ON THE HYERS–ULAM STABILITY OF
APPROXIMATELY PEXIDER MAPPINGS

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Abstract. In this paper we investigate the Hyers-Ulam stability of the Pexider functional equation

$$f_1(x+y) + f_2(x+\sigma(y)) = f_3(x) + f_4(y), \quad x, y \in E,$$

where E is a normed space and $\sigma \colon E \longrightarrow E$ is an involution.

1. Introduction

The stability problem for functional equations was first raised by S. M. Ulam (ref. [24]): Let G_1 be a group and let G_2 be a metric group with a metric $d(.,.)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $f \colon G_1 \longrightarrow G_2$ satisfies the functional inequality $d(f(xy), f(x)f(y)) \leqslant \delta$ for all x and y in G, then there exists a group homomorphism $h \colon G_1 \longrightarrow G_2$ with $d(f(x), h(x)) \leqslant \varepsilon$ for any $x \in G_1$?

The problem for the case of approximately additive mappings was solved by D. H. Hyers [8] when G_1 and G_2 are Banach spaces and the result of Hyers was generalized by Th. M. Rassias (see [17]). Since then, the stability problems of functional equations have been extensively investigated by several mathematicians (cf. [11], [16], [18], [19], [20], [6], and [13]). For the definition of terminologies such as "Hyers-Ulam stability", "Hyers-Ulam-Rassias stability" one is refered to [5], [9], [10], [12] or [21].

According to Corollary 8 in [23], a mapping $f \colon E_1 \longrightarrow E_2$ between vector spaces is a solution of the quadratic equation

$$f(x+y) + f(x + \sigma(y)) = 2f(x) + 2f(y), \qquad (1.1)$$

where σ is an involution of E_1 (i.e. $\sigma(x+y) = \sigma(x) + \sigma(y)$ and $\sigma(\sigma(x)) = x$, for all $x, y \in E_1$) if and only if there exists an additive function $A \colon E_1 \longrightarrow E_2$ and a biadditive symmetric mapping $B \colon E_1 \times E_1 \longrightarrow E_2$ such that $A(\sigma(x)) = A(x)$, $B(\sigma(x), y) = -B(x, y)$ and $f(x) = B(x, x) + A(x)$ for any $x \in E_1$.

The quadratic equation with $\sigma = -I$ has been much studied (cf. Aczél [1]).

A Hyers-Ulam stability theorem for the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad x, y \in E_1 \qquad (1.2)$$

was proved by F. Skof [22] for functions $f : E_1 \longrightarrow E_2$, where E_1 is a normed space and E_2 is a Banach space. P. W. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an abelian group.

In the paper [4], S. Czerwik proved the Hyers-Ulam-Rassias stability of the functional equation (1.2). Recently S. M. Jung [13], S. M. Jung and P. K. Sahoo [14] investigated the Hyers-Ulam-Rassias stability of equation (1.2). Furthermore they proved the Hyers-Ulam-Rassias stability of the functional equation of Pexider type

$$f_1(x+y) + f_2(x-y) = f_3(x) + f_4(y) \tag{1.3}$$

In a previous paper [2], the authors dealt with the Hyers-Ulam stability of the functional equation (1.1) in the case where the control function $\varphi(x, y) = \delta$, for some $\delta > 0$ and $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for $p < 1$ or $p > 2$, (cf. [2]).

In Section 2, the stability problem in the spirit of Găvruta shall be proved i.e. that the mapping φ satisfies the condition (2.1) or (2.2) (see below).

In Section 3, we will investigate the stability of the functional equation

$$f_1(x+y) + f_2(x + \sigma(y)) = f_3(x) + f_4(y). \tag{1.4}$$

The result of this section can be compared with the ones of the papers [13], [14] because we formulate them in the same way by using some ideas from [13] and [14]. It is intriguing to see that several of the methods of [13] and [14] carry over to the more general situations (1.4). However our discussion comprise not only $\sigma = -I$ but also examples like the reflection in hyperplane of $\mathbb{R}^n$ and the symplectic involution of the additive group of 2×2 matrices. In our solutions we take in the account certain types of functions that are not considered in [13] and [14] because they are constants for $\sigma = -I$. See for example the formulas (3.4) and (3.5). Furthermore, our estimates improves the ones obtained in [13] and [14].

Throughout this paper, let $(E_1, \| \ . \ \|)$ be a real normed space and $(E_2, \| \ . \ \|)$ a Banach space.

2. Stability of the quadratic equation

Let us denote by $\varphi : E_1 \times E_1 \longrightarrow [0, \infty)$ a mapping such that

$$\phi(x, y) = \sum_{n=0}^{\infty} 2^{-2(n+1)} [\varphi(2^n x, 2^n y) + (2^n - 1)\varphi(2^{n-1}x + 2^{n-1}\sigma(x), 2^{n-1}y + 2^{n-1}\sigma(y))] < \infty \tag{2.1}$$

for all $x, y \in E_1$, or

$$\psi(x, y) = \sum_{n=0}^{\infty} 2^{2(n)} [\varphi(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}) + (1 - \frac{1}{2^{n+1}})\varphi(\frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}}, \frac{y}{2^{n+2}} + \frac{\sigma(y)}{2^{n+2}})] < \infty \tag{2.2}$$

for all $x, y \in E_1$.

In the following theorem, we will investigate the Hyers-Ulam stability of the functional equation (1.1) when φ satisfies the condition (2.1).

THEOREM 2.1. *Let* $\varphi : E_1 \times E_1 \longrightarrow [0, \infty)$ *be a mapping satisfying the condition (2.1) and let* $f : E_1 \longrightarrow E_2$ *satisfies*

$$\| f(x + y) + f(x + \sigma(y)) - 2f(x) - 2f(y) \| \leqslant \varphi(x, y) \tag{2.3}$$

for all $x, y \in E_1$. *Then, there exists a unique solution* $q : E_1 \longrightarrow E_2$ *of equation (1.1) satisfying*

$$\| f(x) - q(x) \| \leqslant \phi(x, x) \tag{2.4}$$

for all $x \in E_1$.

Proof. If we set $y = x$, in the inequality (2.3), we get

$$\| f(2x) + f(x + \sigma(x)) - 2^2 f(x) \| \leqslant \varphi(x, x). \tag{2.5}$$

If we set $x = y = x + \sigma(x)$ in (2.3) then we obtain

$$\| 2f(2x + 2\sigma(x)) - 4f(x + \sigma(x)) \| \leqslant \varphi(x + \sigma(x), x + \sigma(x)). \tag{2.6}$$

By induction on n, we will prove the following inequality

$$\| f(x) - 2^{-2n}\{f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))\} \| \tag{2.7}$$

$$\leqslant \sum_{i=0}^{n-1} 2^{-2(i+1)}[\varphi(2^i x, 2^i x) + (2^i - 1)\varphi(2^{i-1}x + 2^{i-1}\sigma(x), 2^{i-1}x + 2^{i-1}\sigma(x)].$$

If we replace x, y in (2.3) by $2x$, then we get

$$\| f(4x) + f(2x + 2\sigma(x)) - 4f(2x) \| \leqslant \varphi(2x, 2x). \tag{2.8}$$

Now, in view of (2.5), (2.6) and (2.8), we can easily show that

$$\left\| f(x) - \frac{1}{16}\{f(4x) + 3f(2x + 2\sigma(x))\} \right\|$$

$$\leqslant \frac{1}{16} \| f(4x) + f(2x + 2\sigma(x)) - 4f(2x) \|$$

$$+ \frac{1}{16} \| 2f(2x + 2\sigma(x)) - 4f(x + \sigma(x)) \|$$

$$+ \frac{1}{16} \| 4f(2x) + 4f(x + \sigma(x)) - 16f(x) \|$$

$$\leqslant \frac{\varphi(x, x)}{2^{2(1)}} + \frac{\varphi(2x, 2x)}{2^{2(2)}} + \frac{2-1}{2^{2(2)}}\varphi(x + \sigma(x), x + \sigma(x)),$$

which proves that the assertion (2.7) is true for $n = 2$. Now, we assume that the

assertion (2.7) is true for some n. By using (2.7), (2.5) and (2.6), we obtain

$$
\begin{aligned}
\| f(x) &- 2^{-2(n+1)}\{f(2^{n+1}x) + (2^{n+1} - 1)f(2^n x + 2^n \sigma(x))\} \| \\
&\leqslant \| f(x) - 2^{-2n}\{f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))\} \| \\
&\quad + \frac{2^n - 1}{2^{2(n+1)}} \| 2f(2^n x + 2^n \sigma(x)) - 4f(2^{n-1}x + 2^{n-1}\sigma(x)) \| \\
&\quad + \frac{1}{2^{2(n+1)}} \| f(2.2^n x) + f(2^n x + 2^n \sigma(x)) - 4f(2^n x) \| \\
&\leqslant \sum_{i=0}^{n} 2^{-2(i+1)}[\varphi(2^i x, 2^i x) + (2^i - 1)\varphi(2^{i-1}x + 2^{i-1}\sigma(x), 2^{i-1}x + 2^{i-1}\sigma(x)],
\end{aligned}
$$

which implies that the assertion (2.7) holds for $n + 1$.

First, we show that $q_n(x) = 2^{-2n}\{f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))\}$ is a Cauchy sequence. Let n be an integer, then by (2.5) and (2.6), we see that

$$
\begin{aligned}
\| q_{n+1}&(x) - q_n(x) \| \\
&= \frac{1}{2^{2(n+1)}} \| f(2^{n+1}x) + (2^{n+1} - 1)f(2^n x + 2^n \sigma(x)) \\
&\qquad\qquad - 4[f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))] \| \\
&\leqslant \frac{1}{2^{2(n+1)}} \| f(2.2^n x) + f(2^n x + 2^n \sigma(x)) - 4f(2^n x) \| \\
&\quad + \frac{(2^n - 1)}{2^{2(n+1)}} \| 2f(2^n x + 2^n \sigma(x)) - 4f(2^{n-1}x + 2^{n-1}\sigma(x)) \| \\
&\leqslant \frac{\varphi(2^n x, 2^n x)}{2^{2(n+1)}} + \frac{(2^n - 1)}{2^{2(n+1)}}\varphi(2^{n-1}x + 2^{n-1}\sigma(x), 2^{n-1}x + 2^{n-1}\sigma(x)).
\end{aligned}
$$

Hence, for $n > m$ we get

$$
\| q_n(x) - q_m(x) \| \leqslant \sum_{i=m}^{n-1} \| q_{i+1}(x) - q_i(x) \| \tag{2.9}
$$

$$
\leqslant \sum_{i=m}^{n-1} 2^{-2(i+1)}[\varphi(2^i x, 2^i x) + (2^i - 1)\varphi(2^{i-1}x + 2^{i-1}\sigma(x), 2^{i-1}x + 2^{i-1}\sigma(x))]
$$

for $n > m$. From (2.1) and (2.9), we obtain that the sequence $q_n(x)$ is a Cauchy sequence. Because E_2 is a Banach space, the sequence $q_n(x)$ converges. Denote

$$
q(x) = \lim_{n \to +\infty} 2^{-2n}\{f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))\} \tag{2.10}
$$

for all $x \in E_1$. By the definition of q and the assertion (2.7), we can easily verify the validity of the inequality (2.4).

Now, we will show that q is a solution of equation (1.1). By using the definition of $q_n(x)$ and the inequality (2.3), we get

$$
\begin{aligned}
& \| q_n(x+y) + q_n(x+\sigma(y)) - 2q_n(x) - 2q_n(y) \| \\
& \quad \leqslant \frac{1}{2^{2n}} \| f(2^n x + 2^n y) + f(2^n x + 2^n \sigma(y)) - 2f(2^n x) - 2f(2^n y) \| \\
& \qquad + \frac{2^n - 1}{2^{2n}} \| f(2^{n-1}x + 2^{n-1}y + 2^{n-1}\sigma(x) + 2^{n-1}\sigma(y)) \\
& \qquad + f(2^{n-1}x + 2^{n-1}y + 2^{n-1}\sigma(x) + 2^{n-1}\sigma(y)) \\
& \qquad - 2f(2^{n-1}x + 2^{n-1}\sigma(x)) - 2f(2^{n-1}y + 2^{n-1}\sigma(y)) \| \\
& \quad \leqslant \frac{\varphi(2^n x, 2^n y)}{2^{2n}} + \frac{2^n - 1}{2^{2n}} \varphi(2^{n-1}x + 2^{n-1}\sigma(x), 2^{n-1}y + 2^{n-1}\sigma(y)) \to 0 \text{ as } n \to \infty,
\end{aligned}
$$

which implies that q is a solution of (1.1).

Finally, let $q' : E_1 \longrightarrow E_2$ be another solution of equation (1.1) which satisfies the inequality (2.4). Since q and q' are solutions of equation (1.1), we can easily show that

$$
q(2^n x) + (2^n - 1)q(2^{n-1}x + 2^{n-1}\sigma(x)) = 2^{2n}q(x) \tag{2.11}
$$

and

$$
q'(2^n x) + (2^n - 1)q'(2^{n-1}x + 2^{n-1}\sigma(x)) = 2^{2n}q'(x) \tag{2.12}
$$

for any $n \in \mathbb{N}$ (cf. [2]). Thus, it follows from (2.11), (2.12) and (2.4) that

$$
\begin{aligned}
\| q(x) - q'(x) \| &= \frac{1}{2^{2n}} \| q(2^n x) - q'(2^n x) + (2^n - 1) \\
& \qquad \times [q(2^{n-1}x + 2^{n-1}\sigma(x)) - q'(2^{n-1}x + 2^{n-1}\sigma(x))] \| \\
& \leqslant \frac{1}{2^{2n}} \| p(2^n x) - f(2^n x) \| + \| q'(2^n x) - f(2^n x) \| \\
& \quad + \frac{2^n - 1}{2^{2n}} \| q(2^{n-1}x + 2^{n-1}\sigma(x)) - f(2^{n-1}x + 2^{n-1}\sigma(x)) \| \\
& \quad + \frac{2^n - 1}{2^{2n}} \| q'(2^{n-1}x + 2^{n-1}\sigma(x)) - f(2^{n-1}x + 2^{n-1}\sigma(x)) \| \\
& \leqslant \frac{2\phi(2^n x, 2^n x)}{2^{2n}} + \frac{2\phi(2^{n-1}x + 2^{n-1}\sigma(x), 2^{n-1}x + 2^{n-1}\sigma(x))}{2^n} \\
& \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Consequently, we obtain $q(x) = q'(x)$, for all $x \in E_1$. This completes the proof of theorem. $\square$

In the following theorem, we will investigate the stability problem of the functional equation (1.1) when φ satisfies condition (2.2).

THEOREM 2.2. *Let* $\varphi : E_1 \times E_1 \longrightarrow [0, \infty)$ *be a mapping satisfying the condition (2.2). If a function* $f : E_1 \longrightarrow E_2$ *satisfies the inequality*

$$\| f(x+y) + f(x+\sigma(y)) - 2f(x) - 2f(y) \| \leqslant \varphi(x,y) \qquad (2.13)$$

for all $x,y \in E_1$*, then, there exists a unique solution* $q : E_1 \longrightarrow E_2$ *of equation (1.1) satisfying*

$$\| f(x) - q(x) \| \leqslant \psi(x,x) \qquad (2.14)$$

for all $x \in E_1$*.*

Proof. First we will use the induction on n to prove the following statement

$$\left\| f(x) - 2^{2n}\{f(\tfrac{x}{2^n}) + (\tfrac{1}{2^n} - 1)f(\tfrac{x}{2^{n+1}} + \tfrac{\sigma(x)}{2^{n+1}})\} \right\| \qquad (2.15)$$

$$\leqslant \sum_{i=0}^{n-1} 2^{2(i)} \left[\varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right) + \left(1 - \frac{1}{2^{i+1}}\right) \varphi\left(\frac{x}{2^{i+2}} + \frac{\sigma(x)}{2^{i+2}}, \frac{x}{2^{i+2}} + \frac{\sigma(x)}{2^{i+2}}\right) \right].$$

By replacing x and y by $\frac{x}{2}$, respectively $\frac{x}{4} + \frac{\sigma(x)}{4}$, we immediately get

$$\left\| f(x) + f\left(\frac{x}{2} + \frac{\sigma(x)}{2}\right) - 4f\left(\frac{x}{2}\right) \right\| \leqslant \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \qquad (2.16)$$

and

$$\left\| 2f\left(\frac{x}{4} + \frac{\sigma(x)}{4}\right) - f\left(\frac{x}{2} + \frac{\sigma(x)}{2}\right) \right\| \leqslant \frac{1}{2}\varphi\left(\frac{x}{4} + \frac{\sigma(x)}{4}, \frac{x}{4} + \frac{\sigma(x)}{4}\right). \qquad (2.17)$$

Hence in view of (2.16) and (2.17), we can easily deduce that

$$\left\| f(x) - 4\left[f\left(\frac{x}{2}\right) + \left(\frac{1}{2} - 1\right)f\left(\frac{x}{4} + \frac{\sigma(x)}{4}\right)\right] \right\|$$

$$\leqslant \left\| f(x) + f\left(\frac{x}{2} + \frac{\sigma(x)}{2}\right) - 4f\left(\frac{x}{2}\right) \right\| + \left\| 2f\left(\frac{x}{4} + \frac{\sigma(x)}{4}\right) - f\left(\frac{x}{2} + \frac{\sigma(x)}{2}\right) \right\|$$

$$\leqslant \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \left(1 - \frac{1}{2}\right)\varphi\left(\frac{x}{4} + \frac{\sigma(x)}{4}, \frac{x}{4} + \frac{\sigma(x)}{4}\right),$$

which proves that the assertion (2.15) is true for $n = 1$. Assuming that inequality (2.15) is true for all integers i, $1 \leqslant i \leqslant n$, it follows from (2.15) that

$$\left\| f(x) - 2^{2(n+1)}\left\{f\left(\frac{x}{2^{n+1}}\right) + \left(\frac{1}{2^{n+1}} - 1\right)f\left(\frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}}\right)\right\} \right\|$$

$$\leqslant \left\| f(x) - 2^{2n}\left\{f\left(\frac{x}{2^n}\right) + \left(\frac{1}{2^n} - 1\right)f\left(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}\right)\right\} \right\|$$

$$+ 2^{2n}\left\| f\left(\frac{x}{2^n}\right) + \left(\frac{1}{2^n} - 1\right)f\left(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}\right) \right.$$

$$\left. -4f\left(\frac{x}{2^{n+1}}\right) - 4\left(\frac{1}{2^{n+1}} - 1\right)f\left(\frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}}\right) \right\|$$

$$\leqslant \sum_{i=0}^{n-1} 2^{2(i)} \left[\varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right) + \left(1 - \frac{1}{2^{i+1}}\right) \varphi\left(\frac{x}{2^{i+2}} + \frac{\sigma(x)}{2^{i+2}}, \frac{x}{2^{i+2}} + \frac{\sigma(x)}{2^{i+2}}\right) \right]$$

$$+ 2^{2n} \left\| f\left(\frac{x}{2^n}\right) + f\left(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}\right) - 4f\left(\frac{x}{2^{n+1}}\right) \right\|$$

$$+ 2^{2n}\left(1 - \frac{1}{2^{n+1}}\right)\left\| 2f\left(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}\right) - 4f\left(\frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}}\right)\right\|$$

$$\leqslant \sum_{i=0}^{n-1} 2^{2(i)} \left[\varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right) + \left(1 - \frac{1}{2^{i+1}}\right) \varphi\left(\frac{x}{2^{i+2}} + \frac{\sigma(x)}{2^{i+2}}, \frac{x}{2^{i+2}} + \frac{\sigma(x)}{2^{i+2}}\right) \right]$$

$$+ 2^{2n}\left[\varphi\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}\right) + \left(1 - \frac{1}{2^{n+1}}\right) \varphi\left(\frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}}, \frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}}\right) \right]$$

$$\leqslant \sum_{i=0}^{n} 2^{2(i)} \left[\varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right) + \left(1 - \frac{1}{2^{i+1}}\right) \varphi\left(\frac{x}{2^{i+2}} + \frac{\sigma(x)}{2^{i+2}}, \frac{x}{2^{i+2}} + \frac{\sigma(x)}{2^{i+2}}\right) \right],$$

which proves inequality (2.15). Now, we will show that

$$q_n(x) = 2^{2n}\left\{ f\left(\frac{x}{2^n}\right) + \left(\frac{1}{2^n} - 1\right)f\left(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}\right) \right\} \tag{2.18}$$

is a Cauchy sequence. Let m, n be integers with $n > m$. Then by using the following inequality

$$\| q_{n+1}(x) - q_n(x) \| \leqslant 2^{2n}\left\| f\left(\frac{x}{2^n}\right) + f\left(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}\right) - 4f\left(\frac{x}{2^{n+1}}\right) \right\|$$

$$+ 2^{2n}\left(1 - \frac{1}{2^{n+1}}\right)\left\| 2f\left(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}\right) - 4f\left(\frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}}\right)\right\|$$

$$\leqslant 2^{2n} \left[\varphi\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}\right) + \left(1 - \frac{1}{2^{n+1}}\right) \varphi\left(\frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}}, \frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}}\right) \right],$$

we see that

$$\| q_n(x) - q_m(x) \| \leqslant \sum_{i=m}^{n-1} \| q_{i+1}(x) - q_i(x) \|$$

$$\leqslant \sum_{i=m}^{n-1} 2^{2i} \left[\varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right) + \left(1 - \frac{1}{2^{i+1}}\right) \varphi\left(\frac{x}{2^{i+2}} + \frac{\sigma(x)}{2^{i+2}}, \frac{x}{2^{i+2}} + \frac{\sigma(x)}{2^{i+2}}\right) \right]$$

$$\to 0 \quad \text{as} \quad m \to \infty.$$

Since E_2 is Banach space, we may define a function $q\colon E_1 \longrightarrow E_2$ by

$$q(x) = \lim_{n\to\infty} 2^{2n}\left\{ f\left(\frac{x}{2^n}\right) + \left(\frac{1}{2^n} - 1\right)f\left(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}\right) \right\} \tag{2.19}$$

for any $x \in E_1$. By the definition of q and (2.15), we can easily verify the validity of inequality (2.14).

Similarly as we did in the proof of the precedent theorem, we can prove that q is the unique mapping solution of equation (1.1) satisfying inequality (2.14). This completes the proof of the theorem. $\square$

3.   Stability of equation (1.4)

Throughout this section, δ is a positive number. For any function $f : E_1 \longrightarrow E_2$, we introduce the new functions $F(x) = f(x) - f(0)$, $F^e(x) = \frac{F(x)+F(\sigma(x))}{2}$, $F^o(x) = \frac{F(x)-F(\sigma(x))}{2}$.

We note that $F^e(\sigma(x)) = F^e(x)$, $F^o(\sigma(x)) = -F^o(x)$, for all $x \in E_1$ and $F(0) = F^e(0) = F^o(0) = F^o(x + \sigma(x)) = 0$, for all $x \in E_1$.

In the following theorem, we prove the Hyers-Ulam stability of equation

$$f_1(x+y) + f_2(x+\sigma(y)) = f_3(x) + f_4(y). \tag{3.1}$$

The proof is closely related to and inspired by the work by Jung S.-M. [13] and Jung S.-M. and Sahoo P. K. [14] mentioned in the introduction.

THEOREM 3.1. *If the functions $f_1, f_2, f_3, f_4 : E_1 \longrightarrow E_2$ satisfy the inequality*

$$\| f_1(x+y) + f_2(x+\sigma(y)) - f_3(x) - f_4(y) \| \leqslant \delta, \tag{3.2}$$

for all $x, y \in E_1$, then there exists a unique function $q : E_1 \longrightarrow E_2$ solution of equation (1.1), there exists a function $v : E_1 \longrightarrow E_2$ solution of equation

$$v(x+y) = v(x+\sigma(y)), \quad x, y \in E_1, \tag{3.3}$$

there exist exactly two additive functions $\mathbb{A}_1$, $\mathbb{A}_2 : E_1 \longrightarrow E_2$ such that $\mathbb{A}_i \circ \sigma = -\mathbb{A}_i$ ($i = 1, 2$),

$$\left\| f_1(x) - \frac{1}{2}\mathbb{A}_1(x) - \frac{1}{2}\mathbb{A}_2(x) - \frac{1}{2}v(x) - \frac{1}{2}q(x) - f_1(0) \right\| \leqslant 19\delta, \tag{3.4}$$

$$\left\| f_2(x) + \frac{1}{2}\mathbb{A}_1(x) - \frac{1}{2}\mathbb{A}_2(x) + \frac{1}{2}v(x) - \frac{1}{2}q(x) - f_2(0) \right\| \leqslant 19\delta, \tag{3.5}$$

$$\| f_3(x) - \mathbb{A}_2(x) - q(x) - f_3(0) \| \leqslant 16\delta \tag{3.6}$$

and

$$\| f_4(x) - \mathbb{A}_1(x) - q(x) - f_4(0) \| \leqslant 16\delta, \tag{3.7}$$

for all $x \in E_1$.

Proof. From (3.2), we easily obtain

$$\| F_1(x+y) + F_2(x+\sigma(y)) - F_3(x) - F_4(y) \| \leqslant 2\delta. \tag{3.8}$$

Consequently,

$$\| F_1^e(x+y) + F_2^e(x+\sigma(y)) - F_3^e(x) - F_4^e(y) \| \leqslant 2\delta, \tag{3.9}$$

and

$$\| F_1^o(x+y) + F_2^o(x+\sigma(y)) - F_3^o(x) - F_4^o(y) \| \leqslant 2\delta, \tag{3.10}$$

for all $x, y \in E_1$. Hence, from (3.9) we deduce that

$$\| F_1^e(x) + F_2^e(x) - F_3^e(x) \| \leqslant 2\delta \tag{3.11}$$

$$\| F_1^e(x) + F_2^e(x) - F_4^e(x) \| \leqslant 2\delta, \tag{3.12}$$

for all $x \in E_1$.

From (3.9), (3.11) and (3.12), we obtain

$$
\begin{aligned}
\| F_1^e(x+y) &+ F_2^e(x + \sigma(y)) - (F_1^e + F_2^e)(x) - (F_1^e + F_2^e)(y) \| \\
&\leqslant \| F_1^e(x+y) + F_2^e(x + \sigma(y)) - F_3^e(x) - F_4^e(y) \| \\
&\quad + \| F_1^e(x) + F_2^e(x) - F_3^e(x) \| + \| F_1^e(y) + F_2^e(y) - F_4^e(y) \| \\
&\leqslant 6\delta.
\end{aligned}
\tag{3.13}
$$

If we replace y by $\sigma(y)$ in (3.13), we obtain

$$\| F_1^e(x + \sigma(y)) + F_2^e(x+y) - (F_1^e + F_2^e)(x) - (F_1^e + F_2^e)(y) \| \leqslant 6\delta. \tag{3.14}$$

If we add inequality (3.14) to (3.13), we get

$$\| (F_1^e + F_2^e)(x+y) + (F_1^e + F_2^e)(x + \sigma(y)) - 2(F_1^e + F_2^e)(x) - 2(F_1^e + F_2^e)(y) \| \leqslant 12\delta, \tag{3.15}$$

for all $x, y \in E_1$. Hence by ([2], Theorem 2.1) there exists a unique function q solution of equation (1.1) such that

$$\| (F_1^e + F_2^e)(x) - q(x) \| \leqslant 6\delta, \quad x \in E_1. \tag{3.16}$$

On the other hand, from (3.13) and (3.14), we get

$$\| (F_1^e - F_2^e)(x+y) - (F_1^e - F_2^e)(x + \sigma(y)) \| \leqslant 12\delta. \tag{3.17}$$

Hence, there exists $v \colon E_1 \longrightarrow E_2$ defined by $v(x) = (F_1^e - F_2^e)(\frac{x+\sigma(x)}{2})$ solution of equation

$$v(x+y) = v(x + \sigma(y)), \quad x, y \in E_1 \tag{3.18}$$

that satisfies the inequality

$$\| (F_1^e - F_2^e)(x) - v(x) \| \leqslant 12\delta. \tag{3.19}$$

From (3.16) and (3.19), we get

$$\| 2F_2^e(x) - q(x) + v(x) \| \leqslant 18\delta \tag{3.20}$$

$$\| 2F_1^e(x) - q(x) - v(x) \| \leqslant 18\delta. \tag{3.21}$$

From (3.11), (3.12) and (3.16),

$$\| F_3^e(x) - q(x) \| \leqslant 8\delta \tag{3.22}$$

$$\| F_4^e(x) - q(x) \| \leqslant 8\delta. \tag{3.23}$$

Now, from (3.10) we deduce that

$$\| F_3^o(x) - F_1^o(x) - F_2^o(x) \| \leqslant 2\delta \tag{3.24}$$

and

$$\| F_4^o(x) - F_1^o(x) + F_2^o(x) \| \leqslant 2\delta, \tag{3.25}$$

for all $x \in E_1$ and consequently, we get

$$\| 2F_1^o(x) - F_3^o(x) - F_4^o(x) \| \leqslant 4\delta \tag{3.26}$$

$$\| 2F_2^o(x) - F_3^o(x) + F_4^o(x) \| \leqslant 4\delta, \tag{3.27}$$

for all $x \in E_1$.

By using (3.24) and (3.10), we obtain

$$\| F_3^o(x + y) + F_3^o(x + \sigma(y)) - 2F_3^o(x) \| \tag{3.28}$$
$$\leqslant \| F_3^o(x + y) - F_1^o(x + y) - F_2^o(x + y) \|$$
$$+ \| F_3^o(x + \sigma(y)) - F_1^o(x + \sigma(y)) - F_2^o(x + \sigma(y)) \|$$
$$+ \| F_1^o(x + y) + F_2^o(x + \sigma(y)) - F_3^o(x) - F_4^o(y) \|$$
$$+ \| F_1^o(x + \sigma(y)) + F_2^o(x + y) - F_3^o(x) - F_4^o(\sigma(y)) \|$$
$$\leqslant 8\delta \quad \text{for all } x, y \in E_1.$$

On the other hand, from (3.25) and (3.10), we get

$$\| F_4^o(x + y) + F_4^o(x + \sigma(y)) - 2F_4^o(x) \|$$
$$\leqslant \| F_4^o(x + y) - F_1^o(x + y) + F_2^o(x + y) \| \tag{3.29}$$
$$+ \| F_4^o(x + \sigma(y)) - F_1^o(x + \sigma(y)) + F_2^o(x + \sigma(y)) \|$$
$$+ \| F_1^o(y + x) + F_2^o(y + \sigma(x)) - F_3^o(y) - F_4^o(x) \|$$
$$+ \| F_1^o(\sigma(y) + x) + F_2^o(\sigma(y) + \sigma(x)) - F_3^o(\sigma(y)) - F_4^o(x) \|$$
$$\leqslant 8\delta \quad \text{for all } x, y \in E_1.$$

In the following, we will prove the stability of (3.29). From (3.29), we get

$$\| F_4^o(y + x) + F_4^o(y + \sigma(x)) - 2F_4^o(y) \| \leqslant 8\delta. \tag{3.30}$$

So we obtain

$$\| 2F_4^o(x + y) - 2F_4^o(x) - 2F_4^o(y) \| \leqslant 16\delta. \tag{3.31}$$

Consequently, by using [8] there exists an additive function $\mathbb{A}_1 : E_1 \longrightarrow E_2$ such that

$$\| F_4^o(x) - \mathbb{A}_1(x) \| \leqslant 8\delta. \tag{3.32}$$

Furthermore, $\mathbb{A}_1(\sigma(x)) = -\mathbb{A}_1(x)$, because the inequality (3.32) implies that $\| \mathbb{A}_1(x) + \mathbb{A}_1(\sigma(x)) \| \leqslant 16\delta$ and then $x \longmapsto \mathbb{A}_1(x) + \mathbb{A}_1(\sigma(x))$ is a bounded additive function on E_1, so $\mathbb{A}_1(x) + \mathbb{A}_1(\sigma(x)) = 0$, for all $x \in E_1$.

By using the precedent discussion, we deduce that there exists another additive function $\mathbb{A}_2 : E_1 \longrightarrow E_2$ such that

$$\| F_3^o(x) - \mathbb{A}_2(x) \| \leqslant 8\delta \tag{3.33}$$

for all $x \in E_1$.

Finally, in view of (3.20), (3.21), (3.22), (3.23), (3.32), (3.33), (3.26) and (3.27), we obtain

$$\left\| f_1(x) - \frac{1}{2}\mathbb{A}_1(x) - \frac{1}{2}\mathbb{A}_2(x) - \frac{1}{2}v(x) - \frac{1}{2}q(x) - f_1(0) \right\| \leqslant 19\delta, \tag{3.34}$$

$$\left\| f_2(x) + \frac{1}{2}\mathbb{A}_1(x) - \frac{1}{2}\mathbb{A}_2(x) + \frac{1}{2}v(x) - \frac{1}{2}q(x) - f_2(0) \right\| \leqslant 19\delta, \tag{3.35}$$

$$\| f_3(x) - \mathbb{A}_2(x) - q(x) - f_3(0) \| \leqslant 16\delta \tag{3.36}$$

and

$$\| f_4(x) - \mathbb{A}_1(x) - q(x) - f_4(0) \| \leqslant 16\delta. \tag{3.37}$$

The uniqueness of $q, \mathbb{A}_i$, ($i = 1, 2$) follows by applying some argument used in [13] and [14]. This completes the proof of the theorem. $\square$

COROLLARY 3.2. *If the functions $f_1, f_2, f_3, f_4 : E_1 \longrightarrow E_2$ satisfy the functional equation*

$$f_1(x + y) + f_2(x + \sigma(y)) = f_3(x) + f_4(y) \tag{3.38}$$

for all $x, y \in E_1$, then there exists a biadditive function $B : E_1 \times E_1 \longrightarrow E_2$, an additive function $\mathbb{A}$ such that $\mathbb{B}(x, \sigma(y)) = \mathbb{B}(x, x)$, $\mathbb{A}(\sigma(x)) = \mathbb{A}(x)$, there exists a function $v : E_1 \longrightarrow E_2$ solution of equation

$$v(x + y) = v(x + \sigma(y)), \quad x, y \in E_1, \tag{3.39}$$

there exist $\alpha, \beta, \gamma, \delta \in E_2$ and two additive functions $\mathbb{A}_1, \mathbb{A}_2$ such that $\mathbb{A}_i \circ \sigma = -\mathbb{A}_i$ ($i = 1, 2$),

$$f_1(x) = \frac{1}{2}\mathbb{A}_1(x) + \frac{1}{2}\mathbb{A}_2(x) + \frac{1}{2}v(x) + \frac{1}{2}(\mathbb{B}(x) + \mathbb{A}(x)) + \alpha, \tag{3.40}$$

$$f_2(x) = -\frac{1}{2}\mathbb{A}_1(x) + \frac{1}{2}\mathbb{A}_2(x) - \frac{1}{2}v(x) + \frac{1}{2}(\mathbb{B}(x) + \mathbb{A}(x)) + \beta, \tag{3.41}$$

$$f_3(x) = \mathbb{A}_2(x) + \mathbb{B}(x) + \mathbb{A}(x) + \gamma \qquad (3.42)$$

and

$$f_4(x) = \mathbb{A}_1(x) + \mathbb{B}(x) + \mathbb{A}(x) + \delta \qquad (3.43)$$

for all $x \in E_1$.

Proof. We use [23] and Theorem 3.1 for the particular case $\delta = 0$. $\square$

From Theorem 3.1, we can deduce the results of [14] in the following corollary. Furthermore, we improve the estimates of [14].

COROLLARY 3.3. *If the functions $f_1, f_2, f_3, f_4 : E_1 \longrightarrow E_2$ satisfy the inequality*

$$\| f_1(x+y) + f_2(x-y) - f_3(x) - f_4(y) \| \leqslant \delta, \qquad (3.44)$$

for all $x, y \in E_1$, then there exists a unique function $q : E_1 \longrightarrow E_2$ solution of equation (1.1), there exists $\alpha \in E_2$, there exist exactly two additive functions $\mathbb{A}_1, \mathbb{A}_2 : E_1 \longrightarrow E_2$ such that

$$\left\| f_1(x) - \frac{1}{2}\mathbb{A}_1(x) - \frac{1}{2}\mathbb{A}_2(x) - \frac{1}{2}q(x) - f_1(0) - \alpha \right\| \leqslant 19\delta, \qquad (3.45)$$

$$\left\| f_2(x) + \frac{1}{2}\mathbb{A}_1(x) - \frac{1}{2}\mathbb{A}_2(x) - \frac{1}{2}q(x) - f_2(0) + \alpha \right\| \leqslant 19\delta, \qquad (3.46)$$

$$\| f_3(x) - \mathbb{A}_2(x) - q(x) - f_3(0) \| \leqslant 16\delta \qquad (3.47)$$

and

$$\| f_4(x) - \mathbb{A}_1(x) - q(x) - f_4(0) \| \leqslant 16\delta, \qquad (3.48)$$

for all $x \in E_1$.

The following corollary follows from Theorem 3.1. This result is well known (see for example [25]).

COROLLARY 3.4. *If the functions $f_1, f_2, f_3 : E_1 \longrightarrow E_2$ satisfy the inequality*

$$\| f_1(x+y) - f_2(x) - f_3(y) \| \leqslant \delta, \qquad (3.49)$$

for all $x, y \in E_1$, then there exists a unique additive function $\mathbb{A} : E_1 \longrightarrow E_2$ such that

$$\| f_1(x) - \mathbb{A}(x) - f_1(0) \| \leqslant 38\delta, \qquad (3.50)$$

$$\| f_2(x) - \mathbb{A}(x) - f_2(0) \| \leqslant 16\delta \qquad (3.51)$$

and

$$\| f_3(x) - \mathbb{A}(x) - f_3(0) \| \leqslant 16\delta. \qquad (3.52)$$

for all $x \in E_1$.

In the following corollary, we generalize the stability result obtained in [15] for Drygas functional equation.

COROLLARY 3.5. *If the function $f : E_1 \longrightarrow E_2$ satisfies the inequality*

$$\| f(x+y) + f(x+\sigma(y)) - 2f(x) - f(y) - f(\sigma(y)) \| \leqslant \delta, \qquad (3.53)$$

for all $x, y \in E_1$, then there exists a unique additive function $\mathbb{A}: E_1 \longrightarrow E_2$ and a unique quadratic function $q: E_1 \longrightarrow E_2$ such that $\mathbb{A} \circ \sigma = -\mathbb{A}$ and

$$\| f(x) - q(x) - \mathbb{A}(x) - f(0) \| \leqslant 16\delta \qquad (3.54)$$

for all $x \in E_1$.

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