}(b)-\int_0^{f^{-1}(b)} f(x)\mathrm{d}x\\
&= bf^{-1}(b)+\int_{f^{-1}(b)}^a f(x)\mathrm{d}x\\
&= ab+\int_{f^{-1}(b)}^a \big[f(x)-b\big]\mathrm{d}x.
\end{aligned} \tag{7}$$

From the fourth line in (7), it is easy to see that if $f^{-1}(b)=a$ then equalities in (4) hold.

If $f^{-1}(b)<a$, since $f(x)$ is strictly increasing, then $f(x)-b>0$ for $x\in\big(f^{-1}(b),a\big)$. By the mean value theorem for derivatives, it is obtained that there exists $c=c(x)$ satisfying $f^{-1}(b)<c<x\leqslant a$ such that

$$0<f(x)-b=\big[x-f^{-1}(b)\big]f'(c).$$

Further, by virtue of the monotonicity of $f'(x)$ on $[0,A]$, it is revealed that

$$0<m\triangleq\min\left\{f'(a),f'\big(f^{-1}(b)\big)\right\}<f'(c)<\max\left\{f'(a),f'\big(f^{-1}(b)\big)\right\}\triangleq M.$$

Consequently,

$$0<m\big[x-f^{-1}(b)\big]<f(x)-b<M\big[x-f^{-1}(b)\big].$$

As a result,

$$m\int_{f^{-1}(b)}^a\big[x-f^{-1}(b)\big]\mathrm{d}x<\int_{f^{-1}(b)}^a\big[f(x)-b\big]\mathrm{d}x<M\int_{f^{-1}(b)}^a\big[x-f^{-1}(b)\big]\mathrm{d}x$$

which is equivalent to

$$\frac{m}{2}\left[a-f^{-1}(b)\right]^2<\int_{f^{-1}(b)}^{a}\left[f(x)-b\right]\mathrm{d}x<\frac{M}{2}\left[a-f^{-1}(b)\right]^2. \tag{8}$$

If $f^{-1}(b)>a$, inequalities in (8) can be deduced by a similar argument as above. Substituting (8) into (7) leads to (4). The proof of Theorem 1 is complete.

3. An example

Taking

$$f(x)=\sqrt[4]{x^4+1}-1,$$

$a=3$ and $b=2$ in Theorem 1 and direct calculation gives

$$\int_0^3 f(x)\mathrm{d}x=\int_0^3\sqrt[4]{x^4+1}\,\mathrm{d}x-3,$$

$$\int_0^2 f^{-1}(x)\mathrm{d}x=\int_0^2\sqrt[4]{(x+1)^4-1}\,\mathrm{d}x=\int_1^3\sqrt[4]{x^4-1}\,\mathrm{d}x,$$

$$f'(x)=\frac{x^3}{\sqrt[4]{(x^4+1)^3}},\qquad f'(3)=\frac{27}{\sqrt[4]{82^3}},$$

$$f'\big(f^{-1}(2)\big)=f'\big(\sqrt[4]{80}\big)=\frac{8\sqrt[4]{5^3}}{27},$$

$$m=\frac{8\sqrt[4]{125}}{27},\quad M=\frac{27}{\sqrt[4]{82^3}}$$

and

$$\frac{4\sqrt[4]{125}}{27}\left[3-2\sqrt[4]{5}\right]^2<\int_0^3\sqrt[4]{x^4+1}\,\mathrm{d}x+\int_1^3\sqrt[4]{x^4-1}\,\mathrm{d}x-9<\frac{27}{2\sqrt[4]{82^3}}\left[3-2\sqrt[4]{5}\right]^2$$

which can be computed numerically as

$$9.000042866\cdots<\int_0^3\sqrt[4]{x^4+1}\,\mathrm{d}x+\int_1^3\sqrt[4]{x^4-1}\,\mathrm{d}x<9.000042871\cdots.$$

This refines the following double inequality

$$9<\int_0^3\sqrt[4]{x^4+1}\,\mathrm{d}x+\int_1^3\sqrt[4]{x^4-1}\,\mathrm{d}x<9.0001$$

in [5, Problem 3].

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