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COMPARISON THEOREM FOR TWO–PARAMETER MEANS

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(*communicated by Zs. Pales*)

Abstract. The comparison theorem for

$$R(u,v;r,s;x,y)=\left(\frac{E(r,s;x^v,y^v)}{E(r,s;x^u,y^y)}\right)^{1/(v-u)},\ u\neq v,$$

where E is the Stolarsky mean, is proved. This generalises the results of Leach, Sholander and Páles.

1. Introduction

Stolarsky means,

$$E(r,s;x,y)=\begin{cases}\left(\dfrac{r}{s}\dfrac{y^s-x^s}{y^r-x^r}\right)^{1/(s-r)} & \text{if}\quad sr(s-r)(x-y)\neq 0,\\ \left(\dfrac{1}{r}\dfrac{y^r-x^r}{\log y-\log x}\right)^{1/r} & \text{if}\quad r(x-y)\neq 0,\ s=0,\\ e^{-1/r}\left(y^{y^r}/x^{x^r}\right)^{1/(y^r-x^r)} & \text{if}\quad r=s,\ r(x-y)\neq 0,\\ \sqrt{xy} & \text{if}\quad r=s=0,\\ x & \text{if}\quad x=y\end{cases}$$

appeared first in the literature in 1975 [8]. Leach and Sholander investigated the comparison problem for Stolarsky means. They obtained some partial result in [3] and in 1983 published the comparison theorem in [4]. Three years later Páles developed a new method and restated the theorem as follows:

THEOREM 1.1. (Comparison Theorem for Stolarsky means, [6]) *The inequality*

$$E(a,b;x,y)\leqslant E(c,d;x,y)$$

holds for all $x,y>0$ *if and only if*

$$a+b\leqslant c+d \tag{1.1}$$

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and

$$e(a,b) \leqslant e(c,d) \tag{1.2}$$

where

$$e(r,s) = \begin{cases} 0 & r = s = 0, \\ \dfrac{|r|-|s|}{r-s} & \min(a,b,c,d) < 0 < \max(a,b,c,d), \\ \begin{cases} \dfrac{r-s}{\log(r/s)} & rs > 0, \\ 0 & rs = 0 \end{cases} & \textit{otherwise.} \end{cases} \tag{1.3}$$

His method led him also to the solution of similar problem for Gini means

$$G(r,s;x,y) = \begin{cases} \left(\dfrac{x^s+y^s}{x^r+y^r}\right)^{1/(s-r)} & \text{if} \quad s \neq r, \\ \exp\left(\dfrac{x^r \log x + y^r \log y}{x^r+y^r}\right) & \text{if} \quad s = r. \end{cases}$$

THEOREM 1.2. (Comparison Theorem for Gini means, [5]) *The inequality*

$$G(a,b;x,y) \leqslant G(c,d;x,y)$$

holds for all $x,y > 0$ *if and only if*

$$a+b \leqslant c+d$$

and

$$m(a,b) \leqslant m(c,d) \tag{1.4}$$

where

$$m(r,s) = \begin{cases} 0 & r = s = 0, \\ \dfrac{|r|-|s|}{r-s} & \min(a,b,c,d) < 0 < \max(a,b,c,d), \\ \min(r,s) & \min(a,b,c,d) \geqslant 0, \\ \max(r,s) & \max(a,b,c,d) \leqslant 0. \end{cases} \tag{1.5}$$

Recently Czinder and Páles found another way to approach the Stolarsky and Gini comparison problem (see [1, 2]).

It is natural to ask if similar results can be obtained for other two-parameter families of means like Heronian

$$N(r,s;x,y) = \left(\frac{x^s + (\sqrt{xy})^s + y^s}{x^r + (\sqrt{xy})^r + y^r}\right)^{1/s-r} \tag{1.6}$$

or centroidal

$$T(r,s;x,y) = \left(\frac{x^{2s} + (xy)^s + y^{2s}}{x^s+y^s} \Big/ \frac{x^{2r} + (xy)^r + y^{2r}}{x^r+y^r}\right)^{1/s-r} \tag{1.7}$$

means. In a private communication Ed Neuman conjectured that the comparison theorem for Heronian and Gini means may be the same. This paper shows that his conjecture was true. We also give a new proof of Theorem 1.2.

In the paper [7] Páles solved the comparison problem for Stolarsky and Gini means in case x and y vary over a compact interval $[\alpha,\beta]$. He proved that for $D=E$ or $D=G$ the inequality $D(a,b;x,y)\leqslant D(c,d;x,y)$ holds if and only if (1.1) is satisfied and $D(a,b;\alpha,\beta)\leqslant D(c,d;\alpha,\beta)$. The main theorem from his paper will be also used in the proof of our results:

THEOREM 1.3. ([7] Theorem 2) *Let f be an even function satisfying $f''(0)>0$, $f''(x)>0$, $f'''(x)<0$ and $(xf'''(x)/f''(x))'<0$ for $x>0$. Define*

$$F_{a,b}(t)=\begin{cases}\dfrac{f(at)-f(bt)}{a-b} & a\neq b\\ tf'(at) & a=b.\end{cases} \tag{1.8}$$

Then the inequality $F_{a,b}(t)\leqslant F_{c,d}(t)$ holds for all $t\in[-s,s]$ if and only if $a+b\leqslant c+d$ and $F_{a,b}(s)\leqslant F_{c.d}(s)$.

The following characterization of convex function will be useful:

PROPERTY 1.4. *Function f is (strictly) convex if and only if the function*

$$g(p,q)=\frac{f(p)-f(q)}{p-q},\quad p\neq q$$

is (strictly) increasing in both variables.

2. Comparison theorem for S-means

For real α we define the S-means as

$$S(\alpha;r,s;x,y)=\frac{E(r,s;x^{\alpha+1},y^{\alpha+1})}{E(r,s;x^{\alpha},y^{\alpha})}. \tag{2.1}$$

If there is no risk of confusion we will use shorter notation: $S(\alpha;r,s)$ or $S(\alpha)$.

It is not obvious that S is indeed a mean, so let us prove this first.

LEMMA 2.1. *If $x\leqslant y$ then $x\leqslant S(\alpha;r,s;x,y)\leqslant y$.*

Proof. Stolarsky means are increasing in all variables and homogeneous of degree 1 in x and y, so for fixed $p,q>0$ we have

$$x=\frac{E(r,s;px,qx)}{E(r,s;p,q)}\leqslant\frac{E(r,s;px,qy)}{E(r,s;p,q)}\leqslant\frac{E(r,s;py,qy)}{E(r,s;p,q)}=y.$$

Setting $p=x^{\alpha},q=y^{\alpha}$ we obtain required inequalities. □

The reader will find more about weighted Stolarsky means $E(r,s;px,qy)/E(r,s;p,q)$ in [9].

As one can easily see S-means contain all families of means mentioned so far:

$$E = S(0), \quad N = S(1/2), \quad G = S(1), \text{ and } T = S(2).$$

The main result of this chapter is the following

THEOREM 2.2. *For* $\alpha > 0$ *the inequality*

$$S(\alpha; a, b; x, y) \leqslant S(\alpha; c, d; x, y) \tag{2.2}$$

holds for all $x, y > 0$ *if and only if* (1.1) *and* (1.4) *are satisfied.*

REMARK. As S is symmetric with respect to r and s we can and do assume that $a \leqslant b$ and $c \leqslant d$.

Note also that due to homogeneity of S we can further assume that $y = 1$.

The inequality

$$S(\alpha; a, b; x, 1) \leqslant S(\alpha; c, d; x, 1) \tag{2.3}$$

is equivalent to

$$H(a, b; c, d; x^{\alpha+1}) \leqslant H(a, b; c, d; x^{\alpha}), \tag{2.4}$$

where

$$H(x) = H(a, b; c, d; x) = \log \frac{E(a, b; x, 1)}{E(c, d; x, 1)}. \tag{2.5}$$

The idea of the proof is the following: we will show that if (2.4) holds then (1.1) and (1.4) hold also. Then we shall show that if (1.1) and (1.4) are satisfied then the function H is increasing on the interval $(0, 1)$, which yields (2.4) in this interval. Since $H(x) = H(1/x)$, then (2.4) is valid for all $x > 0$.

Differentiating (2.5) we obtain

$$H'(x) = x^{-1} h(a, b; c, d; x) \tag{2.6}$$

where

$$h(x) = h(a, b; c, d; x)$$

$$= \frac{\dfrac{bx^b}{x^b - 1} - \dfrac{ax^a}{x^a - 1}}{b - a} - \frac{\dfrac{dx^d}{x^d - 1} - \dfrac{cx^c}{x^c - 1}}{d - c} \tag{2.7}$$

$$= \left(\frac{\dfrac{b}{x^b - 1} - \dfrac{a}{x^a - 1}}{b - a} - \frac{\dfrac{d}{x^d - 1} - \dfrac{c}{x^c - 1}}{d - c} \right) \tag{2.8}$$

$$= \left(\frac{\dfrac{b}{2}\dfrac{x^b + 1}{x^b - 1} - \dfrac{a}{2}\dfrac{x^a + 1}{x^a - 1}}{b - a} - \frac{\dfrac{d}{2}\dfrac{x^d + 1}{x^d - 1} - \dfrac{c}{2}\dfrac{x^c + 1}{x^c - 1}}{d - c} \right) \tag{2.9}$$

$$= \frac{1}{\log x} \left(\frac{k\left(\dfrac{b \log x}{2}\right) - k\left(\dfrac{a \log x}{2}\right)}{b - a} - \frac{k\left(\dfrac{d \log x}{2}\right) - k\left(\dfrac{c \log x}{2}\right)}{d - c} \right) \tag{2.10}$$

and $k(t) = \dfrac{t}{\tanh t}$. (We use the convention that $\frac{x^0-1}{0} = \log x$). In case $a = b$ or $c = d$ the function h is defined as appropriate limits.

The expressions in numerators of (2.7), (2.8) and (2.9) differ by an affine term that gets annihilated by the difference of divided differences.

The last representation is nothing but the difference of functions defined in Theorem 1.3.

LEMMA 2.3. *The function $k(t) = t/\tanh t$ satisfies assumptions of Theorem 1.3.*

Proof. Obviously k is even. For positive t we have

$$\begin{aligned} k''(t) &= 2\frac{t\cosh t - \sinh t}{\sinh^3 t} > 0, & (2.11)\\ k''(0) &= \frac{2}{3}, \\ k'''(t) &= 2\frac{3\sinh t\cosh t - t(1+2\cosh^2 t)}{\sinh^4 t} \\ &= \frac{3\sinh(2t) - 2t[2+\cosh(2t)]}{\sinh^4 t} < 0, & (2.12)\end{aligned}$$

because

$$3\sinh(t) - t[2+\cosh(t)] = \sum_{k=2}^{\infty}\left(\frac{3}{2k+1} - 1\right)\frac{t^{2k+1}}{(2k)!} < 0.$$

It is still to be shown that $tk'''(t)/k''(t)$ is decreasing.

$$\begin{aligned} \frac{tk'''(t)}{k''(t)} &= t\frac{3\sinh t\cosh t - t(1+2\cosh^2 t)}{t\cosh t\sinh t - \sinh^2 t} \\ &= -2t\frac{4t + 2t\cosh(2t) - 3\sinh(2t)}{2t\sinh(2t) + 2 - 2\cosh(2t)} \equiv -2tu(2t). \end{aligned}$$

From (2.11) and (2.12) it follows that $u(t) > 0$, so is is enough to show that u increases:

$$\begin{aligned} u'(t) &= \frac{\cosh^2 t - 8\cosh t + 7 + 6t\sinh t - t^2(1+2\cosh t)}{(t\sinh t + 2 - 2\cosh t)^2} \\ &= \frac{1}{(t\sinh t + 2 - 2\cosh t)^2}\sum_{k=4}^{\infty}\frac{2^{2k-1} - 8(k-1)^2}{(2k)!}t^{2k} > 0. \end{aligned}$$

□

Lemmas that follow demonstrate the behavior of h in the interval $(0,1)$.

LEMMA 2.4. *Let $0 < x < 1$.*

If $a \leqslant c$ and $b \leqslant d$, then $h(x;a,b;c,d) \geqslant 0$, with equality if and only if $a = c$ and $b = d$.

If $a \geqslant c$ and $b \geqslant d$, then $h(x;a,b;c,d) \leqslant 0$, with equality if and only if $a = c$ and $b = d$.

Proof. From Lemma 2.3 we know that k is strictly convex, so the proof follows immediately from Property 1.4. □

LEMMA 2.5. *$h(x)$ is positive for $x \in (1-\varepsilon, 1)$, if $a+b < c+d$, and negative, if $a+b > c+d$.*

Proof. Substitution $(1+t)^a - 1 = at + \frac{a(a-1)}{2}t^2 + \frac{a(a-1)(a-2)}{6}t^3 + o(t^3)$ in (2.8) leads after some elementary transformations to

$$h(1+t) = \frac{((a+b)-(c+d))t + o(t)}{12 + O(t)},$$

which completes the proof. □

LEMMA 2.6. *$h(x)$ is positive near 0, if $m(a,b) < m(c,d)$, and negative in case $m(a,b) > m(c,d)$.*

If $m(a,b) = m(c,d)$ then $h(x)$ is positive for small x, if $a+b < c+d$, and negative if $a+b > c+d$.

Finally, if $m(a,b) = m(c,d)$ and $a+b = c+d$, then $h \equiv 0$.

Proof. Consider three cases:

Case 1: $\min(a,b,c,d) > 0$,
$m(a,b) < m(c,d)$ means that $a < c$. If $b \leqslant d$ then h is positive by Lemma 2.4, else $a < c \leqslant d < b$ and we have (using the formula (2.7))

$$\lim_{x \to 0} \frac{h(x)}{x^a} = \frac{1}{(b-a)} > 0.$$

Similarly, if $m(a,b) > m(c,d)$, then $a > c$ and $b \geqslant d$ (in which case Lemma 2.4 shows that h is negative) or $c < a \leqslant b < d$, and then

$$\lim_{x \to 0} \frac{h(x)}{x^c} = -\frac{1}{(d-c)} < 0.$$

In case $a = c$ the required result follows directly from Lemma 2.4.

Case 2: $\max(a,b,c,d) < 0$,
This case is similar to the previous one.

Case 3: $\min(a,b,c,d) < 0 < \max(a,b,c,d)$
From (2.9) we see that

$$\lim_{x \to 0} h(x) = \frac{1}{2}\frac{|a|-|b|}{b-a} - \frac{1}{2}\frac{|c|-|d|}{d-c} = \frac{m(c,d)-m(a,b)}{2},$$

which completes the proof in case $m(a,b) \neq m(c,d)$.
The only remaining case is $m(a,b) = m(c,d)$. Geometrically this means that the lines passing through appropriate points on the graph of the function $|x|$ are parallel, and this is possible only if $a, c < 0 < b, d$.

If $m(a,b) = m(c,d) = 0$, then these lines are parallel to the x-axis so we have $a = -b$ and $c = -d$. In this case we easily see that $h(x) \equiv 0$.

Let $m(a,b) = m(c,d) \neq 0$. Then there exists a positive constant k such that

$$|a| = k\,|b| \qquad \text{and} \qquad |c| = k\,|d|. \tag{2.13}$$

The condition $m(a,b) = m(c,d)$ can be rewritten as

$$\frac{a+b}{|a|+|b|} = \frac{c+d}{|c|+|d|}. \tag{2.14}$$

Suppose that $\min(|a|,|b|,|c|,|d|) = |a|$. Then $a+b$ and $c+d$ are positive and (2.14) and (2.13) imply that $a+b < c+d$. Similar reasoning shows that

$$a+b < c+d \quad \Leftrightarrow \quad \min(|a|,|b|,|c|,|d|) \in \{|a|,|d|\} \tag{2.15}$$

$$a+b > c+d \quad \Leftrightarrow \quad \min(|a|,|b|,|c|,|d|) \in \{|b|,|c|\}. \tag{2.16}$$

On the other hand $m(a,b) = m(c,d)$ means that $bc = ad$. Taking the limit in (2.7) we see that

$$\lim_{x\to 0} h(x) = \frac{-a}{b-a} - \frac{-c}{d-c} = \frac{bc-ad}{(b-a)(d-c)} = 0$$

so we can apply the l'Hospital's rule obtaining

$$\lim_{x\to 0} \frac{h(x)}{x^{\min(|a|,|b|,|c|,|d|)}} = \begin{cases} -a/(b-a) > 0 & \text{if } |a| = \min(|a|,|b|,|c|,|d|), \\ -b/(b-a) < 0 & \text{if } |b| = \min(|a|,|b|,|c|,|d|), \\ c/(d-c) < 0 & \text{if } |c| = \min(|a|,|b|,|c|,|d|), \\ d/(d-c) > 0 & \text{if } |d| = \min(|a|,|b|,|c|,|d|). \end{cases} \tag{2.17}$$

Comparing (2.17), (2.15) and (2.16) we get the required result. □

Now we are ready to prove the announced result.

Proof of Theorem 2.2. Suppose (2.4) holds for all $x \in (0,1)$. Then, by the Mean Value Theorem

$$H(a,b;c,d;x^{\alpha+1}) - H(a,b;c,d;x^{\alpha}) = H'(\xi)(x^{\alpha+1} - x^{\alpha})$$

with $x^{\alpha+1} < \xi < x^{\alpha}$. Thus $h(\xi) \geqslant 0$. If x tends to 1 so does ξ, hence h takes nonnegative values arbitrarily close to 1. By Lemma 2.5 the condition (1.1) must hold.

Similarly, for x close to 0 corresponding ξ is also close to 0, so by Lemma 2.6 the condition (1.4) is satisfied.

Conversely, if (1.1) and (1.4) hold then by Lemma 2.6 or $h(x) \equiv 0$, in which case $H(x) \equiv 1$ (because $H(1) = 1$) and (2.4) is evident, or $h(x) > 0$ for small x. But in this case from (2.10) we deduce that

$$\frac{k(bt) - k(at)}{b-a} < \frac{k(dt) - k(ct)}{d-c} \tag{2.18}$$

for arbitrarily large t. Now (1.1), Lemma 2.3 and Theorem 1.3 imply that the above inequality is valid for all t, which means that $h(x) > 0$ for all $x < 1$. Thus $H(x)$ increases and this completes the proof. □

To prove the comparison theorem for $-1/2 < \alpha < 0$ we use the following

LEMMA 2.7. *For* $\alpha \neq -1/2$ *the identity*

$$S(\alpha;r,s) = (xy)^{-\alpha} S^{2\alpha+1}\left(\frac{-\alpha}{2\alpha+1};(2\alpha+1)r,(2\alpha+1)s\right) \tag{2.19}$$

holds

Proof. Let $\mu = 2\alpha+1$ and $\nu = -\alpha/(2\alpha+1)$. Then $-\alpha = \mu\nu$, and $\alpha+1 = \mu(\nu+1)$ and from homogeneity of E we obtain

$$\begin{aligned} S(\alpha;r,s;x,y) &= \frac{E(r,s;x^{\alpha+1},y^{\alpha+1})}{E(r,s;x^{\alpha},y^{\alpha})} = \frac{E(r,s;x^{\alpha+1},y^{\alpha+1})}{(xy)^{\alpha}E(r,s;x^{-\alpha},y^{-\alpha})} \\ &= (xy)^{-\alpha}\frac{E(r,s;x^{\mu(\nu+1)},y^{\mu(\nu+1)})}{E(r,s;x^{\mu\nu},y^{\mu\nu})} \\ &= (xy)^{-\alpha}S(\nu;r,s;x^{\mu},y^{\mu}) \\ &= (xy)^{-\alpha}S^{2\alpha+1}\left(\frac{-\alpha}{(2\alpha+1)};(2\alpha+1)r,(2\alpha+1)r;x,y\right). \end{aligned}$$

□

Note that the conditions (1.1) and (1.4) are invariant under multiplication by a positive constant, so the Lemma 2.7 yields the

COROLLARY 2.8. *If* $-1/2 < \alpha < 0$ *then* $S(\alpha;a,b) \leqslant S(\alpha;c,d)$ *if and only if the conditions* (1.1) *and* (1.4) *hold.*

Finally for $\alpha < -1/2$ we use the easily verifiable identity

$$S(-1/2+\alpha;r,s;x,y)S(-1/2-\alpha;r,s;x,y) = xy$$

to see that

COROLLARY 2.9. *If* $\alpha < -1/2$ *and* $\alpha \neq -1$ *then* $S(\alpha;a,b) \leqslant S(\alpha;c,d)$ *if and only if the inequalities in* (1.1) *and* (1.4) *are reversed.*

3. One step further

The definition of $S(\alpha)$ in case $r \neq s$ can be rewritten as

$$S(\alpha;r,s;x,y) = \left(\frac{E(\alpha,\alpha+1;x^s,y^s)}{E(\alpha,\alpha+1;x^r,y^r)}\right)^{\frac{1}{s-r}}, \tag{3.1}$$

which leads to the obvious generalization:

$$R(u,v;r,s;x,y) = \left(\frac{E(u,v;x^s,y^s)}{E(u,v;x^r,y^r)}\right)^{\frac{1}{s-r}}. \tag{3.2}$$

Since E is analytic the above definition can be extended by continuity to the case $r = s$. The family R contains some known means, for example

$$R(1,n+1;0,1;x,y) = \left(\frac{x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n}{n+1}\right)^{1/n}$$

and their two-parameter generalizations:

$$R(1,n+1;r,s;x,y)=\left(\frac{x^{ns}+x^{(n-1)s}y^{s}+\cdots+x^{s}y^{(n-1)s}+y^{ns}}{x^{nr}+x^{(n-1)r}y^{r}+\cdots+x^{r}y^{(n-1)r}+y^{nr}}\right)^{1/n(s-r)}.$$

Since

$$E(u,v;x,y)=\left(\frac{L(x^{v},y^{v})}{L(x^{u},y^{u})}\right)^{\frac{1}{v-u}}$$

where L is the logarithmic mean, we conclude that

$$\begin{aligned}R(u,v;r,s;x,y) &= \left(\frac{E(u,v;x^{s},y^{s})}{E(u,v;x^{r},y^{r})}\right)^{\frac{1}{s-r}}\\ &= \left(\frac{L(x^{sv},y^{sv})}{L(x^{su},y^{su})}\frac{L(x^{ru},y^{ru})}{L(x^{rv},y^{rv})}\right)^{\frac{1}{v-u}\frac{1}{s-r}} \qquad (3.3)\\ &= R(r,s;u,v,x,y).\end{aligned}$$

For $u\neq v$, $u=(v-u)\frac{u}{v-u}$ and $v=(v-u)(\frac{u}{v-u}+1)$, so using (3.3) we obtain

$$\begin{aligned}R(u,v;r,s;x,y) &= \left(\frac{E(r,s;x^{v},y^{v})}{E(r,s;x^{u},y^{u})}\right)^{\frac{1}{v-u}} \qquad (3.4)\\ &= S(\tfrac{u}{v-u};r,s;x^{v-u},y^{v-u})^{\frac{1}{v-u}}.\end{aligned}$$

THEOREM 3.1. (Comparison theorem for R-means) *In case $u\neq v$ the inequality*

$$R(u,v;a,b;x,y)\leqslant R(u,v;c,d;x,y)$$

holds for all $x,y>0$ if and only if either

$$u+v=0$$

or

$$u+v>0,\quad a+b\leqslant c+d\quad \textit{and}\quad w(a,b)\leqslant w(c,d)$$

or

$$u+v<0,\quad a+b\geqslant c+d\quad \textit{and}\quad w(a,b)\geqslant w(c,d)$$

where

$$w(r,s)=\begin{cases} e(r,s) & \textit{if}\quad uv=0,\\ m(r,s) & \textit{if}\quad uv\neq 0.\end{cases} \qquad (3.5)$$

Proof. By (3.4)

$$R(u,v;a,b;x,y)\leqslant R(u,v;c,d;x,y)$$

is equivalent to

$$S\left(\frac{u}{v-u};a,b;x^{v-u},y^{v-u}\right)^{\frac{1}{v-u}}\leqslant S\left(\frac{u}{v-u};c,d;x^{v-u},y^{v-u}\right)^{\frac{1}{v-u}}.$$

Now it is enough to note that $\frac{u}{v-u} + \frac{1}{2} = \frac{u+v}{2(v-u)}$ and that the condition $\frac{u}{v-u} \in \{0, -1\}$ is equivalent to $uv = 0$ to derive the theorem from the results of the previous section. $\square$

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