Mathematical
Inequalities
& Applications
Volume 11, Number 4 (2008), 701–716

ON SOME HIGHER–DIMENSIONAL HILBERT'S AND HARDY–HILBERT'S INTEGRAL INEQUALITIES WITH PARAMETERS

MARIO KRNIĆ, JOSIP PEČARIĆ AND PREDRAG VUKOVIĆ

(*communicated by I. Perić*)

Abstract. By introducing some parameters and a norm $|x|_\alpha$, $x \in \mathbf{R}_+{}^n$, we give higher-dimensional Hilbert's and Hardy-Hilbert's integral inequalities in non-conjugate case. Further, we prove that their constant factors are the best possible, in the conjugate case, when the parameters satisfy appropriate conditions. We also compare our results with some known results.

1. Introduction

Hilbert's and Hardy-Hilbert's type inequalities (see [7]) are very significant weight inequalities which play an important role in many fields of mathematics. Similar inequalities, in operator form, appear in harmonic analysis where one investigate properties of boundedness of such operators. This is the reason why Hilbert's inequality is so popular and represents a field of interest for numerous mathematicians: since Hilbert till nowadays.

During the past century such inequalities were generalized in many different directions and also numerous mathematicians reproved them using various technics. Some possibilities of generalizing such inequalities are, for example, by various choices of non-negative measures, kernels, sets of integration, extension to multi-dimensional case etc. Several generalizations involve very important notions such as Hilbert's transform, Laplace transform, singular integrals, Weyl operators.

At present, because of the requirement of higher-dimensional harmonic analysis and higher-dimensional operator theory, multiple Hilbert-type inequality is researched (see [2],[4], [5] and [8]). In this paper we refer to a recent paper of Yong (see [8]):

THEOREM A. *If* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$, $\beta>0$, $s>0$, $a\in\mathbf{R}$, $b\in\mathbf{R}$, $0<n-ap<\beta s$, $0<n-bq<\beta s$, $f\geqslant 0$, $g\geqslant 0$ *and*

$$0<\int_{\mathbf{R}_+{}^n}|\mathbf{x}|_\alpha{}^{(n-\beta s)+p(b-a)}f^p(\mathbf{x})d\mathbf{x}<+\infty,$$

Mathematics subject classification (2000): 26D15.

Key words and phrases: Inequalities, Beta function, Gamma function, Hilbert's inequality, Hardy–Hilbert's inequality, conjugate exponents, non-conjugate exponents, the best possible constant.

$$0 < \int_{\mathbf{R}_+{}^n} |\mathbf{y}|_\alpha{}^{(n-\beta s)+q(a-b)} g^q(\mathbf{y}) d\mathbf{y} < +\infty,$$

then the following two inequalities hold and are equivalent:

$$\int_{\mathbf{R}_+{}^n} \int_{\mathbf{R}_+{}^n} \frac{f(\mathbf{x})g(\mathbf{y})}{\left(|\mathbf{x}|_\alpha{}^\beta + |\mathbf{y}|_\alpha{}^\beta\right)^s} d\mathbf{x} d\mathbf{y} < C_{\alpha,\beta,s}(a,b,p,q) \times \left(\int_{\mathbf{R}_+{}^n} |\mathbf{x}|_\alpha{}^{(n-\beta s)+p(b-a)} f^p(\mathbf{x}) d\mathbf{x}\right)^{\frac{1}{p}} \left(\int_{\mathbf{R}_+{}^n} |\mathbf{y}|_\alpha{}^{(n-\beta s)+q(a-b)} g^q(\mathbf{y}) d\mathbf{y}\right)^{\frac{1}{q}} \quad (1)$$

and

$$\int_{\mathbf{R}_+{}^n} |\mathbf{y}|_\alpha{}^{\frac{(n-\beta s)+q(a-b)}{1-q}} \left[\int_{\mathbf{R}_+{}^n} \frac{f(\mathbf{x})}{\left(|\mathbf{x}|_\alpha{}^\beta + |\mathbf{y}|_\alpha{}^\beta\right)^s} d\mathbf{x}\right]^p d\mathbf{y} < C^p{}_{\alpha,\beta,s}(a,b,p,q) \int_{\mathbf{R}_+{}^n} |\mathbf{x}|_\alpha{}^{(n-\beta s)+p(b-a)} f^p(\mathbf{x}) d\mathbf{x}, \quad (2)$$

with the constant $C_{\alpha,\beta,s}(a,b,p,q)$ *defined by*

$$C_{\alpha,\beta,s}(a,b,p,q) = \frac{\Gamma^n(\frac{1}{\alpha})}{\beta\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B^{\frac{1}{p}}\left(\frac{n-ap}{\beta}, s - \frac{n-ap}{\beta}\right) B^{\frac{1}{q}}\left(\frac{n-bq}{\beta}, s - \frac{n-bq}{\beta}\right),$$

where Γ *is gamma function,* B *is beta function and* $|\cdot|_\alpha$ *is the* $\alpha-$ *norm.*

We usually call inequality (1) Hilbert's type inequality, and inequality (2) Hardy-Hilbert's type inequality. These inequalities are equivalent. Note that the inequalities (1) and (2) include two conjugate exponents p and q.

Further, paper [8] discusses the problem of the best possible constant. However, the best possible constant is obtained there in some special cases.

Hilbert's and Hardy-Hilbert's inequalities with two non-conjugate parameters were firstly researched by Hardy, Littlewood and Pólya (see [7], chapter IX, page 254). They obtained the inequality without the specific value for the constant factor. The value of that constant factor was given by Levin (see [11]), but the paper did not decide whether that was the best possible constant. That question is a hard problem and still remains open.

After that, the described inequalities with non-conjugate parameters were researched and generalized by Bonsall (see [1]). He obtained some multidimensional results and established general conditions of non-conjugate exponents. Now, we present these notions and definitions: Let p_i, $i = 1,2,\ldots,k$, be the real parameters which satisfy

$$\sum_{i=1}^{k} \frac{1}{p_i} \geqslant 1 \quad \text{and} \quad p_i > 1, \quad i = 1,2,\ldots k. \quad (3)$$

Further, the parameters $p_i{}'$, $i = 1,2,\ldots,k$ are defined by the equations

$$\frac{1}{p_i} + \frac{1}{p_i'} = 1, \quad i = 1,2,\ldots k. \quad (4)$$

Since $p_i > 1$, $i = 1,2,\dots,k$, it is obvious that $p_i' > 1$, $i = 1,2,\dots,k$. We define

$$\lambda := \frac{1}{k-1}\sum_{i=1}^{k}\frac{1}{p_i'}. \tag{5}$$

It is easy to deduce that $0 < \lambda \leqslant 1$. Also, we introduce the parameters q_i, $i = 1,2,\dots k$, defined by the relations

$$\frac{1}{q_i} = \lambda - \frac{1}{p_i'}, \quad i = 1,2,\dots,k. \tag{6}$$

In order to obtain our results we need to require

$$q_i > 0 \quad i = 1,2,\dots,k. \tag{7}$$

It is easy to see that the above conditions do not automatically apply (7). Further, it follows

$$\lambda = \sum_{i=1}^{k}\frac{1}{q_i} \quad \text{and} \quad \frac{1}{q_i} + 1 - \lambda = \frac{1}{p_i}, \quad i = 1,2,\dots,k.$$

Of course, if $\lambda = 1$, then $\sum_{i=1}^{k}\frac{1}{p_i} = 1$, so the conditions (3)–(6) reduce to the case of conjugate parameters.

Our main aim in this paper is to generalize Theorem A. Our generalization will include the integrals taken over the set $(\mathbf{R}_+{}^n)^k$, where $k \geqslant 2$, with k non-conjugate parameters. The techniques that will be used in the proofs are mainly based on classical real analysis, especially on the well known Hölder's inequality and on Fubini's theorem.

Conventions. Throughout this paper we suppose that all the functions are non-negative and measurable, so that all integrals converge. We also introduce the notations:

$$\mathbf{R}_+{}^n = \{\mathbf{x} = (x_1,x_2,\dots,x_n);\ x_1,x_2,\dots,x_n > 0\},$$

$$|\mathbf{x}|_\alpha = (x_1{}^\alpha + x_2{}^\alpha + \cdots + x_n{}^\alpha)^{\frac{1}{\alpha}},\ \alpha > 0,$$

and we agree that $|\mathbf{x}|_\alpha < \delta$ represents the set $\{\mathbf{x} \in \mathbf{R}_+{}^n;\ |\mathbf{x}|_\alpha < \delta\}$.

2. Preliminaries

The following lemma can be found in [6]:

LEMMA 1. *Let $p_i > 0$, $a_i > 0$, $\alpha_i > 0$ and $\Psi(u)$ be a measurable function. Then the following integral equality holds*

$$\begin{aligned}\int\cdots\int_{x_1,x_2,\dots,x_n>0,(\frac{x_1}{a_1})^{\alpha_1}+\cdots+(\frac{x_n}{a_n})^{\alpha_n}\leqslant 1}&\Psi\left(\left(\frac{x_1}{a_1}\right)^{\alpha_1}+\cdots+\left(\frac{x_n}{a_n}\right)^{\alpha_n}\right)\\ &\times x_1{}^{p_1-1}\cdots x_n{}^{p_n-1}dx_1\cdots dx_n\\ =\frac{a_1{}^{p_1}\cdots a_n{}^{p_n}\Gamma\left(\frac{p_1}{\alpha_1}\right)\cdots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\alpha_1\cdots\alpha_n\Gamma\left(\frac{p_1}{\alpha_1}+\cdots+\frac{p_n}{\alpha_n}\right)}&\int_0^1\Psi(u)u^{\frac{p_1}{\alpha_1}+\cdots+\frac{p_n}{\alpha_n}-1}du,\end{aligned} \tag{8}$$

where Γ *is gamma function.*

For the sake of technical reasons, we introduce real parameters $A_{ij},\ i,j=1,2,\dots,k$ satisfying

$$\sum_{i=1}^{k} A_{ij}=0, \quad j=1,2,\dots,k. \tag{9}$$

We also define

$$\alpha_i=\sum_{j=1}^{k} A_{ij}, \quad i=1,2,\dots,k. \tag{10}$$

The main results in this paper will be based on the following result:

LEMMA 2. *We define the weight function by*

$$\omega_{\alpha,\beta,s}(\mathbf{x_i})=\int_{(\mathbf{R}_+{}^n)^{k-1}} \frac{\prod_{j=1,j\neq i}^{k}|\mathbf{x_j}|_\alpha{}^{q_iA_{ij}}}{(\sum_{j=1}^{k}|\mathbf{x_j}|_\alpha{}^{\beta})^s} d\mathbf{x_1}d\mathbf{x_2}\cdots d\mathbf{x_{i-1}}d\mathbf{x_{i+1}}\cdots d\mathbf{x_k}, \tag{11}$$

where $\alpha,\beta,s,q_i>0$, $A_{ij}>-\frac{n}{q_i}$ *and* $\frac{\beta s-(k-1)n}{q_i}>\alpha_i-A_{ii}$, $i=1,2,\dots,k$. *Then,*

$$\omega_{\alpha,\beta,s}(\mathbf{x_i})=K_i|\mathbf{x_i}|_\alpha{}^{(k-1)n-\beta s+q_i\alpha_i-q_iA_{ii}} \tag{12}$$

where the constant K_i *is defined by*

$$K_i=\frac{\Gamma^{(k-1)n}\left(\frac{1}{\alpha}\right)\Gamma\left(s-\frac{(k-1)n+q_i\alpha_i-q_iA_{ii}}{\beta}\right)}{\beta^{k-1}\alpha^{(n-1)(k-1)}\Gamma^{k-1}\left(\frac{n}{\alpha}\right)\Gamma(s)}\prod_{j=1,j\neq i}^{k}\Gamma\left(\frac{n+q_iA_{ij}}{\beta}\right). \tag{13}$$

Proof. By applying Fubini's theorem we have

$$\begin{aligned}\omega_{\alpha,\beta,s}(\mathbf{x_i}) &= \int_{(\mathbf{R}_+{}^n)^{k-2}}\prod_{j=2,j\neq i}^{k}|\mathbf{x_j}|_\alpha{}^{q_iA_{ij}}\left(\int_{\mathbf{R}_+{}^n}\frac{|\mathbf{x_1}|_\alpha{}^{q_iA_{i1}}}{(\sum_{j=1}^{k}|\mathbf{x_j}|_\alpha{}^{\beta})^s}d\mathbf{x_1}\right)\\ &\qquad\times d\mathbf{x_2}\cdots d\mathbf{x_{i-1}}d\mathbf{x_{i+1}}\cdots d\mathbf{x_k}.\end{aligned} \tag{14}$$

Now, the inner integral can be transformed in the following way:

$$\begin{aligned}&\int_{\mathbf{R}_+{}^n}\frac{|\mathbf{x_1}|_\alpha{}^{q_iA_{i1}}}{(\sum_{j=1}^{k}|\mathbf{x_j}|_\alpha{}^{\beta})^s}d\mathbf{x_1}\\ &=\lim_{r\to\infty}\int\cdots\int_{\substack{x_{11},\dots,x_{1n}>0,\\ x_{11}{}^\alpha+\cdots+x_{1n}{}^\alpha<r^\alpha}}\frac{\left[r\left(\left(\frac{x_{11}}{r}\right)^\alpha+\cdots+\left(\frac{x_{1n}}{r}\right)^\alpha\right)^{\frac{1}{\alpha}}\right]^{q_iA_{i1}}}{\left[r^\beta\left(\left(\frac{x_{11}}{r}\right)^\alpha+\cdots+\left(\frac{x_{1n}}{r}\right)^\alpha\right)^{\frac{\beta}{\alpha}}+\sum_{j=2}^{k}|\mathbf{x_j}|_\alpha{}^{\beta}\right]^s}\\ &\qquad\times x_{11}{}^{1-1}\cdots x_{1n}{}^{1-1}dx_{11}\dots dx_{1n},\end{aligned} \tag{15}$$

where $\mathbf{x_1}=(x_{11},\dots,x_{1n})$.

By using the integral equality from Lemma 1 we have

$$\begin{aligned}\int_{\mathbf{R}_+{}^n} \frac{|\mathbf{x_1}|_\alpha{}^{q_iA_{i1}}}{(\sum_{j=1}^k |\mathbf{x_j}|_\alpha{}^\beta)^s} d\mathbf{x_1} &= \lim_{r\to\infty} \frac{r^n\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^n\Gamma\left(\frac{n}{\alpha}\right)} \int_0^1 \frac{(ru^{\frac{1}{\alpha}})^{q_iA_{i1}}}{\left(r^\beta u^{\frac{\beta}{\alpha}} + \sum_{j=2}^k |\mathbf{x_j}|_\alpha{}^\beta\right)^s} u^{\frac{n}{\alpha}-1} du \\ &= \lim_{r\to\infty} \frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)} \int_0^r \frac{t^{n+q_iA_{i1}-1}}{(t^\beta + \sum_{j=2}^k |\mathbf{x_j}|_\alpha{}^\beta)^s} dt \\ &= \frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)} \int_0^\infty \frac{t^{n+q_iA_{i1}-1}}{(t^\beta + \sum_{j=2}^k |\mathbf{x_j}|_\alpha{}^\beta)^s} dt.\end{aligned}$$

Finally, by using the substitution $t^\beta = u\left(\sum_{j=2}^k |\mathbf{x_j}|_\alpha{}^\beta\right)$ one obtains

$$\begin{aligned}\int_{\mathbf{R}_+{}^n} \frac{|\mathbf{x_1}|_\alpha{}^{q_iA_{i1}}}{(\sum_{j=1}^k |\mathbf{x_j}|_\alpha{}^\beta)^s} d\mathbf{x_1} &= \frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\beta\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)} \Big(\sum_{j=2}^k |\mathbf{x_j}|_\alpha{}^\beta\Big)^{\frac{n+q_iA_{i1}}{\beta}-s} \int_0^\infty \frac{u^{\frac{n+q_iA_{i1}}{\beta}-1}}{(1+u)^s} du \\ &= \frac{\Gamma^n\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{n+q_iA_{i1}}{\beta}\right)\Gamma\left(s-\frac{n+q_iA_{i1}}{\beta}\right)}{\beta\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)\Gamma(s)} \left(\sum_{j=2}^k |\mathbf{x_j}|_\alpha{}^\beta\right)^{\frac{n+q_iA_{i1}}{\beta}-s}. \quad (16)\end{aligned}$$

Now, we substitute (16) in (14) and we again use Fubini's theorem. In that second step we have, as above in (16),

$$\begin{aligned}&\int_{\mathbf{R}_+{}^n} \frac{|\mathbf{x_2}|_\alpha{}^{q_iA_{i2}}}{(\sum_{j=2}^k |\mathbf{x_j}|_\alpha{}^\beta)^{s-\frac{n+q_iA_{i1}}{\beta}}} d\mathbf{x_2} \\ &= \frac{\Gamma^n\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{n+q_iA_{i2}}{\beta}\right)\Gamma\left(s-\frac{2n+q_iA_{i1}+q_iA_{i2}}{\beta}\right)}{\beta\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)\Gamma\left(s-\frac{n+q_iA_{i1}}{\beta}\right)} \left(\sum_{j=3}^k |\mathbf{x_j}|_\alpha{}^\beta\right)^{\frac{2n+q_iA_{i1}+q_iA_{i2}}{\beta}-s}.\end{aligned}$$

If we continue with the described procedure, for the variables $\mathbf{x_3}$, $\mathbf{x_4}$, ..., $\mathbf{x_{i-1}}$, $\mathbf{x_{i+1}}$, ..., $\mathbf{x_k}$, we obtain equality (12) and the proof is completed. □

3. Main results

In this section we shall use the relations (11) and (12) to obtain the higher-dimensional generalization of Theorem A over the set $(\mathbf{R}_+{}^n)^k$, $k \geqslant 2$, with non-conjugate exponents. Our main result is following:

THEOREM 1. *Let $k \geqslant 2$ be an integer and p_i, p_i', q_i, $i = 1,2,\dots,k$, be real numbers satisfying (3)-(7). Further, let A_{ij}, $i,j = 1,2,\dots,k$ be real parameters satisfying (9) and (10). Then, for any non-negative measurable functions f_i, $i =$*

$1,2,\ldots,k$, *the following inequalities hold and are equivalent*

$$\int_{(\mathbf{R}_+{}^n)^k} \frac{\prod_{i=1}^k f_i(\mathbf{x_i})}{\left(\sum_{i=1}^k |\mathbf{x_i}|_\alpha{}^\beta\right)^{\lambda s}} d\mathbf{x_1} d\mathbf{x_2}\cdots d\mathbf{x_k} \leqslant K \prod_{i=1}^k \left[\int_{\mathbf{R}_+{}^n} |\mathbf{x_i}|_\alpha{}^{\frac{p_i(k-1)n-p_i\beta s}{q_i}+p_i\alpha_i} f_i^{p_i}(\mathbf{x_i}) d\mathbf{x_i}\right]^{\frac{1}{p_i}} \tag{17}$$

and

$$\left\{\int_{\mathbf{R}_+{}^n} |\mathbf{x_k}|_\alpha{}^{-\frac{p_k'}{q_k}[(k-1)n-\beta s]-p_k'\alpha_k} \times \left[\int_{(\mathbf{R}_+{}^n)^{k-1}} \frac{\prod_{i=1}^{k-1} f_i(\mathbf{x_i})}{\left(\sum_{i=1}^k |\mathbf{x_i}|_\alpha{}^\beta\right)^{\lambda s}} d\mathbf{x_1} d\mathbf{x_2}\cdots d\mathbf{x_{k-1}}\right]^{p_k'} d\mathbf{x_k}\right\}^{\frac{1}{p_k'}} \leqslant K \prod_{i=1}^{k-1} \left[\int_{\mathbf{R}_+{}^n} |\mathbf{x_i}|_\alpha{}^{\frac{p_i(k-1)n-p_i\beta s}{q_i}+p_i\alpha_i} f_i^{p_i}(\mathbf{x_i}) d\mathbf{x_i}\right]^{\frac{1}{p_i}}, \tag{18}$$

for any $s, q_i > 0$, $A_{ij} > -\frac{n}{q_i}$, $\alpha_i - A_{ii} < \frac{\beta s-(k-1)n}{q_i}$, *where the constant* K *is given by the formula*

$$K = \frac{\Gamma^{\lambda(k-1)n}\left(\frac{1}{\alpha}\right)}{\beta^{\lambda(k-1)}\alpha^{\lambda(n-1)(k-1)}\Gamma^{\lambda(k-1)}\left(\frac{n}{\alpha}\right)\Gamma^\lambda(s)} \prod_{i=1}^k \Gamma^{\frac{1}{q_i}}\left(s - \frac{(k-1)n+q_i\alpha_i-q_iA_{ii}}{\beta}\right) \times \prod_{i,j=1, i\neq j}^k \Gamma^{\frac{1}{q_i}}\left(\frac{n+q_iA_{ij}}{\beta}\right).$$

Proof. We start with inequality (17). The left-hand side of the inequality (17) can easily be transformed in the following way

$$\int_{(\mathbf{R}_+{}^n)^k} \frac{\prod_{i=1}^k f_i(\mathbf{x_i})}{\left(\sum_{i=1}^k |\mathbf{x_i}|_\alpha{}^\beta\right)^{\lambda s}} d\mathbf{x_1} d\mathbf{x_2}\cdots d\mathbf{x_k} = \int_{(\mathbf{R}_+{}^n)^k} \prod_{i=1}^k \left[\frac{|\mathbf{x_i}|_\alpha{}^{p_iA_{ii}} \prod_{j=1,j\neq i}^k |\mathbf{x_j}|_\alpha{}^{q_iA_{ij}}}{\left(\sum_{i=1}^k |\mathbf{x_i}|_\alpha{}^\beta\right)^s} F_i^{p_i-q_i}(\mathbf{x_i}) f_i^{p_i}(\mathbf{x_i})\right]^{\frac{1}{q_i}} \times \left[\prod_{i=1}^k |\mathbf{x_i}|_\alpha{}^{p_iA_{ii}} (F_if_i)^{p_i}(\mathbf{x_i})\right]^{1-\lambda} d\mathbf{x_1} d\mathbf{x_2}\cdots d\mathbf{x_k},$$

where $F_i(\mathbf{x_i}) = \left[\omega_{\alpha,\beta,s}(\mathbf{x_i})\right]^{\frac{1}{q_i}}$, and the weight function $\omega_{\alpha,\beta,s}(\mathbf{x_i})$ is as defined in Lemma 2. It follows from Lemma 2 that

$$F_i(\mathbf{x_i}) = K_i^{\frac{1}{q_i}} |\mathbf{x_i}|_\alpha{}^{\frac{(k-1)n-\beta s}{q_i}+\alpha_i-A_{ii}}, \tag{19}$$

where the constant K_i is defined by formula (13).

Further, since $\sum_{i=1}^{k}\frac{1}{q_i}+1-\lambda=1$, $q_i>1$ and $0<\lambda\leqslant 1$, we can apply Hölder's inequality with conjugate parameters $q_1,q_2,\dots,q_k$ and $\frac{1}{1-\lambda}$, to the above transformation. Namely, we obtain

$$\int_{(\mathbf{R_+}^n)^k}\frac{\prod_{i=1}^{k}f_i(\mathbf{x_i})}{\left(\sum_{i=1}^{k}|\mathbf{x_i}|_\alpha{}^\beta\right)^{\lambda s}}d\mathbf{x_1}d\mathbf{x_2}\cdots d\mathbf{x_k}$$
$$\leqslant \prod_{i=1}^{k}\left[\int_{\mathbf{R_+}^n}|\mathbf{x_i}|_\alpha{}^{p_iA_{ii}}(F_if_i)^{p_i}(\mathbf{x_i})d\mathbf{x_i}\right]^{\frac{1}{q_i}}\prod_{i=1}^{k}\left[\int_{\mathbf{R_+}^n}|\mathbf{x_i}|_\alpha{}^{p_iA_{ii}}(F_if_i)^{p_i}(\mathbf{x_i})d\mathbf{x_i}\right]^{1-\lambda}$$
$$=\prod_{i=1}^{k}\left[\int_{\mathbf{R_+}^n}|\mathbf{x_i}|_\alpha{}^{p_iA_{ii}}(F_if_i)^{p_i}(\mathbf{x_i})d\mathbf{x_i}\right]^{\frac{1}{p_i}},$$

since $\frac{1}{q_i}+1-\lambda=\frac{1}{p_i}$. Finally, by using definition (19) of the functions F_i, $i=1,2,\dots,k$, one obtains the inequality (17).

Let us show that the inequalities (17) and (18) are equivalent. Suppose that the inequality (17) is valid. If we put the function $f_k:\mathbf{R}^n\mapsto\mathbf{R}$, defined by

$$f_k(\mathbf{x_k})=|\mathbf{x_k}|_\alpha{}^{-\frac{p_k{}'}{q_k}[(k-1)n-\beta s]-p_k{}'\alpha_k}\left[\int_{(\mathbf{R_+}^n)^{k-1}}\frac{\prod_{i=1}^{k-1}f_i(\mathbf{x_i})}{\left(\sum_{i=1}^{k}|\mathbf{x_i}|_\alpha{}^\beta\right)^{\lambda s}}d\mathbf{x_1}d\mathbf{x_2}\cdots d\mathbf{x_{k-1}}\right]^{\frac{p_k{}'}{p_k}}$$

in inequality (17), we obtain

$$I^{p_k'}(\mathbf{x_k})\leqslant K\prod_{i=1}^{k-1}\left[\int_{\mathbf{R_+}^n}|\mathbf{x_i}|_\alpha{}^{\frac{p_i(k-1)n-p_i\beta s}{q_i}+p_i\alpha_i}f_i^{p_i}(\mathbf{x_i})d\mathbf{x_i}\right]^{\frac{1}{p_i}}I^{\frac{p_k'}{p_k}}(\mathbf{x_k})$$

where $I(\mathbf{x_k})$ denotes the left-hand side of inequality (18). That gives inequality (18).

It remains to prove that inequality (17) is a consequence of inequality (18). For this purpose, let's suppose that inequality (18) is valid. Then the left-hand side of inequality (17) can be transformed in the following way:

$$\int_{(\mathbf{R_+}^n)^k}\frac{\prod_{i=1}^{k}f_i(\mathbf{x_i})}{\left(\sum_{i=1}^{k}|\mathbf{x_i}|_\alpha{}^\beta\right)^{\lambda s}}d\mathbf{x_1}d\mathbf{x_2}\cdots d\mathbf{x_k}$$
$$=\int_{\mathbf{R_+}^n}|\mathbf{x_k}|_\alpha{}^{\frac{(k-1)n-\beta s}{q_k}+\alpha_k}f_k(\mathbf{x_k})$$
$$\times\left[|\mathbf{x_k}|_\alpha{}^{-\frac{(k-1)n-\beta s}{q_k}-\alpha_k}\int_{(\mathbf{R_+}^n)^{k-1}}\frac{\prod_{i=1}^{k-1}f_i(\mathbf{x_i})}{\left(\sum_{i=1}^{k}|\mathbf{x_i}|_\alpha{}^\beta\right)^{\lambda s}}d\mathbf{x_1}d\mathbf{x_2}\cdots d\mathbf{x_{k-1}}\right]d\mathbf{x_k}.$$

Applying Hölder's inequality to the conjugate parameters p_k and p'_k we have

$$\int_{(\mathbf{R}_+{}^n)^k} \frac{\prod_{i=1}^k f_i(\mathbf{x_i})}{\left(\sum_{i=1}^k |\mathbf{x_i}|_\alpha{}^\beta\right)^{\lambda s}} d\mathbf{x_1} d\mathbf{x_2}\cdots d\mathbf{x_k}$$

$$\leqslant \left[\int_{\mathbf{R}_+{}^n} |\mathbf{x_k}|_\alpha{}^{\frac{p_k(k-1)n-p_k\beta s}{q_k}+p_k\alpha_k} f_k^{p_k}(\mathbf{x_k})d\mathbf{x_k}\right]^{\frac{1}{p_k}} \cdot I(\mathbf{x_k}),$$

and the result follows from (18). Hence, we have shown that inequalities (17) and (18) are equivalent. Since the first inequality is valid, the second one is also valid. That completes the proof. □

REMARK 1. It is easy to see that Theorem 1 is the generalization of Theorem A from the Introduction. Namely, let us define, for $k = 2$, $p_1 = p$, $p_2 = q$, $A_{11} = b$, $A_{12} = -a$, $A_{21} = -b$, $A_{22} = a$. Since $\lambda = 1$ and $p_i = q_i$, $\alpha_i = A_{i1} + A_{i2}$, $i = 1,2$, in the conjugate case, and since $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, inequalities (17) and (18) become (1) and (2) respectively, as well as the constant K from Theorem 1 becomes $C_{\alpha,\beta,s}(a,b,p,q)$ from Theorem A.

REMARK 2. Further, if we put $n = 1$ and $\beta = 1$ in Theorem 1, we obtain the result from our paper [4], and in the conjugate case from [2].

It is very interesting to investigate under which conditions the equality in (17) and (18) holds. The answer to that question is given in the following result:

THEOREM 2. *The equality in (17) and (18) holds if and only if at least one of the functions f_i is equal to zero.*

Proof. Observe that the equality in inequality (17) holds if and only if it holds in Hölder's inequality. By using the notation from Theorem 1, it means that the functions $|\mathbf{x_i}|_\alpha{}^{p_iA_{ii}} \prod_{j=1,j\neq i}^k |\mathbf{x_j}|_\alpha{}^{q_iA_{ij}} \left(\sum_{j=1}^k |\mathbf{x_j}|_\alpha{}^\beta\right)^{-s} F_i^{p_i-q_i}(\mathbf{x_i}) f_i^{p_i}(\mathbf{x_i})$, $i = 1,2,\ldots,k$, and $\prod_{i=1}^k |\mathbf{x_i}|_\alpha{}^{p_iA_{ii}} (F_if_i)^{p_i}(\mathbf{x_i})$ are effectively proportional. So, if we suppose that the functions f_i, $i = 1,2,\ldots,k$ are not equal to zero, a straightforward computation (see also [4], Remark 1) leads to the following condition

$$\left(\sum_{j=1}^k |\mathbf{x_j}|_\alpha{}^\beta\right)^{-s} = C\prod_{j=1}^k |\mathbf{x_j}|_\alpha{}^{(k-1)n-\beta s+q_i\alpha_i-q_iA_{ii}},$$

where C is an appropriate constant, and that is a contradiction. So the equality in Theorem 1 holds if and only if at least one of the functions f_i is identically equal to zero. Otherwise, for non-negative and non-zero functions, the inequalities (17) and (18) are strict. □

The following remark deals with the reversed sign of inequalities (17) and (18):

REMARK 3. If the parameters p_i, $i = 1,2,\ldots,k$ are chosen in such a way that

$$q_j > 0, \text{ for some } j \in \{1,2,\ldots n\}, \quad q_i < 0, i \neq j \quad \text{and} \quad \lambda < 1 \tag{20}$$

or

$$q_i < 0, \quad i = 1, 2, \dots, n \tag{21}$$

then the exponents from the proof of Theorem 1 fulfill the conditions for the reverse Hölder's inequality (for details see e.g. [12], Chapter V), which gives the reversed sign of inequalities (17) and (18).

4. The best possible constants in the conjugate case

In this section we consider the inequalities in Theorem 1. In such a way we shall obtain the best possible constants for some general cases. As we have already mentioned, obtaining the best possible constants in the case of non-conjugate parameters seems to be a very difficult problem and it remains still open.

It follows easily that the constant K from the previous theorem, in the conjugate case ($\lambda = 1$, $p_i = q_i$), becomes

$$\begin{aligned} K \;=\; & \frac{\Gamma^{(k-1)n}\left(\frac{1}{\alpha}\right)}{\beta^{(k-1)}\alpha^{(n-1)(k-1)}\Gamma^{(k-1)}\left(\frac{n}{\alpha}\right)\Gamma(s)} \prod_{i=1}^{k} \Gamma^{\frac{1}{p_i}}\left(s - \frac{(k-1)n + p_i\alpha_i - p_i A_{ii}}{\beta}\right) \\ & \times \prod_{i,j=1, i\neq j}^{k} \Gamma^{\frac{1}{p_i}}\left(\frac{n + p_i A_{ij}}{\beta}\right). \end{aligned}$$

However, we shall deal with appropriate forms of the inequalities obtained in the previous section in the conjugate case. The main idea is to simplify the above constant K, i.e. to obtain the constant without the conjugate exponents. For that sake, it is natural to consider real parameters A_{ij} satisfying the following relation

$$\beta s - (k-1)n + p_i A_{ii} - p_i\alpha_i = n + p_j A_{ji}, \quad j \neq i, \quad i, j \in \{1, 2, \dots, k\}. \tag{22}$$

In that case the constant K above is simplified to the following form:

$$K^* = \frac{\Gamma^{(k-1)n}\left(\frac{1}{\alpha}\right)}{\beta^{k-1}\alpha^{(n-1)(k-1)}\Gamma^{k-1}\left(\frac{n}{\alpha}\right)\Gamma(s)} \prod_{i=1}^{k} \Gamma\left(\frac{n + \widetilde{A}_i}{\beta}\right), \tag{23}$$

where

$$\widetilde{A}_i = p_j A_{ji}, \quad j \neq i \quad \text{and} \quad -n < \widetilde{A}_i. \tag{24}$$

It is easy to see that the parameters $\widetilde{A}_i$ satisfy the relation

$$\sum_{i=1}^{k} \widetilde{A}_i = \beta s - kn. \tag{25}$$

Further, by using (9) and (24) we have

$$\begin{aligned} A_{ii} &= -A_{1i} - A_{2i} - \cdots - A_{i-1i} - A_{i+1i} - \cdots - A_{ki} \\ &= -\frac{\widetilde{A}_i}{p_1} - \frac{\widetilde{A}_i}{p_2} - \cdots - \frac{\widetilde{A}_i}{p_{i-1}} - \frac{\widetilde{A}_i}{p_{i+1}} - \cdots - \frac{\widetilde{A}_i}{p_k} \\ &= \widetilde{A}_i\left(\frac{1}{p_i} - 1\right). \end{aligned} \tag{26}$$

Further, by using (24), (25) and (26), the inequalities (17) and (18) with the parameters A_{ij}, satisfying relation (22), become

$$\int_{(\mathbf{R}_+{}^n)^k} \frac{\prod_{i=1}^k f_i(\mathbf{x_i})}{\left(\sum_{i=1}^k |\mathbf{x_i}|_\alpha{}^\beta\right)^s} d\mathbf{x_1} d\mathbf{x_2}\cdots d\mathbf{x_k} \leqslant K^* \prod_{i=1}^k \left[\int_{\mathbf{R}_+{}^n} |\mathbf{x_i}|_\alpha{}^{-n-p_i\widetilde{A_i}} f_i^{p_i}(\mathbf{x_i}) d\mathbf{x_i}\right]^{\frac{1}{p_i}} \quad (27)$$

and

$$\left\{\int_{\mathbf{R}_+{}^n} |\mathbf{x_k}|_\alpha{}^{(1-p_k')(-n-p_k\widetilde{A_k})} \times \left[\int_{(\mathbf{R}_+{}^n)^{k-1}} \frac{\prod_{i=1}^{k-1} f_i(\mathbf{x_i})}{\left(\sum_{i=1}^k |\mathbf{x_i}|_\alpha{}^\beta\right)^s} d\mathbf{x_1} d\mathbf{x_2}\cdots d\mathbf{x_{k-1}}\right]^{p_k'} d\mathbf{x_k}\right\}^{\frac{1}{p_k'}}$$
$$\leqslant K^* \prod_{i=1}^{k-1} \left[\int_{\mathbf{R}_+{}^n} |\mathbf{x_i}|_\alpha{}^{-n-p_i\widetilde{A_i}} f_i^{p_i}(\mathbf{x_i}) d\mathbf{x_i}\right]^{\frac{1}{p_i}}. \quad (28)$$

We shall see that the constant K^* in (27) and (28) is the best possible in the sense that we can't replace that constant K^* in inequalities (27) and (28) with the smaller constant, so that the inequalities are fulfilled for all non-negative measurable functions.

THEOREM 3. *The constant K^* is the best possible in both inequalities (27) and (28).*

Proof. Let's suppose that the constant factor K^* given by (23) is not the best possible in the inequality (27). Then, there exist a positive constant $K_1 < K^*$, such that (27) is still valid when we replace K^* by K_1.

We define the real functions $\widetilde{f_{i,\varepsilon}} : \mathbf{R}^n \mapsto \mathbf{R}$ by the formulas

$$\widetilde{f_{i,\varepsilon}}(\mathbf{x_i}) = \begin{cases} 0, & |\mathbf{x_i}|_\alpha < 1 \\ |\mathbf{x_i}|_\alpha{}^{\widetilde{A_i}-\frac{\varepsilon}{p_i}}, & |\mathbf{x_i}|_\alpha \geqslant 1 \end{cases}, \quad i = 1,\dots,k,$$

where $0 < \varepsilon < \min_{1\leqslant i\leqslant k}\{p_i + p_i\widetilde{A_i}\}$. Now, we shall put these functions in inequality (27). By using the $n-$ dimensional spherical coordinates, the right-hand side of the inequality (27) becomes

$$K_1 \prod_{i=1}^k \left[\int_{|\mathbf{x_i}|_\alpha\geqslant 1} |\mathbf{x_i}|_\alpha^{-n-\varepsilon} d\mathbf{x_i}\right]^{\frac{1}{p_i}} = \frac{K_1|\mathbf{S}^{n-1}|_\alpha}{2^n}\int_1^\infty t^{-1-\varepsilon}dt = \frac{K_1|\mathbf{S}^{n-1}|_\alpha}{2^n\varepsilon}, \quad (29)$$

where $\mathbf{S}^{n-1}$ denotes unit sphere in $\mathbf{R}^n$, and let $|\mathbf{S}^{n-1}|_\alpha = \frac{2^n\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}$ be its area in view of $\alpha-$ norm.

Further, let J denotes the left-hand side of the inequality (27), for above choice of the functions $\widetilde{f_{i,\varepsilon}}$. By applying $n-$ spherical coordinates and the substitutions $y_i = t_i^\beta$,

$i=1,\dots,k,$ in $J,$ we find

$$J=\int_{|\mathbf{x_1}|_\alpha\geqslant 1}\cdots\int_{|\mathbf{x_k}|_\alpha\geqslant 1}\frac{\prod_{i=1}^k|\mathbf{x_i}|_\alpha^{\widetilde{A}_i-\frac{\varepsilon}{p_i}}}{\left(\sum_{i=1}^k|\mathbf{x_i}|_\alpha^\beta\right)^s}d\mathbf{x_1}\dots d\mathbf{x_k}$$

$$=\frac{|\mathbf{S}^{n-1}|_\alpha^k}{2^{kn}}\int_1^\infty\cdots\int_1^\infty\frac{\prod_{i=1}^k t_i^{n-1+\widetilde{A}_i-\frac{\varepsilon}{p_i}}}{\left(\sum_{i=1}^k t_i^\beta\right)^s}dt_1\dots dt_k$$

$$=\frac{|\mathbf{S}^{n-1}|_\alpha^k}{\beta^k2^{kn}}\int_1^\infty y_1^{\frac{n+\widetilde{A}_1-\frac{\varepsilon}{p_1}}{\beta}-1}\left(\int_1^\infty\cdots\int_1^\infty\frac{\prod_{i=2}^k y_i^{\frac{n+\widetilde{A}_i-\frac{\varepsilon}{p_i}}{\beta}-1}}{\left(\sum_{i=1}^k y_i\right)^s}dy_2\dots dy_k\right)dy_1. \quad (30)$$

By using the substitutions $u_i=\frac{y_i}{y_1},$ $i=2,\dots,k,$ in (30), and the identity (25), we have

$$J=\frac{|\mathbf{S}^{n-1}|_\alpha^k}{\beta^k2^{kn}}\int_1^\infty y_1^{\frac{n+\widetilde{A}_1-\frac{\varepsilon}{p_1}}{\beta}-1}\cdot y_1^{\frac{1}{\beta}\left[(k-1)n+\sum_{i=2}^k(\widetilde{A}_i-\frac{\varepsilon}{p_i})\right]-s}$$

$$\times\left(\int_{\frac{1}{y_1}}^\infty\cdots\int_{\frac{1}{y_1}}^\infty\frac{\prod_{i=2}^k u_i^{\frac{n+\widetilde{A}_i-\frac{\varepsilon}{p_i}}{\beta}-1}}{\left(1+\sum_{i=2}^k u_i\right)^s}du_2\dots du_k\right)dy_1$$

$$=\frac{|\mathbf{S}^{n-1}|_\alpha^k}{\beta^k2^{kn}}\int_1^\infty y_1^{-1-\frac{\varepsilon}{\beta}}\left(\int_{\frac{1}{y_1}}^\infty\cdots\int_{\frac{1}{y_1}}^\infty\frac{\prod_{i=2}^k u_i^{\frac{n+\widetilde{A}_i-\frac{\varepsilon}{p_i}}{\beta}-1}}{\left(1+\sum_{i=2}^k u_i\right)^s}du_2\dots du_k\right)dy_1.$$

Now, it is easy to see that the following inequality holds

$$J\geqslant\frac{|\mathbf{S}^{n-1}|_\alpha^k}{\beta^k2^{kn}}\int_1^\infty y_1^{-1-\frac{\varepsilon}{\beta}}\left(\int_0^\infty\cdots\int_0^\infty\frac{\prod_{i=2}^k u_i^{\frac{n+\widetilde{A}_i-\frac{\varepsilon}{p_i}}{\beta}-1}}{\left(1+\sum_{i=2}^k u_i\right)^s}du_2\dots du_k\right)dy_1$$

$$-\frac{|\mathbf{S}^{n-1}|_\alpha^k}{\beta^k2^{kn}}\int_1^\infty y_1^{-1-\frac{\varepsilon}{\beta}}\sum_{j=2}^k I_j(y_1)dy_1, \quad (31)$$

where for $j=2,\dots,k,$ $I_j(y_1),$ is defined by

$$I_j(y_1)=\int_{D_j}\frac{\prod_{i=2}^k u_i^{\frac{n+\widetilde{A}_i-\frac{\varepsilon}{p_i}}{\beta}-1}}{\left(1+\sum_{i=2}^k u_i\right)^s}du_2\dots du_k,$$

satisfying $D_j = \{(u_2,\dots,u_k); 0 < y_j < \frac{1}{y_1}, 0 < u_l < \infty, l \neq j\}$. Without losing generality, we only estimate the integral $I_2(y_1)$. For $k=2$ we have

$$I_2(y_1) = \int_0^{\frac{1}{y_1}} \frac{u_2^{\frac{n+\widetilde{A_2}-\frac{\varepsilon}{p_2}}{\beta}-1}}{(1+u_2)^s} du_2 \leqslant \int_0^{\frac{1}{y_1}} u_2^{\frac{n+\widetilde{A_2}-\frac{\varepsilon}{p_2}}{\beta}-1} du_2 = \beta\left(n+\widetilde{A_2}-\frac{\varepsilon}{p_2}\right)^{-1} y_1^{\frac{\frac{\varepsilon}{p_2}-n-\widetilde{A_2}}{\beta}},$$

and for $k>2$ we find

$$I_2(y_1) \leqslant \left(\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=3}^k u_i^{\frac{n+\widetilde{A_i}-\frac{\varepsilon}{p_i}}{\beta}-1}}{\left(1+\sum_{i=3}^k u_i\right)^s} du_3\dots du_k\right) \cdot \int_0^{\frac{1}{y_1}} u_2^{\frac{n+\widetilde{A_2}-\frac{\varepsilon}{p_2}}{\beta}-1} du_2$$

$$\leqslant \beta\left(n+\widetilde{A_2}-\frac{\varepsilon}{p_2}\right)^{-1} y_1^{\frac{\frac{\varepsilon}{p_2}-n-\widetilde{A_2}}{\beta}} \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=3}^k u_i^{\frac{n+\widetilde{A_i}-\frac{\varepsilon}{p_i}}{\beta}-1}}{\left(1+\sum_{i=3}^k u_i\right)^{s-\frac{1}{\beta}(n+\widetilde{A_1}+\varepsilon-\frac{\varepsilon}{p_1})}} \times du_3\dots du_k$$

$$= \beta\left(n+\widetilde{A_2}-\frac{\varepsilon}{p_2}\right)^{-1} y_1^{\frac{\frac{\varepsilon}{p_2}-n-\widetilde{A_2}}{\beta}} \frac{1}{\Gamma(s-\frac{1}{\beta}(n+\widetilde{A_1}+\varepsilon-\frac{\varepsilon}{p_1}))} \times \prod_{i=2}^k \Gamma\left(\frac{n+\widetilde{A_i}-\frac{\varepsilon}{p_i}}{\beta}+1\right),$$

where we used the well known formula for gamma function (see, for instance, [3], Lemma 5.1.):

$$\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{k-1} u_i^{a_i-1}}{(1+\sum_{i=1}^{k-1} u_i)^{\sum_{i=1}^k a_i}} du_1\dots du_{k-1} = \frac{\prod_{i=1}^k \Gamma(a_i)}{\Gamma(\sum_{i=1}^k a_i)}.$$

Hence, we have $I_j(y_1) \leqslant y_1^{\frac{\frac{\varepsilon}{p_j}-n-\widetilde{A_2}}{\beta}} O_j(1)$, for $\varepsilon \to 0^+$, $j \in \{2,\dots,k\}$, and consequently

$$\int_1^\infty y_1^{-1-\frac{\varepsilon}{\beta}} \sum_{j=2}^k I_j(y_1)dy_1 \leqslant O(1). \tag{32}$$

Now, from (29), (31) and (32) we get $K^* \leqslant K_1$. This contradiction follows that the constant K^* in(27) is the best possible.

Finally, equivalence of inequalities (27) and (28) means that the constant K^* is also the best possible in the inequality (28). That completes the proof. □

REMARK 4. In two-dimensional conjugate case, by using the notation from Remark 1, relation (22) reduces to $pa+qb = 2n-\beta s$, what is the condition for the best possible constant in paper [8]. So, Theorem 3 is the generalization of the appropriate result from [8]. Further, Theorem 3 also generalizes our result from [2].

5. Some examples

Our aim in this section is to apply the results from Theorem 1, to some special choices of real parameters $A_{ij}, \quad i,j=1,2,\dots,k$. In such a way, we shall obtain generalizations of the numerous versions of multiple Hilbert's and Hardy-Hilbert's inequality, previously known from the literature. Further, in the conjugate case we shall obtain the best possible constants in some cases.

At the beginning, let us define the real parameters A_{ij}, $i,j=1,2,\dots,k$, by $A_{ii}=(nk-s)\frac{\lambda q_i-1}{q_i^2}$ and $A_{ij}=(s-nk)\frac{1}{q_iq_j}$, $i\neq j$, $i,j=1,2,\dots,k$. Then we have

$$\sum_{i=1}^{k}A_{ij}=\sum_{i\neq j}\frac{s-nk}{q_iq_j}+(nk-s)\left(\frac{\lambda q_j-1}{q_j^{\,2}}\right)=\frac{s-nk}{q_j}\left(\sum_{i=1}^{k}\frac{1}{q_i}-\lambda\right)=0,$$

for $j=1,2,\dots,k$. Clearly, the parameters A_{ij} are symmetric and it follow that $\alpha_i=\sum_{j=1}^{n}A_{ij}=0$, for $i=1,2,\dots,k$. In such a way we obtain the following result:

COROLLARY 1. *Let $k\geqslant 2$ be an integer and p_i,p_i',q_i, $i=1,2,\dots,k$, be real numbers satisfying (3)-(7). Then, for any non-negative measurable functions f_i, $i=1,2,\dots,k$, the following inequalities hold and are equivalent*

$$\int_{(\mathbf{R}_+{}^n)^k}\frac{\prod_{i=1}^{k}f_i(\mathbf{x_i})}{\left(\sum_{i=1}^{k}|\mathbf{x_i}|_\alpha{}^\beta\right)^{\lambda s}}d\mathbf{x_1}d\mathbf{x_2}\cdots d\mathbf{x_k}\leqslant L\prod_{i=1}^{k}\left[\int_{\mathbf{R}_+{}^n}|\mathbf{x_i}|_\alpha{}^{\frac{p_i(k-1)n-p_i\beta s}{q_i}}f_i^{p_i}(\mathbf{x_i})d\mathbf{x_i}\right]^{\frac{1}{p_i}} \tag{33}$$

and

$$\left\{\int_{\mathbf{R}_+{}^n}|\mathbf{x_k}|_\alpha{}^{-\frac{p_k'}{q_k}[(k-1)n-\beta s]}\times\left[\int_{(\mathbf{R}_+{}^n)^{k-1}}\frac{\prod_{i=1}^{k-1}f_i(\mathbf{x_i})}{\left(\sum_{i=1}^{k}|\mathbf{x_i}|_\alpha{}^\beta\right)^{\lambda s}}d\mathbf{x_1}d\mathbf{x_2}\cdots d\mathbf{x_{k-1}}\right]^{p_k'}d\mathbf{x_k}\right\}^{\frac{1}{p_k'}}\leqslant L\prod_{i=1}^{k-1}\left[\int_{\mathbf{R}_+{}^n}|\mathbf{x_i}|_\alpha{}^{\frac{p_i(k-1)n-p_i\beta s}{q_i}}f_i^{p_i}(\mathbf{x_i})d\mathbf{x_i}\right]^{\frac{1}{p_i}}, \tag{34}$$

where $nk-\beta s<n\min\{p_i,q_j,i,j=1,2,\dots k\}$ and the constant L is defined by the formula

$$L=\frac{\Gamma^{\lambda(k-1)n}\left(\frac{1}{\alpha}\right)}{\beta^{\lambda(k-1)}\alpha^{\lambda(n-1)(k-1)}\Gamma^{\lambda(k-1)}\left(\frac{n}{\alpha}\right)\Gamma^{\lambda}(s)}\prod_{i=1}^{k}\Gamma^{\frac{1}{q_i}}\left(\frac{\beta s+n(p_i-k)}{\beta p_i}\right)\times\prod_{i=1}^{k}\Gamma^{\lambda-\frac{1}{q_i}}\left(\frac{\beta s+n(q_i-k)}{\beta q_i}\right).$$

The equality in both inequalities holds if and only if at least one of the functions f_i, $i = 1,2,\dots,k$, is equal to zero. Further, the constant L is the best possible in the conjugate case.

Proof. The corollary follows directly by putting the real parameters $A_{ii} = (nk - s)\frac{\lambda q_i - 1}{q_i^2}$ and $A_{ij} = (s - nk)\frac{1}{q_i q_j}$, $i \neq j$, $i,j = 1,2,\dots,k$, in inequalities (17) and (18). Straightforward computation shows that such parameters A_{ij}, in the conjugate case, satisfy equation (22). Hence, the constant L is the best possible. □

REMARK 5. By using the same notation as in Remark 1 and by defining $A_{11} = A_{22} = \frac{2n-s}{pq}$, $A_{12} = A_{21} = \frac{s-2n}{pq}$ in two-dimensional conjugate case, the constant L becomes $L = B\left(\frac{np+s-2n}{p}, \frac{nq+s-2n}{q}\right)$, so the inequalities (33) and (34) are generalizations of those from the paper [8]. Further, by putting $n = 1$ and $\beta = 1$ we obtain appropriate results from [2] and [4].

REMARK 6. Similarly as in the previous corollary, if we define the parameters A_{ij} by $A_{ii} = \frac{n(\lambda q_i - 1)}{\lambda q_i^2}$ and $A_{ij} = -\frac{n}{\lambda q_i q_j}$, $i \neq j$, $i,j \in \{1,2,\dots k\}$, then we have

$$\sum_{i=1}^{k} A_{ij} = \sum_{i\neq j} -\frac{n}{\lambda q_i q_j} + \frac{n(\lambda q_j - 1)}{\lambda {q_j}^2} = -\frac{n}{\lambda q_j}\left(\sum_{i=1}^{k}\frac{1}{q_i} - \lambda\right) = 0,$$

for $j = 1,2,\dots,k$. Since the parameters A_{ij} are symmetric, one obtains $\alpha_i = \sum_{j=1}^{n} A_{ij} = 0$, for $j = 1,2,\dots,k$. So, by putting these parameters in Theorem 1 we obtain the same inequalities as in the Corollary 1, with the constant L replaced with

$$L' = \frac{\Gamma^{\lambda(k-1)n}\left(\frac{1}{\alpha}\right)}{\beta^{\lambda(k-1)}\alpha^{\lambda(n-1)(k-1)}\Gamma^{\lambda(k-1)}\left(\frac{n}{\alpha}\right)\Gamma^{\lambda}(s)}\prod_{i=1}^{k}\Gamma^{\frac{1}{q_i}}\left(s - \frac{(k-1)n}{\beta} + \frac{n}{\lambda\beta p_i'}\right) \times \prod_{i=1}^{k}\Gamma^{\lambda-\frac{1}{q_i}}\left(\frac{n}{\lambda\beta p_i'}\right),$$

where $s + \frac{n}{\lambda\beta p_i'} > \frac{(k-1)n}{\beta}$. Similarly as before, if $\beta s = (k-1)n$, then the defined parameters satisfy constraint (22) in the conjugate case, the constant L' becomes

$$L'' = \frac{\Gamma^{(k-1)n}\left(\frac{1}{\alpha}\right)}{\beta^{(k-1)}\alpha^{(n-1)(k-1)}\Gamma^{(k-1)}\left(\frac{n}{\alpha}\right)\Gamma(s)}\prod_{i=1}^{k}\Gamma\left(\frac{n}{\beta p_i'}\right),$$

and that is the best possible constant.

On the other hand, if we define the parameters by $A_{ii} = A_i$, $A_{ii+1} = -A_{i+1}$, $A_{ij} = 0$, where $|i-j| > 1$ and the indices are taken modulo k, then we have

$$\sum_{i=1}^{k} A_{ij} = A_{j-1j} + A_{jj} = -A_j + A_j = 0,$$

so we obtain the following result:

COROLLARY 2. *Let $k \geqslant 2$ be an integer and p_i, p_i', q_i, $i = 1,2,\dots,k$, be real numbers satisfying (3)-(7). Then, for any non-negative measurable functions f_i, $i = 1,2,\dots,k$, the following inequalities hold and are equivalent*

$$\int_{(\mathbf{R}_+{}^n)^k} \frac{\prod_{i=1}^k f_i(\mathbf{x_i})}{\left(\sum_{i=1}^k |\mathbf{x_i}|_\alpha{}^\beta\right)^{\lambda s}} d\mathbf{x_1} d\mathbf{x_2}\cdots d\mathbf{x_k} \leqslant M \prod_{i=1}^k \left[\int_{\mathbf{R}_+{}^n} |\mathbf{x_i}|_\alpha{}^{\frac{p_i(k-1)n - p_i\beta s}{q_i} + p_i(A_i - A_{i+1})} f_i^{p_i}(\mathbf{x_i}) d\mathbf{x_i}\right]^{\frac{1}{p_i}} \tag{35}$$

and

$$\left\{\int_{\mathbf{R}_+{}^n} |\mathbf{x_k}|_\alpha{}^{-\frac{p_k{}'}{q_k}[(k-1)n-\beta s] - p_k{}'(A_k - A_1)} \times \left[\int_{(\mathbf{R}_+{}^n)^{k-1}} \frac{\prod_{i=1}^{k-1} f_i(\mathbf{x_i})}{\left(\sum_{i=1}^k |\mathbf{x_i}|_\alpha{}^\beta\right)^{\lambda s}} d\mathbf{x_1} d\mathbf{x_2}\cdots d\mathbf{x_{k-1}}\right]^{p_k{}'} d\mathbf{x_k}\right\}^{\frac{1}{p_k{}'}} \leqslant M \prod_{i=1}^{k-1} \left[\int_{(\mathbf{R}_+)^n} |\mathbf{x_i}|_\alpha{}^{\frac{p_i(k-1)n - p_i\beta s}{q_i} + p_i(A_i - A_{i+1})} f_i^{p_i}(\mathbf{x_i}) d\mathbf{x_i}\right]^{\frac{1}{p_i}}, \tag{36}$$

for any $A_i \in \left(\frac{(k-1)n-\beta s}{q_{i-1}}, \frac{n}{q_{i-1}}\right)$, with the constant M defined by the formula

$$M = \frac{\Gamma^{\lambda(k-1)n}\left(\frac{1}{\alpha}\right)\Gamma^{\lambda(k-2)}\left(\frac{n}{\beta}\right)}{\beta^{\lambda(k-1)}\alpha^{\lambda(n-1)(k-1)}\Gamma^{\lambda(k-1)}\left(\frac{n}{\alpha}\right)\Gamma^{\lambda}(s)} \prod_{i=1}^k \Gamma^{\frac{1}{q_i}}\left(\frac{\beta s + q_i A_{i+1} - (k-1)n}{\beta}\right) \times \prod_{i=1}^k \Gamma^{\frac{1}{q_i}}\left(\frac{n - q_i A_{i+1}}{\beta}\right).$$

The equality in both inequalities holds if and only if at least one of the functions f_i, $i = 1,2,\dots,k$, is equal to zero.

Clearly, by putting $n = 1$ and $\beta = 1$ in (35) and (36) we obtain appropriate inequalities from the papers [2] and [4].

REFERENCES

[1] F.F. BONSALL, *Inequalities with non-conjugate parameters*, Quart. Journ. Math. Oxford, **(2)**, **2** (1951), 135–150

[2] Y. BICHENG, I. BRNETIĆ, M. KRNIĆ AND J. PEČARIĆ, *Generalization of Hilbert and Hardy-Hilbert integral inequality*, Math. Inequal. Appl. **8(2)**, (2005), 259–272

[3] Y. BICHENG, T.M. RASSIAS, *On the way of Weight Coefficents and Research for the Hilbert-type inequalities*, Math. Inequal. Appl. **6(4)** (2003), 625–658

[4] I. BRNETIĆ, M. KRNIĆ, J. PEČARIĆ, *Multiple Hilbert and Hardy-Hilbert inequalities with non-conjugate parameters*, Bull. Austral. Math. Soc. **71** (2005), 447–457

[5] A. ČIŽMEŠIJA, I. PERIĆ, P. VUKOVIĆ, *Inequalities of the Hilbert type in* $\mathbf{R}^{\mathbf{n}}$ *with non-conjugate exponents*, Proc. Edinb. Math. Soc. **51** (2008), 11–26

[6] G.M. FIHTENGOLZ, *A Course in Differential and Integral Calculus*, Izdevateljstvo Nauka, Moskva, 1966.

[7] G.H. HARDY, J.E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge Univ. Press, Cambridge 1952.

[8] Y. HONG, *On multiple Hardy–Hilbert's integral inequalities with some parameters*, Jour. Ineq. & Appl, Vol. 2006, Art. ID 94960, (2006)

[9] M. KRNIĆ, J. PEČARIĆ, *General Hilbert's and Hardy's inequalities*, Math. Inequal. Appl. **8(1)** (2005)

[10] M. KRNIĆ, G. MINGZHE, J. PEČARIĆ, G. XUEMEI, *On the best constant in Hilbert's inequality*, Math. Inequal. Appl. **8(2)** (2005)

[11] V. LEVIN, *On the Two Parameter Extension and Analogue of Hilbert's Inequality*, J. London Math. Soc. **11** (1936), 119–124

[12] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and new inequalities in analysis,* Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

(Received July 8, 2007)

Mario Krnić
Faculty of Electrical Engineering and Computing
University of Zagreb
Unska 3
Zagreb
CROATIA
e-mail: mario.krnic@fer.hr

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6
10000 Zagreb
CROATIA
e-mail: pecaric@hazu.hr

Predrag Vuković
Faculty of Teacher Education
University of Zagreb
Ante Starčevića 55
40000 Čakovec
CROATIA
e-mail: predrag.vukovic@vus-ck.hr