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EXTENSIONS OF A BONNESEN–STYLE INEQUALITY TO MINKOWSKI SPACES

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Abstract. Various definitions of surface area and volume are possible in finite dimensional normed linear spaces (= Minkowski spaces). Using a Bonnesen-style inequality, we investigate the ratio of the Holmes-Thompson surface area of the unit ball to its volume. In particular, in the planar case a stronger lower bound for this ratio is established when the area is defined in the sense of Holmes-Thompson, or is given by the definition of mass. From this we obtain some (characteristic) properties of Radon curves.

1. Introduction

In their paper [1], J.C. Alvarez and C. Duran asked whether, besides the Euclidean plane, there are other Minkowski (i.e., normed) planes for which the ratio of the Minkowski length of the unit circle to its Holmes-Thompson area (see [12] and Chapter 5 of [28]) equals 2.

R.D. Holmes and A.C. Thompson investigated the ratio

$$\omega(B) = \frac{\epsilon_{d-1}}{d\epsilon_d} \frac{\mu_B^{HT}(\partial B)}{\mu_B^{HT}(B)},$$

where ϵ_d is the volume of the d-dimensional standard Euclidean unit ball, and $\mu_B^{HT}(\partial B)$ and $\mu_B^{HT}(B)$ stand for the Holmes-Thompson definitions of surface area and volume, respectively (cf. again [12] and [28], Chapter 5). They established certain bounds on ω. Namely, if B is the unit ball of a d-dimensional Minkowski (= normed linear) space, then

$$\frac{1}{2} \leqslant \omega(B) \leqslant \frac{d}{2}$$

with equality on the right if B is a cube or a cross-polytope. When $d=2$, we have

$$\frac{1}{2} \leqslant \omega(B) = \frac{1}{\pi} \frac{\mu(\partial B)}{\mu(B)} \leqslant 1.$$

Holmes and Thompson subsequently raised the following question: "What is the lower bound for $\omega(B)$ in d-dimensional space?"

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When $d = 2$, it has been proved that $\dfrac{\mu_B(\partial B)}{\mu_B^{HT}(B)} \geqslant 2$ with equality if and only if B is an ellipse, see [20]. In other words, the ratio 2 is achieved only for Minkowski planes that are affinely equivalent to the Euclidean plane.

One of the aims of this paper is to investigate and improve such inequalities by using a Bonnesen-style inequality. We will also show the importance of this ratio for higher dimensional Minkowski spaces. More precisely, we discuss the relationship between this ratio problem and Petty's conjectured projection inequality. Furthermore, we will investigate this ratio also for other definitions of area in Minkowski planes. In particular, we will prove a stronger lower bound for the ratio when the area is defined in the sense of mass.

We will also give new characterizations of Radon curves in terms of maximally inscribed and minimally circumscribed parallelograms of the unit disc.

2. Definitions and preliminaries

Recall that a *convex body* K is a compact, convex set with nonempty interior, and that K is said to be *centered* if it is symmetric with respect to the origin o of $\mathbb{R}^d$.

Let $(\mathbb{R}^d, \ ||\cdot||) =: \mathbb{M}^d$ be a d-dimensional real normed linear space, i.e., a *Minkowski space* with *unit ball* B which is a centered convex body. The *unit sphere* of $\mathbb{M}^d$ is the boundary of the unit ball and denoted by ∂B. By K° we denote the *polar reciprocal* of a convex body K, and so B° is written for the polar reciprocal (or dual) of the unit ball B.

Let λ be the *Lebesgue measure* induced by the standard Euclidean structure in $\mathbb{R}^d$. We will refer to this measure as volume (area in $\mathbb{R}^2$) and denote it by $\lambda(\cdot)$. The measure λ gives rise to consider a dual measure λ^* on the family of convex subsets of the dual space $\mathbb{R}^{d*}$.

However, using the standard basis we will identify $\mathbb{R}^d$ and $\mathbb{R}^{d*}$, and in that case λ and λ^* coincide in $\mathbb{R}^d$.

A Minkowski space $\mathbb{M}^d$ possesses a *Haar measure* μ, and this measure is unique up to multiplication of the Lebesgue measure with a positive constant, i.e.,

$$\mu = \sigma_B \lambda. \tag{1}$$

Choosing the 'correct' multiple, which can depend on orientation, is not as easy as it seems at first glance. Also these two measures μ and λ must coincide in the standard Euclidean space. The following notions of measure are well known.

DEFINITION 1. Let $\mathbb{M}^2$ be a Minkowski plane with unit disc B. For a convex body K in $\mathbb{M}^2$

i) the *Holmes-Thompson area* of K is defined by

$$\mu_B^{HT}(K) = \frac{\lambda(K)\lambda(B^\circ)}{\pi}$$

(for the Holmes Thompson volume $\sigma_B = \dfrac{\lambda(B^\circ)}{\epsilon_d}$ in $\mathbb{M}^d$, where, again, ϵ_d is the volume of the standard Euclidean ball), *ii*) the *Busemann area* of K is defined by

$$\mu_B^{Bus}(K) = \frac{\pi}{\lambda(B)}\lambda(K),$$

iii) the *Benson area* of K is defined by

$$\mu_B^{Ben}(K) = \frac{4}{\lambda(P)}\lambda(K),$$

where P is a parallelogram of minimal area circumscribed about B, and

iv) the *mass definition of area* of K is given by

$$\mu_B^{mass}(K) = \frac{2}{\lambda(C)}\lambda(K),$$

where C is a parallelogram of maximal area inscribed to B.

These definitions coincide with the standard notion of area if the plane under consideration is Euclidean.

By I_B we denote the polar reciprocal of B with respect to the Euclidean unit circle rotated through 90°. It turns out that I_B plays the central role regarding the solution of the *isoperimetric problem* in Minkowski planes. More precisely, among all convex bodies with area $\lambda(I_B)$ those with minimum Minkowski perimeter are the translates of I_B. Dually, the same applies to bodies of maximal area with given perimeter. For more details see [28], p. 119-121, and in general the geometry of unit balls and their duals is discussed and applied in [24], [28], [18], [17], and [19].

The curve ∂B is called a *Radon curve* if $B = \alpha I_B$ for some positive α. Centrally symmetric closed convex curves that are touched at each of their points by some circumscribed parallelogram of smallest area are called *equiframed curves*. The set of equiframed curves properly contains the set of Radon curves. See [7], [11], [13], [16], [17], [18], and [27] for various results and more background material on these two classes of curves.

Recall that if K is a convex body in $\mathbb{M}^d$, then the *Minkowski surface area* (*Minkowski length* in $\mathbb{M}^2$) of the boundary of K can also be defined in terms of *mixed volumes* $V(\cdot,\cdot)$, i.e.,

$$\mu_B(\partial K) = dV(K[d-1], I_B), \tag{2}$$

where I_B is (up to a constant) the solution of the isoperimetric problem for the given Haar measure. For definitions and many results related to mixed volumes we refer to Chapter 5 of [26]. In particular, one of the fundamental results on mixed volumes, *Minkowski's inequality*, states that

$$V^d(K[d-1], L) \geqslant \lambda^{d-1}(K)\lambda(L)$$

with equality if and only if K and L are homothetic.

3. The isoperimetrices

Let $\mathbb{M}^d$ be a Minkowski space with unit ball B. Assigning a Haar measure to $\mathbb{M}^d$, one can also define I_B for this measure. Among the homothetic images of I_B we want to specify a unique one, denoted by $\hat{I}_B$. It is called the *isoperimetrix* of $\mathbb{M}^d$ and determined by $\mu_B(\partial\hat{I}_B) = d\mu_B(\hat{I}_B)$.

PROPOSITION 2. *The equality $\mu_B(\partial\hat{I}_B) = d\mu_B(\hat{I}_B)$ holds if and only if $\hat{I}_B = \sigma_B^{-1}I_B$.*

Proof. Let $\mu_B(\partial\hat{I}_B) = d\mu_B(\hat{I}_B)$ and $\hat{I}_B = \alpha I_B$ for some positive α. Then we have $\mu_B(\partial(\alpha I_B)) = d\mu_B(\alpha I_B)$. Using (1), (2), and properties of mixed volumes, we get $\alpha = \sigma_B^{-1}$.

Let $\hat{I}_B = \sigma_B^{-1}I_B$. Then we have the sequence of equalities $\mu_B(\partial\hat{I}_B) = \sigma_B^{1-d}\mu_B(\partial I_B) = d\sigma_B^{1-d}\lambda(I_B) = d\sigma_B\lambda(\hat{I}_B) = d\mu_B(\hat{I}_B)$. □

We denote the isoperimetrices of $\mathbb{M}^2$ (as well as of $\mathbb{M}^d$) with respect to the definitions of Benson, Busemann, Holmes-Thompson and that referring to the notion of mass by $\hat{I}_B^{Ben}$, $\hat{I}_B^{Bus}$, $\hat{I}_B^{HT}$, and $\hat{I}_B^{mass}$, respectively.

Thus we have $\hat{I}_B^{HT} = \dfrac{\pi}{\lambda(B^\circ)}I_B$, $\hat{I}_B^{Bus} = \dfrac{\lambda(B)}{\pi}I_B$, $\hat{I}_B^{Ben} = \dfrac{\lambda(P)}{4}I_B$, and $\hat{I}_B^{mass} = \dfrac{\lambda(C)}{2}I_B$ in $\mathbb{M}^2$.

From the *Blaschke-Santaló inequality* (cf. Section 7.4 of [26]) we have

$$\lambda(\hat{I}^{Bus}) \leqslant \lambda(B) \leqslant \lambda(\hat{I}_B^{HT}) \tag{3}$$

with equality on either side if and only if B is an ellipse.

We can also show the following inclusion properties between different types of isoperimetrices in $\mathbb{M}^2$.

i. Since $\dfrac{\lambda(B)}{\pi} \leqslant \dfrac{\pi}{\lambda(B^\circ)}$ with equality if and only if B is an ellipse, we have $\hat{I}_B^{Bus} \subseteq \hat{I}_B^{HT}$.

ii. Since $\mu_B^{HT}(B) \leqslant \mu_B^{Ben}(B)$ with equality if and only if B is an ellipse (see [2]), we have $\dfrac{\lambda(P)}{4} \leqslant \dfrac{\pi}{\lambda(B^\circ)}$. This yields $\hat{I}_B^{Ben} \subseteq \hat{I}_B^{HT}$.

iii. Since $\dfrac{2}{\pi} \leqslant \dfrac{\lambda(C)}{\lambda(B)}$ with equality if and only if B is an ellipse (cf. [16]), we have $\hat{I}_B^{Bus} \subseteq I_B^{mass}$.

iv. Since $\lambda(P) \leqslant 2\lambda(C)$ with equality if and only if B is a Radon curve (see the next section), $\hat{I}_B^{Ben} \subseteq \hat{I}_B^{mass}$ is obtained.

There are no such inclusion properties between $\hat{I}_B^{mass}$ and $\hat{I}_B^{HT}$ as well as between $\hat{I}_B^{Bus}$ and $\hat{I}_B^{Ben}$.

4. The ratio problem in $\mathbb{M}^2$ and Radon curves

Let K and L be convex bodies in $\mathbb{R}^d$. Then the *relative inradius* $r(K,L)$ and the *relative circumradius* $R(K,L)$ of K with respect to L are defined by

$$r(K,L) := \sup\{\alpha : \exists x \in \mathbb{R}^d,\ \alpha L + x \subseteq K\}$$

and

$$R(K,L) := \inf\{\alpha : \exists x \in \mathbb{R}^d,\ \alpha L + x \supseteq K\},$$

respectively; see [26], p. 135, and [23]. It is interesting to choose L so that it is equal to $\hat{I}_B$. In that case sharp bounds on $r(B,\hat{I}_B)$ and $R(B,\hat{I}_B)$ for some isoperimetrices are known. Namely, it is known that $\frac{2\epsilon_{d-1}}{d\epsilon_d} \leqslant r(B,\hat{I}_B^{HT}) \leqslant 1$ with equality on the left if and only if B is a cube or cross-polytope, and on the right if and only if B is an ellipsoid, see [15] and [28]. In $\mathbb{M}^2$, for $R(B,\hat{I}_B^{HT})$ we have $R(B,\hat{I}_B^{HT}) \geqslant \frac{3}{\pi}$ with equality if and only if B is a regular hexagon (see [20] and [28]). In $\mathbb{M}^d$ we have $R(B,\hat{I}_B^{Bus}) \leqslant \frac{d\epsilon_d}{2\epsilon_{d-1}}$ with equality if and only if B is a parallelotope (see [15]).

The *Bonnesen-style inequality* in the Euclidean plane that we use here states that

$$\lambda(K) - 2tV(K,L) + t^2\lambda(L) \leqslant 0, \quad r(K,L) \leqslant t \leqslant R(K,L).$$

Equality holds at $t = r(K,L)$ if and only if $K = \{x : x + r(K,L)L \subseteq K\} + r(K,L)L$, and at $t = R(K,L)$ if and only if $K = L$ (see also [26], pp. 324-325, and [23]). This inequality was proved by T. Bonnesen for the case that L is the standard Euclidean disc, and a generalization was established by W. Blaschke (see again [26], p. 324). It can also be found in [9], and it yields a strengthened form of the relative isoperimetric inequality in $\mathbb{M}^2$. See also [10] and [21] for more about this Bonnesen-style inequality and its applications.

Setting $K = B$ and $L = \hat{I}_B$ in the Bonnesen-style inequality, we get

$$\lambda(B) - 2tV(B,\hat{I}_B) + t^2\lambda(\hat{I}_B) \leqslant 0.$$

This gives us

$$\lambda(B) - 2t\sigma_B^{-1}V(B,I_B) + t^2\lambda(\hat{I}_B) \leqslant 0,$$

and therefore

$$t\mu_B(\partial B) \geqslant \mu_B(B) + t^2\sigma_B\lambda(\hat{I}_B)$$

as well as

$$\frac{\mu_B(\partial B)}{\mu_B(B)} \geqslant \frac{1}{t} + t\frac{\lambda(\hat{I}_B)}{\lambda(B)}. \tag{4}$$

THEOREM 3. *If B is the unit disc of a Minkowski plane $\mathbb{M}^2$, then*

$$\frac{\mu_B(\partial B)}{\mu_B^{HT}(B)} \geqslant t + \frac{1}{t}, \quad r(B,\hat{I}_B^{HT}) \leqslant t \leqslant R(B,\hat{I}_B^{HT}),$$

with equality if and only if B is an ellipse.

Proof. The result follows from (3) and (4). The equality case holds when $t = r(B,\hat{I}_B^{HT})$ and $\lambda(B) = \lambda(\hat{I}_B^{HT})$, and this holds if and only if B is an ellipse. □

Thus, $\mu_B(\partial B) = 2\mu_B^{HT}(B)$ with equality if and only if B is an ellipse.

In [2] (see also [16] and [3]) it was proved that if B is the unit disc of $\mathbb{M}^2$, then

$$2\mu_B^{mass}(B) \leqslant \mu_B(\partial B) \leqslant 2\mu_B^{Ben}(B), \tag{5}$$

with equality on the left if and only if ∂B is a Radon curve and on the right if and only if ∂B is an equiframed curve.

From (5) one can easily deduce that if P is a parallelogram of minimal area circumscribed about B, and C is a parallelogram of maximal area inscribed to B, then

$$2\lambda(C) \geqslant \lambda(P) \tag{6}$$

with equality if and only if ∂B is a Radon curve.

LEMMA 4. *If B is the unit disc of the normed plane $\mathbb{M}^2$, then*

$$\mu_B^2(\partial B) = 4\lambda(B)\lambda(B^\circ)$$

if and only if ∂B is a Radon curve.

Proof. If ∂B is a Radon curve (i.e., if $B = \alpha I_B$), then

$$\mu_B^2(\partial B) = 4V^2(\alpha I_B, I_B) = 4\alpha^2\lambda^2(I_B) = 4\lambda(B)\lambda(B^\circ).$$

If $\mu_B^2(\partial B) = 4\lambda(B)\lambda(B^\circ)$, then $V^2(B, I_B) = \lambda(B)\lambda(I_B)$. Thus, Minkowski's inequality implies that B and I_B are homothetic. □

THEOREM 5. *If B is the unit disc of a Minkowski plane $\mathbb{M}^2$ and P is a parallelogram of minimal area circumscribed about B, then*

$$\lambda^2(P) \leqslant \frac{16\lambda(B)}{\lambda(B^\circ)},$$

with equality if and only if B is a Radon curve.

Proof. Since $\mu_B(\partial B) \leqslant 2\mu_B^{Ben}(B)$, it follows from (4) that $\lambda(\hat{I}_B^{Ben}) \leqslant \lambda(B)$. Hence the result follows from the definition of $\hat{I}_B^{Ben}$. The equality case follows from Lemma 4 and $\mu_B(\partial B) = 2\mu_B^{Ben}(B)$; it holds if and only if B is an equiframed curve. □

COROLLARY 6. *If B is a Radon curve (i.e., $B = \alpha I_B$) and P is a parallelogram of minimal area circumscribed about B, then $\lambda(P) = 4\alpha$.*

THEOREM 7. *If B is the unit disc of a normed plane $\mathbb{M}^2$ and C is a parallelogram of maximal area inscribed to B, then*

$$\lambda^2(C) \geqslant \frac{4\lambda(B)}{\lambda(B^\circ)},$$

with equality if and only if B is a Radon curve.

Proof. Let P_1 be a parallelogram of minimal area circumscribed about B°. Then Theorem 5 implies

$$\lambda^2(P_1) \leqslant \frac{16\lambda(B^\circ)}{\lambda(B)}. \tag{7}$$

Since $C \subseteq B$ and $B^\circ \subseteq P_1$, we have $B^\circ \subseteq C^\circ$ and $P_1^\circ \subseteq B$. Therefore also $\lambda(C^\circ) \geqslant \lambda(P_1)$ and $\lambda(C) \geqslant \lambda(P_1^\circ)$. The *Mahler-Reisner inequality* (see, e.g., [22]) implies $\lambda(C)\lambda(C^\circ) = \lambda(P_1)\lambda(P_1^\circ) = 8$. Therefore $\lambda(C^\circ) = \lambda(P_1)$ and $\lambda(C) = \lambda(P_1^\circ)$. Hence the result follows from $\lambda(C)\lambda(P_1) = 8$ and (7).

The equality case follows from Lemma 4 and the fact that $\mu_B(\partial B) = 2\mu_B^{mass}(B)$ if and only if B is a Radon curve. □

COROLLARY 8. *If B is the unit disc of a Minkowski plane $\mathbb{M}^2$, then*

$$\frac{\mu_B(\partial B)}{\mu_B^{mass}(B)} \geqslant t + \frac{1}{t}, \quad r(B, \hat{I}_B^{mass}) \leqslant t \leqslant R(B, \hat{I}_B^{mass}),$$

with equality if and only if B is a Radon curve.

Proof. Since $\lambda(\hat{I}_B^{mass}) \geqslant \lambda(B)$ with equality if and only if B is a Radon curve, the result follows from (4). □

COROLLARY 9. *If B is a Radon curve (i.e., $B = \alpha I_B$) and C is a parallelogram of maximal area inscribed to B, then $\lambda(C) = 2\alpha$.*

Combining Theorems 5 and 7, we get

$$\lambda(\hat{I}_B^{Ben}) \leqslant \lambda(B) \leqslant \lambda(\hat{I}_B^{mass})$$

with equality on either side if and only if B is a Radon curve.

5. The ratio problem in $\mathbb{M}^d$

Petty's conjectured projection inequality states that if K is a convex body in $\mathbb{R}^d$, then

$$\epsilon_d^{-2}\lambda(\Pi K)\lambda^{1-d}(K) \geqslant (\frac{\epsilon_{d-1}}{\epsilon_d})^d$$

with equality if and only if K is an ellipsoid. (For the definition of the *projection body* ΠK of K and this famous inequality see [26], p. 296 and Section 7.4.) In [14], E. Lutwak describes this inequality as one of the major open problems in the area of affine isoperimetric inequalities. E.g., R. Schneider [25] shows applications of Petty's conjectured projection inequality in Stochastic Geometry, and N. S. Brannen [5] proved that this inequality holds for 3-dimensional convex cylindrical bodies.

It turns out (see [28]) that if B is the unit ball of $\mathbb{M}^d$ and I_B^{HT} the solution of the isoperimetric problem for the Holmes-Thompson definition of measure, then

$$I_B^{HT} = \frac{\Pi B^\circ}{\epsilon_{d-1}},$$

where ΠB° is the projection body of B°. Thus $\hat{I}_B^{HT} = \frac{\epsilon_d}{\lambda(B^\circ)} I_B^{HT}$. Setting $K = B^\circ$ in Petty's conjectured projection inequality, we obtain

$$\epsilon_d^{d-2}\lambda(I_B^{HT}) \geqslant \lambda^{d-1}(B^\circ).$$

Using this inequality and the Blaschke-Santaló inequality, we get

$$(\mu_B^{HT}(B))^d \leqslant \frac{\lambda^{d-1}(B)\lambda^{d-1}(B^\circ)}{\epsilon_d^{d-2}} \leqslant \lambda^{d-1}(B)\lambda(I_B^{HT}).$$

It follows from Minkowski's inequality that

$$\mu_B^{HT}(\partial B) \geqslant d\mu_B^{HT}(B)$$

with equality if and only if B is an ellipsoid.

Thus, if there exists the unit ball B of $\mathbb{M}^d$ such that $\mu_B^{HT}(\partial B) < d\mu_B^{HT}(B)$, then this body will contradict Petty's conjectured projection inequality.

There exist some extensions of the Bonnesen-style inequality to d-dimensional space (see [23] and [6]), such as

$$dr(K,L)V(K[d-1],L) \geqslant \lambda(K), \tag{8}$$

again using mixed volumes. Unfortunately, there are no stronger inequalities over the class of centrally symmetric bodies to prove this ratio problem.

Setting $K = B$ and $L = \hat{I}_B^{HT}$ in (8), we get

$$r(B,\hat{I}_B^{HT})\mu_B^{HT}(\partial B) \geqslant \mu_B^{HT}(B).$$

From a property of mixed volumes we also have

$$\lambda(B) = V(B[d-1],B) \geqslant r(B,\hat{I}_B)V(B[d-1],\hat{I}_B).$$

For $\hat{I}_B = \hat{I}_B^{HT}$ this yields

$$d\mu_B^{HT}(B) \geqslant r(B,\hat{I}_B)\mu_B^{HT}(\partial B).$$

PROPOSITION 10. *Let B be the unit ball of a normed linear space $\mathbb{M}^d$. Then*

$$\mu_B^{HT}(\partial B) = d\mu_B^{HT}(B)$$

holds if and only if B is an ellipsoid.

Proof. Obviously, if B is an ellipsoid, then equality holds.

Assume $\mu_B^{HT}(\partial B) = d\mu_B^{HT}(B)$, that is, $\lambda(B) = V(B[d-1],\hat{I}_B^{HT})$. The Favard Theorem (see [8] or [23]) states that $\lambda(K) = V(K[d-1],L)$ holds if and only if K is a $(d-1)$-*tangent body of* L. Recall that a convex body K is a $(d-1)$-tangent body of L if and only if through each boundary point of K there exists a supporting hyperplane of K that also supports L (see [4], p. 19, or [26], p. 75-76 and p. 136, for the definition of tangent bodies). Thus, B is a $(d-1)$-tangent body of $\hat{I}_B^{HT}$. This means that $\hat{I}_B^{HT} \subseteq B$, and it holds if and only if B is an ellipsoid. □

We ask the following question, the affirmative answer of which would solve this ratio problem easily.

PROBLEM. Let B be a centered convex body in $\mathbb{R}^d$. Is it then true that

$$V(B[d-1],\Pi B^\circ) \geqslant (\frac{\epsilon_{d-1}}{\epsilon_d})\lambda(B)\lambda(B^\circ)?$$

Obviously, if $B \subseteq \hat{I}_B^{HT}$, then this is true. By Proposition 10 we see that equality holds if and only if B is an ellipsoid.

We also give the following partial answer to this ratio problem.

PROPOSITION 11. *If B is the unit ball of a Minkowski space $\mathbb{M}^d$ such that $\mu_B^{HT}(\partial B) \geqslant d\epsilon_d$, then*

$$\frac{\mu_B^{HT}(\partial B)}{\mu_B^{HT}(B)} \geqslant d.$$

Proof. Since for any positive real number α the equalities $(\alpha B)^\circ = \alpha^{-1}B^\circ$ and $I_{\alpha B}^{HT} = \alpha^{1-d}I_B^{HT}$ hold, the quantities $\mu_B^{HT}(\partial B)$ and $\mu_B^{HT}(B)$ are unchanged by dilation. Therefore we may assume that $\lambda(B^\circ) = \epsilon_d$. From the Blaschke-Santaló inequality we get $\lambda(B) \leqslant \epsilon_d$. Hence the result follows. □

As we have seen, Petty's conjectured projection inequality would completely solve this problem. We show that this conjecture is equivalent to another open problem (Minkowski's isoperimetric problem) over the class of centered convex bodies.

THEOREM 12. *Let B be the unit ball of $\mathbb{M}^d$. Then Petty's conjectured projection inequality is true for all centered convex bodies if and only if*

$$\frac{\mu_B^d(\partial I_B^{HT})}{\mu_B^{d-1}(I_B^{HT})} \geqslant d^d\epsilon_d.$$

Proof. Assume that the conjecture is valid for all centered convex bodies in $\mathbb{R}^d$. Then, setting $K = B^\circ$ in the conjecture, we get

$$\lambda(\hat{I}_B^{HT})\lambda(B^\circ) \geqslant \epsilon_d^2. \tag{9}$$

Therefore

$$\frac{\mu_B^d(\partial I_B^{HT})}{\mu_B^{d-1}(I_B^{HT})} = \frac{\mu_B^d(\partial \hat{I}_B^{HT})}{\mu_B^{d-1}(\hat{I}_B^{HT})} = d^d\mu_B(\hat{I}_B^{HT}) \geqslant d^d\epsilon_d.$$

Conversely, assume that $\frac{\mu_B^d(\partial I_B^{HT})}{\mu_B^{d-1}(I_B^{HT})} \geqslant d^d\epsilon_d$. Since $I_{\alpha B}^{HT} = \alpha^{1-d}I_B^{HT}$ for all positive reals α, the quantity $\frac{\mu_B^d(\partial I_B^{HT})}{\mu_B^{d-1}(I_B^{HT})}$ is unchanged by dilation. Therefore we may assume that $\lambda(B^\circ) = \epsilon_d$. Then we get

$$\frac{\mu_B^d(\partial I_B^{HT})}{\mu_B^{d-1}(I_B^{HT})} = d^d\lambda(I_B^{HT}) \geqslant d^d\epsilon_d.$$

This is (9), since $I_B^{HT} = \hat{I}_B^{HT}$. □

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