

GENERALIZATION OF THE PEČARIĆ-RAJIĆ INEQUALITY IN NORMED LINEAR SPACES

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Abstract. In this paper we establish a generalization of the recent *Pečarić-Rajić inequality* by providing upper and lower bounds for the norm of the linear combination $\sum_{j=1}^n \alpha_j x_j$ where $\alpha_j \in \mathbb{K}$ and $x_j \in X$ for $j \in \{1, \dots, n\}$ with $n \geq 2$. Applications for two vectors that are related to the *Massera-Schäffer*, *Dunkl-Williams* and *Maligranda-Mercer inequalities* are given. Some bounds for the quantity $\|x/\|y\| - y/\|x\|\|$ with $x, y \in X \setminus \{0\}$, are also provided.

1. Introduction

In the recent paper [13], J. Pečarić and R. Rajić proved the following inequality for n nonzero vectors $x_k, k \in \{1, \dots, n\}$ in the real or complex normed linear space $(X, \|\cdot\|)$:

$$\begin{aligned} \max_{k \in \{1, \dots, n\}} \left\{ \frac{1}{\|x_k\|} \left[\left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n \|x_j\| - \|x_k\| \right] \right\} \\ \leq \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \leq \min_{k \in \{1, \dots, n\}} \left\{ \frac{1}{\|x_k\|} \left[\left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n \|x_j\| - \|x_k\| \right] \right\} \end{aligned} \quad (1.1)$$

and showed that this inequality implies the following *refinement of the generalised triangle inequality* obtained by M. Kato et al. in [8]:

$$\begin{aligned} \min_{k \in \{1, \dots, n\}} \{ \|x_k\| \} \left[n - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \right] \\ \leq \sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \leq \max_{k \in \{1, \dots, n\}} \{ \|x_k\| \} \left[n - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \right]. \end{aligned} \quad (1.2)$$

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The inequality (1.2) can be also obtained as a particular case of the author's result established in [1]

$$\begin{aligned} & \max_{1 \leq j \leq n} \{ \|x_j\| \} \left[\sum_{j=1}^n \|x_j\|^{p-1} - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\|^p \right] \\ & \geq \sum_{j=1}^n \|x_j\|^p - n^{1-p} \left\| \sum_{j=1}^n x_j \right\|^p \geq \min_{1 \leq j \leq n} \{ \|x_j\| \} \left[\sum_{j=1}^n \|x_j\|^{p-1} - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\|^p \right], \end{aligned} \quad (1.3)$$

where $p \geq 1$ and $n \geq 2$.

Notice that, in [1], a more general inequality for convex functions has been obtained as well.

In [13], Pečarić and Rajić also observed that, for $n = 2$, $x_1 = x$ and $x_2 = -y$ their result reduces to

$$\frac{\|x - y\| - \left| \|x\| - \|y\| \right|}{\min \{ \|x\|, \|y\| \}} \leq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{\|x - y\| + \left| \|x\| - \|y\| \right|}{\max \{ \|x\|, \|y\| \}} \quad (1.4)$$

which holds for each nonzero vector $x, y \in X$.

The second inequality in (1.4) has been obtained by L. Maligranda in [9]. It provides a refinement of the *Massera-Schäffer inequality* [10]

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2 \|x - y\|}{\max \{ \|x\|, \|y\| \}},$$

which, in its turn, is a refinement of the *Dunkl-Williams inequality* [7]

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4 \|x - y\|}{\|x\| + \|y\|}.$$

The first inequality in (1.4) was obtained by P.R. Mercer in [11].

The main aim of this paper is to establish a generalization of the *Pečarić-Rajić inequality* (1.1) by providing upper and lower bounds for the norm of the linear combination $\sum_{j=1}^n \alpha_j x_j$ where $\alpha_j \in \mathbb{K}$ and $x_j \in X$ for $j \in \{1, \dots, n\}$ with $n \geq 2$. Applications for two vectors that are related to the *Maligranda-Mercer inequalities* (1.4) are given. Some bounds for the dual quantity $\|x/\|y\| - y/\|x\|\|$ with $x, y \in X \setminus \{0\}$, are also provided.

2. A General Norm Inequality for n Vectors

We can state the following result

THEOREM 1. *Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field \mathbb{K} . If $\alpha_j \in \mathbb{K}$ and $x_j \in X$ for $j \in \{1, \dots, n\}$ with $n \geq 2$, then*

$$\begin{aligned} \max_{k \in \{1, \dots, n\}} \left\{ |\alpha_k| \left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n |\alpha_j - \alpha_k| \|x_j\| \right\} \\ \leq \left\| \sum_{j=1}^n \alpha_j x_j \right\| \leq \min_{k \in \{1, \dots, n\}} \left\{ |\alpha_k| \left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n |\alpha_j - \alpha_k| \|x_j\| \right\}. \end{aligned} \quad (2.1)$$

Proof. Observe that, for any fixed $k \in \{1, \dots, n\}$ we have

$$\sum_{j=1}^n \alpha_j x_j = \alpha_k \sum_{j=1}^n x_j + \sum_{j=1}^n (\alpha_j - \alpha_k) x_j. \quad (2.2)$$

Taking the norm in (2.2) and utilizing the triangle inequality we have successively

$$\begin{aligned} \left\| \sum_{j=1}^n \alpha_j x_j \right\| &\leq \left\| \alpha_k \sum_{j=1}^n x_j \right\| + \left\| \sum_{j=1}^n (\alpha_j - \alpha_k) x_j \right\| \\ &\leq |\alpha_k| \left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n |\alpha_j - \alpha_k| \|x_j\|, \end{aligned}$$

which, on taking the minimum over $k \in \{1, \dots, n\}$, produces the second inequality in (2.1).

Since, obviously, by (2.2) we have

$$\sum_{j=1}^n \alpha_j x_j = \alpha_k \sum_{j=1}^n x_j - \sum_{j=1}^n (\alpha_k - \alpha_j) x_j$$

then on utilizing the continuity property of the norm we also have

$$\begin{aligned} \left\| \sum_{j=1}^n \alpha_j x_j \right\| &\geq \left\| \left\| \alpha_k \sum_{j=1}^n x_j \right\| - \left\| \sum_{j=1}^n (\alpha_k - \alpha_j) x_j \right\| \right\| \\ &\geq \left\| \alpha_k \sum_{j=1}^n x_j \right\| - \left\| \sum_{j=1}^n (\alpha_k - \alpha_j) x_j \right\| \\ &\geq |\alpha_k| \left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n |\alpha_k - \alpha_j| \|x_j\| \end{aligned}$$

which, on taking the maximum over $k \in \{1, \dots, n\}$ produces the first part of (2.1) and the theorem is completely proved. \square

REMARK 1. If some information is available about the location of the scalars $\alpha_j \neq 0, j \in \{1, \dots, n\}$ namely, if

$$\left| \frac{\alpha_j}{\alpha_k} - 1 \right| \leq \rho \quad \text{for each } j, k \in \{1, \dots, n\}$$

for a given $\rho > 0$, then we get from the second part of (2.1) that

$$\left\| \sum_{j=1}^n \alpha_j x_j \right\| \leq \min_{k \in \{1, \dots, n\}} \{|\alpha_k|\} \left[\left\| \sum_{j=1}^n x_j \right\| + \rho \sum_{j=1}^n \|x_j\| \right].$$

If $x_j \in X$ for $j \in \{1, \dots, n\}$ are such that

$$\left\| \sum_{j=1}^n x_j \right\| - \rho \sum_{j=1}^n \|x_j\| \geq 0$$

then the following nontrivial lower bound can be stated as well

$$\max_{k \in \{1, \dots, n\}} \{|\alpha_k|\} \left[\left\| \sum_{j=1}^n x_j \right\| - \rho \sum_{j=1}^n \|x_j\| \right] \leq \left\| \sum_{j=1}^n \alpha_j x_j \right\|.$$

COROLLARY 1. Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field \mathbb{K} . If $x_j \in X$ for $j \in \{1, \dots, n\}$ with $n \geq 2$, then

$$\begin{aligned} & \max_{k \in \{1, \dots, n\}} \left\{ \|x_k\| \left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n \|x_j\| - \|x_k\| \|x_j\| \right\} \\ & \leq \left\| \sum_{j=1}^n \|x_j\| x_j \right\| \leq \min_{k \in \{1, \dots, n\}} \left\{ \|x_k\| \left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n \|x_j\| - \|x_k\| \|x_j\| \right\}. \end{aligned} \quad (2.3)$$

The proof is obvious by Theorem 1 on choosing $\alpha_k = \|x_k\|$, $k \in \{1, \dots, n\}$.

From (2.3) we can deduce some upper and lower bounds for the nonnegative quantity $\sum_{j=1}^n \|x_j\|^2 - \left\| \sum_{j=1}^n \|x_j\| x_j \right\|$ as follows:

COROLLARY 2. If $x_j \in X$ for $j \in \{1, \dots, n\}$ with $n \geq 2$, then

$$\begin{aligned} & (0 \leq) \min_{k \in \{1, \dots, n\}} \{\|x_k\|\} \left(\sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \right) \\ & \leq \sum_{j=1}^n \|x_j\|^2 - \left\| \sum_{j=1}^n \|x_j\| x_j \right\| \\ & \leq \max_{k \in \{1, \dots, n\}} \{\|x_k\|\} \left(\sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \right). \end{aligned} \quad (2.4)$$

Proof. Assume that $\min_{k \in \{1, \dots, n\}} \{\|x_k\|\} = \|x_{k_0}\|$ with $k_0 \in \{1, \dots, n\}$. Then, on utilizing the second inequality in (2.3) we have

$$\begin{aligned} \left\| \sum_{j=1}^n \|x_j\| x_j \right\| &\leq \|x_{k_0}\| \left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n \left| \|x_j\| - \|x_{k_0}\| \right| \|x_j\| \\ &= \|x_{k_0}\| \left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n \|x_j\|^2 - \|x_{k_0}\| \sum_{j=1}^n \|x_j\| \end{aligned}$$

which is clearly equivalent to the first inequality in (2.4).

The second part follows likewise and the details are omitted. \square

REMARK 2. If $x_j \in X \setminus \{0\}$ for $j \in \{1, \dots, n\}$ with $n \geq 2$, then from (2.1) for $\alpha_k = 1/\|x_k\|$, $k \in \{1, \dots, n\}$ we deduce the Pečarić-Rajić inequality (1.1).

3. Inequalities for Two Vectors

The case of two vectors may be of interest for applications in the Geometry of Banach Spaces.

We start with the following result:

PROPOSITION 1. *For any two vectors $x, y \in X$ and two scalars $\alpha, \beta \in \mathbb{K}$ we have the double inequality*

$$\begin{aligned} &\frac{1}{2} [(|\alpha| + |\beta|) \|x + y\| - |\alpha - \beta| (\|x\| + \|y\|)] \\ &\quad + \frac{1}{2} [(|\alpha| - |\beta|) \|x + y\| + |\alpha - \beta| (\|x\| - \|y\|)] \\ &\leq \|\alpha x + \beta y\| \\ &\leq \frac{1}{2} [(|\alpha| + |\beta|) \|x + y\| + |\alpha - \beta| (\|x\| + \|y\|)] \\ &\quad - \frac{1}{2} [(|\alpha| - |\beta|) \|x + y\| - |\alpha - \beta| (\|x\| - \|y\|)]. \end{aligned} \tag{3.1}$$

Proof. If we apply Theorem 1 for $n = 2$, $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $x_1 = x$ and $x_2 = y$ we have

$$\begin{aligned} &\max \{ |\alpha| \|x + y\| - |\alpha - \beta| \|y\|, |\beta| \|x + y\| - |\alpha - \beta| \|x\| \} \\ &\leq \|\alpha x + \beta y\| \\ &\leq \min \{ |\alpha| \|x + y\| + |\alpha - \beta| \|y\|, |\beta| \|x + y\| + |\alpha - \beta| \|x\| \}. \end{aligned} \tag{3.2}$$

We utilize the properties that

$$\max \{a, b\} = \frac{1}{2} (a + b + |a - b|), \quad \min \{a, b\} = \frac{1}{2} (a + b - |a - b|),$$

for any $a, b \in \mathbb{R}$ and since

$$\begin{aligned} & \max \{ |\alpha| \|x + y\| - |\alpha - \beta| \|y\|, |\beta| \|x + y\| - |\alpha - \beta| \|x\| \} \\ &= \frac{1}{2} [(|\alpha| + |\beta|) \|x + y\| - |\alpha - \beta| (\|x\| + \|y\|)] \\ & \quad + \frac{1}{2} | (|\alpha| - |\beta|) \|x + y\| + |\alpha - \beta| (\|x\| - \|y\|) | \end{aligned}$$

and

$$\begin{aligned} & \min \{ |\alpha| \|x + y\| + |\alpha - \beta| \|y\|, |\beta| \|x + y\| + |\alpha - \beta| \|x\| \} \\ &= \frac{1}{2} [(|\alpha| + |\beta|) \|x + y\| + |\alpha - \beta| (\|x\| + \|y\|)] \\ & \quad - \frac{1}{2} | (|\alpha| - |\beta|) \|x + y\| - |\alpha - \beta| (\|x\| - \|y\|) |, \end{aligned}$$

hence by (3.2) we deduce the desired result (3.1). \square

The following particular cases are of interest.

COROLLARY 3. *Under the assumptions of Proposition 1 and if $|\alpha| = |\beta| = 1$, then*

$$\| |\alpha x + \beta y| - \|x + y\| \leq |\alpha - \beta| \min \{ \|x\|, \|y\| \}, \quad (3.3)$$

for any $x, y \in X$.

COROLLARY 4. *Under the assumptions of Proposition 1 and if $\|x\| = \|y\| = 1$, then*

$$\left| \| |\alpha x + \beta y| - (|\alpha| + |\beta|) \cdot \left\| \frac{x + y}{2} \right\| \right| \leq |\alpha - \beta| - \left| |\alpha| - |\beta| \right| \cdot \left\| \frac{x + y}{2} \right\|, \quad (3.4)$$

for any $\alpha, \beta \in \mathbb{K}$.

4. Dual Versions of the Maligranda-Mercer Inequality

In this section we provide two dual versions of the *Maligranda-Mercer inequality*:

$$\frac{\|x - y\| - \left| \|x\| - \|y\| \right|}{\min \{ \|x\|, \|y\| \}} \leq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{\|x - y\| + \left| \|x\| - \|y\| \right|}{\max \{ \|x\|, \|y\| \}} \quad (4.1)$$

namely, we obtain upper and lower bounds for the quantity

$$\left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\|$$

in the case when the vectors x and y are nonzero in the normed linear space $(X, \|\cdot\|)$.

THEOREM 2. *For any $x, y \in X \setminus \{0\}$ we have*

$$\begin{aligned} 0 & \leq \frac{\|x - y\|}{\min \{ \|x\|, \|y\| \}} - \frac{\left| \|x\| - \|y\| \right|}{\max \{ \|x\|, \|y\| \}} \\ & \leq \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \leq \frac{\|x - y\|}{\max \{ \|x\|, \|y\| \}} + \frac{\left| \|x\| - \|y\| \right|}{\min \{ \|x\|, \|y\| \}}. \end{aligned} \quad (4.2)$$

Proof. We use the inequality (3.1) for $\alpha = 1/\|y\|$ and $\beta = 1/\|x\|$. Firstly, we observe that

$$\begin{aligned}
 I &:= \frac{1}{2} [(|\alpha| + |\beta|) \|x + y\| - |\alpha - \beta| (\|x\| + \|y\|)] \\
 &\quad + \frac{1}{2} [(|\alpha| - |\beta|) \|x + y\| + |\alpha - \beta| (\|x\| - \|y\|)] \\
 &= \frac{1}{2} \left[\left(\frac{\|x\| + \|y\|}{\|x\| \|y\|} \right) \|x + y\| - \frac{\|x\| - \|y\|}{\|x\| \|y\|} (\|x\| + \|y\|) \right] \\
 &\quad + \frac{1}{2} \left[\left(\frac{\|x\| - \|y\|}{\|x\| \|y\|} \right) \|x + y\| + \frac{\|x\| - \|y\|}{\|x\| \|y\|} (\|x\| - \|y\|) \right] \\
 &= \frac{1}{2} \left(\frac{\|x\| + \|y\|}{\|x\| \|y\|} \right) (\|x + y\| - \|x\| - \|y\|) \\
 &\quad + \frac{1}{2} \frac{\|x\| - \|y\|}{\|x\| \|y\|} (\|x + y\| + \|x\| - \|y\|)
 \end{aligned} \tag{4.3}$$

and since

$$\|x + y\| + \|\|x\| - \|y\|\| = \|x + y\| + \|\|x\| - \|y\|\|,$$

we get from (4.3) that

$$\begin{aligned}
 I &= \frac{1}{2} \|x + y\| \left[\frac{\|x\| + \|y\|}{\|x\| \|y\|} + \frac{\|x\| - \|y\|}{\|x\| \|y\|} \right] \\
 &\quad - \frac{1}{2} \|\|x\| - \|y\|\| \left[\frac{\|x\| + \|y\|}{\|x\| \|y\|} - \frac{\|x\| - \|y\|}{\|x\| \|y\|} \right].
 \end{aligned} \tag{4.4}$$

Moreover, it is clear that

$$\frac{1}{2} \left[\frac{\|x\| + \|y\|}{\|x\| \|y\|} + \frac{\|x\| - \|y\|}{\|x\| \|y\|} \right] = \max \left\{ \frac{1}{\|x\|}, \frac{1}{\|y\|} \right\} = \frac{1}{\min \{\|x\|, \|y\|\}}$$

and

$$\frac{1}{2} \left[\frac{\|x\| + \|y\|}{\|x\| \|y\|} - \frac{\|x\| - \|y\|}{\|x\| \|y\|} \right] = \min \left\{ \frac{1}{\|x\|}, \frac{1}{\|y\|} \right\} = \frac{1}{\max \{\|x\|, \|y\|\}}$$

and then, by (4.4) we deduce

$$I = \frac{\|x + y\|}{\min \{\|x\|, \|y\|\}} - \frac{\|\|x\| - \|y\|\|}{\max \{\|x\|, \|y\|\}}. \tag{4.5}$$

Secondly, if we define J by

$$\begin{aligned}
 J &:= \frac{1}{2} [(|\alpha| + |\beta|) \|x + y\| + |\alpha - \beta| (\|x\| + \|y\|)] \\
 &\quad - \frac{1}{2} [(|\alpha| - |\beta|) \|x + y\| - |\alpha - \beta| (\|x\| - \|y\|)]
 \end{aligned}$$

then for $\alpha = 1/\|y\|$ and $\beta = 1/\|x\|$ we get in a similar manner the equality

$$J = \frac{\|x+y\|}{\max\{\|x\|, \|y\|\}} + \frac{\||x\| - \|y\||}{\min\{\|x\|, \|y\|\}}. \quad (4.6)$$

Finally, by making use of the representations (4.5) and (4.6) we deduce from the inequality (3.1) that

$$\begin{aligned} 0 &\leq \frac{\|x+y\|}{\min\{\|x\|, \|y\|\}} - \frac{\||x\| - \|y\||}{\max\{\|x\|, \|y\|\}} \\ &\leq \left\| \frac{x}{\|y\|} + \frac{y}{\|x\|} \right\| \leq \frac{\|x+y\|}{\max\{\|x\|, \|y\|\}} + \frac{\||x\| - \|y\||}{\min\{\|x\|, \|y\|\}}. \end{aligned} \quad (4.7)$$

which is clearly equivalent with (4.2). \square

The second result looks slightly different:

THEOREM 3. For any two nonzero vectors $x, y \in X$ we have

$$\left\| \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| - \frac{\|x+y\|}{\min\{\|x\|, \|y\|\}} \right\| \leq \frac{\|x\| + \|y\|}{\max\{\|x\|, \|y\|\}} (\leq 2). \quad (4.8)$$

Proof. For $\alpha = \frac{1}{\|y\|}$ and $\beta = -\frac{1}{\|x\|}$ in the left side of (3.2), we have

$$\begin{aligned} &\max \left\{ \frac{1}{\|y\|} \|x+y\| - \frac{\|x\| + \|y\|}{\|x\| \|y\|} \cdot \|y\|, \frac{1}{\|x\|} \|x+y\| - \frac{\|x\| + \|y\|}{\|x\| \|y\|} \cdot \|x\| \right\} \\ &= \frac{1}{2} \left[\frac{\|x+y\| (\|x\| + \|y\|)}{\|x\| \|y\|} - \frac{(\|x\| + \|y\|)^2}{\|x\| \|y\|} \right] \\ &\quad + \frac{1}{2} \left| \frac{\|x+y\| (\|x\| - \|y\|)}{\|x\| \|y\|} + \frac{(\|x\| + \|y\|) (\|x\| - \|y\|)}{\|x\| \|y\|} \right| \\ &= \frac{1}{2} \cdot \frac{\|x+y\| (\|x\| + \|y\|)}{\|x\| \|y\|} - \frac{1}{2} \cdot \frac{(\|x\| + \|y\|)^2}{\|x\| \|y\|} \\ &\quad + \frac{1}{2} \cdot \frac{\||x\| - \|y\|| \|x+y\|}{\|x\| \|y\|} + \frac{1}{2} \cdot \frac{\|x\| - \|y\| (\|x\| + \|y\|)}{\|x\| \|y\|} \\ &= \frac{1}{2} \cdot \frac{\|x+y\|}{\|x\| \|y\|} [\|x\| + \|y\| + \||x\| - \|y\||] \\ &\quad - \frac{1}{2} \cdot \frac{\|x\| + \|y\|}{\|x\| \|y\|} [\|x\| + \|y\| - \||x\| - \|y\||] \\ &= \|x+y\| \max \left\{ \frac{1}{\|x\|}, \frac{1}{\|y\|} \right\} - (\|x\| + \|y\|) \min \left\{ \frac{1}{\|x\|}, \frac{1}{\|y\|} \right\} \\ &= \frac{\|x+y\|}{\min\{\|x\|, \|y\|\}} - \frac{\|x\| + \|y\|}{\max\{\|x\|, \|y\|\}}. \end{aligned}$$

On utilizing the first inequality in (3.2) we then conclude that

$$\frac{\|x+y\|}{\min\{\|x\|, \|y\|\}} - \frac{\|x\| + \|y\|}{\max\{\|x\|, \|y\|\}} \leq \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\|, \quad x, y \in X \setminus \{0\}. \quad (4.9)$$

We also have

$$\begin{aligned}
 & \min \left\{ \frac{\|x+y\|}{\|y\|} - \frac{\|x\| + \|y\|}{\|x\| \|y\|} \cdot \|y\|, \frac{\|x+y\|}{\|x\|} - \frac{\|x\| + \|y\|}{\|x\| \|y\|} \cdot \|x\| \right\} \\
 &= \frac{1}{2} \left[\frac{\|x+y\| (\|x\| + \|y\|)}{\|x\| \|y\|} + \frac{(\|x\| + \|y\|)^2}{\|x\| \|y\|} \right] \\
 &\quad - \frac{1}{2} \left| \frac{\|x+y\| (\|x\| - \|y\|)}{\|x\| \|y\|} - \frac{(\|x\| + \|y\|) (\|x\| - \|y\|)}{\|x\| \|y\|} \right| \\
 &= \frac{1}{2} \cdot \frac{\|x+y\| (\|x\| + \|y\|)}{\|x\| \|y\|} + \frac{1}{2} \cdot \frac{(\|x\| + \|y\|)^2}{\|x\| \|y\|} \\
 &\quad - \frac{1}{2} \cdot \frac{\|x\| - \|y\|}{\|x\| \|y\|} [\|x\| + \|y\| - \|x+y\|] \\
 &= \frac{1}{2} \cdot \frac{\|x+y\|}{\|x\| \|y\|} [\|x\| + \|y\| + \|x\| - \|y\|] \\
 &\quad + \frac{1}{2} \cdot \frac{\|x\| + \|y\|}{\|x\| \|y\|} [\|x\| + \|y\| - \|x\| - \|y\|] \\
 &= \|x+y\| \max \left\{ \frac{1}{\|x\|}, \frac{1}{\|y\|} \right\} + (\|x\| + \|y\|) \min \left\{ \frac{1}{\|x\|}, \frac{1}{\|y\|} \right\} \\
 &= \frac{\|x+y\|}{\min \{\|x\|, \|y\|\}} + \frac{\|x\| + \|y\|}{\max \{\|x\|, \|y\|\}}.
 \end{aligned}$$

On utilizing the second inequality in (3.2) we deduce

$$\left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \leq \frac{\|x+y\|}{\min \{\|x\|, \|y\|\}} + \frac{\|x\| + \|y\|}{\max \{\|x\|, \|y\|\}}. \tag{4.10}$$

The desired result (4.8) is clearly equivalent with (4.9) and (4.10) and the proof is complete. \square

5. Bounds for the Čebyšev Functional

For $\beta := (\beta_1, \dots, \beta_n) \in \mathbb{K}^n$ and $\mathbf{y} := (y_1, \dots, y_n) \in X^n$ we consider the *un-weighted Čebyšev functional* defined by

$$C_n(\beta, \mathbf{y}) := \frac{1}{n} \sum_{j=1}^n \beta_j y_j - \frac{1}{n} \sum_{j=1}^n \beta_j \cdot \frac{1}{n} \sum_{j=1}^n y_j.$$

This functional plays an important role in providing error bounds for approximating $\frac{1}{n} \sum_{j=1}^n \beta_j y_j$ by the simpler quantities $\frac{1}{n} \sum_{j=1}^n \beta_j$ and $\frac{1}{n} \sum_{j=1}^n y_j$.

We remark that, this functional has been considered previously by the author and some bounds have been established. We recall here some simple results.

With the above assumptions for X, α and \mathbf{y} , we have

$$\|C_n(\alpha, \mathbf{y})\| \leq \begin{cases} \frac{1}{12} (n^2 - 1) \max_{j \in \{1, \dots, n-1\}} |\Delta \alpha_j| \max_{j \in \{1, \dots, n-1\}} \|\Delta y_j\|, & [6]; \\ \frac{1}{2} \cdot \left(1 - \frac{1}{n}\right) \sum_{j=1}^{n-1} |\Delta \alpha_j| \sum_{j=1}^{n-1} \|\Delta y_j\|, & [3]; \\ \frac{1}{6} \frac{n^2 - 1}{n} \left(\sum_{j=1}^{n-1} |\Delta \alpha_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^{n-1} \|\Delta y_j\|^q \right)^{\frac{1}{q}}, & \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, & [2], \end{cases} \quad (5.1)$$

where $\Delta z_j = z_{j+1} - z_j$ is the forward difference. Here the constants $\frac{1}{12}, \frac{1}{2}$ and $\frac{1}{6}$ are best possible in the sense that they cannot be replaced by smaller quantities.

In [5] we also have established that

$$\|C_n(\alpha, \mathbf{y})\| \leq \frac{1}{n^2} \times \begin{cases} \max_{i \in \{1, \dots, n-1\}} \left| \det \left(\begin{array}{cc} i & n \\ \sum_{k=1}^i \alpha_k & \sum_{k=1}^n \alpha_k \end{array} \right) \right| \cdot \sum_{j=1}^{n-1} \|\Delta y_j\|; \\ \left(\sum_{i=1}^{n-1} \left| \det \left(\begin{array}{cc} i & n \\ \sum_{k=1}^i \alpha_k & \sum_{k=1}^n \alpha_k \end{array} \right) \right|^q \right)^{\frac{1}{q}} \cdot \left(\sum_{j=1}^{n-1} \|\Delta y_j\|^p \right)^{\frac{1}{p}} \\ \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} \left| \det \left(\begin{array}{cc} i & n \\ \sum_{k=1}^i \alpha_k & \sum_{k=1}^n \alpha_k \end{array} \right) \right| \cdot \max_{j \in \{1, \dots, n-1\}} \|\Delta y_j\|. \end{cases} \quad (5.2)$$

and

$$\|C_n(\alpha, \mathbf{y})\| \leq \frac{1}{n} \times \begin{cases} \max_{i \in \{1, \dots, n-1\}} \left| \frac{1}{n} \sum_{k=1}^n \alpha_k - \frac{1}{i} \sum_{k=1}^i \alpha_k \right| \cdot \sum_{i=1}^{n-1} i \|\Delta y_i\|; \\ \left(\sum_{i=1}^{n-1} i \left| \frac{1}{n} \sum_{k=1}^n \alpha_k - \frac{1}{i} \sum_{k=1}^i \alpha_k \right|^q \right)^{\frac{1}{q}} \cdot \left(\sum_{i=1}^{n-1} i \|\Delta y_i\|^p \right)^{\frac{1}{p}} \\ \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} i \left| \frac{1}{n} \sum_{k=1}^n \alpha_k - \frac{1}{i} \sum_{k=1}^i \alpha_k \right| \cdot \max_{i \in \{1, \dots, n-1\}} \|\Delta y_i\|. \end{cases} \quad (5.3)$$

Finally, we recall the following result from [4]:

If there exists the complex numbers $a, A \in \mathbb{C}$ such that

$$\operatorname{Re} [(A - \alpha_i) (\overline{\alpha_i} - \overline{a})] \geq 0 \quad \text{for each } i \in \{1, \dots, n\}$$

or, equivalently,

$$\left| \alpha_i - \frac{a+A}{2} \right| \leq \frac{1}{2} |A-a| \quad \text{for each } i \in \{1, \dots, n\},$$

then one has the inequality:

$$\|C_n(\beta, \mathbf{y})\| \leq \frac{1}{2} |A-a| \cdot \frac{1}{n} \sum_{i=1}^n \left\| y_i - \frac{1}{n} \sum_{j=1}^n y_j \right\|. \tag{5.4}$$

The constant $\frac{1}{2}$ in the right hand side of the inequality (5.4) is best possible in the sense that it cannot be replaced by a smaller quantity.

For many other results that hold for n -tuples β and \mathbf{y} of real numbers we recommend the chapters devoted to Grüss and Čebyšev inequalities from the books [12] and [14].

In the following we provide other upper and lower bounds for $\|C_n(\beta, \mathbf{y})\|$.

PROPOSITION 2. For any $\beta := (\beta_1, \dots, \beta_n) \in \mathbb{K}^n$ and $\mathbf{y} := (y_1, \dots, y_n) \in X^n$ we have

$$\begin{aligned} \|C_n(\beta, \mathbf{y})\| &\leq \min_{k \in \{1, \dots, n\}} \left\{ \frac{1}{n} \sum_{j=1}^n |\beta_j - \beta_k| \left\| y_j - \frac{1}{n} \sum_{\ell=1}^n y_\ell \right\| \right\} \\ &\leq \begin{cases} \min_{k \in \{1, \dots, n\}} \left\{ \max_{j \in \{1, \dots, n\}} \{|\beta_j - \beta_k|\} \right\} \frac{1}{n} \sum_{j=1}^n \left\| y_j - \frac{1}{n} \sum_{\ell=1}^n y_\ell \right\|; \\ \min_{k \in \{1, \dots, n\}} \left\{ \left[\frac{1}{n} \sum_{j=1}^n |\beta_j - \beta_k|^p \right]^{\frac{1}{p}} \right\} \left[\frac{1}{n} \sum_{j=1}^n \left\| y_j - \frac{1}{n} \sum_{\ell=1}^n y_\ell \right\|^q \right]^{\frac{1}{q}} \\ \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \min_{k \in \{1, \dots, n\}} \left\{ \frac{1}{n} \sum_{j=1}^n |\beta_j - \beta_k| \right\} \max_{j \in \{1, \dots, n\}} \left\{ \left\| y_j - \frac{1}{n} \sum_{\ell=1}^n y_\ell \right\| \right\}. \end{cases} \end{aligned} \tag{5.5}$$

Proof. We observe that

$$C_n(\beta, \mathbf{y}) = \frac{1}{n} \sum_{j=1}^n \beta_j \left(y_j - \frac{1}{n} \sum_{\ell=1}^n y_\ell \right).$$

Now, on applying the second inequality in Theorem 1 for $\alpha_j = \beta_j$ and $x_j = y_j - \frac{1}{n} \sum_{\ell=1}^n y_\ell$ we deduce the first part of (5.5). The second part is obvious by the Hölder inequality. \square

The following results can be stated as well:

PROPOSITION 3. For any $\beta := (\beta_1, \dots, \beta_n) \in \mathbb{K}^n$ and $\mathbf{y} := (y_1, \dots, y_n) \in X^n$ we have

$$\begin{aligned} & \max_{k \in \{1, \dots, n\}} \left\{ \left| \beta_k - \frac{1}{n} \sum_{\ell=1}^n \beta_\ell \right| \left\| \frac{1}{n} \sum_{j=1}^n y_j - z \right\| - \frac{1}{n} \sum_{j=1}^n |\beta_j - \beta_k| \|y_j - z\| \right\} \\ & \leq \|C_n(\beta, \mathbf{y})\| \\ & \leq \min_{k \in \{1, \dots, n\}} \left\{ \left| \beta_k - \frac{1}{n} \sum_{\ell=1}^n \beta_\ell \right| \left\| \frac{1}{n} \sum_{j=1}^n y_j - w \right\| + \frac{1}{n} \sum_{j=1}^n |\beta_j - \beta_k| \|y_j - w\| \right\} \end{aligned} \quad (5.6)$$

for any $z, w \in X$.

Proof. Follows from Theorem 1 on noticing that

$$C_n(\beta, \mathbf{y}) = \frac{1}{n} \sum_{j=1}^n \left(\beta_j - \frac{1}{n} \sum_{\ell=1}^n \beta_\ell \right) (y_j - t)$$

for any $t \in X$. □

REMARK 3. As a particular case, one can state the following inequality

$$\begin{aligned} & \max_{k \in \{1, \dots, n\}} \left\{ \left| \beta_k - \frac{1}{n} \sum_{\ell=1}^n \beta_\ell \right| \left\| \frac{1}{n} \sum_{j=1}^n y_j \right\| - \frac{1}{n} \sum_{j=1}^n |\beta_j - \beta_k| \|y_j\| \right\} \\ & \leq \|C_n(\beta, \mathbf{y})\| \\ & \leq \min_{k \in \{1, \dots, n\}} \left\{ \left| \beta_k - \frac{1}{n} \sum_{\ell=1}^n \beta_\ell \right| \left\| \frac{1}{n} \sum_{j=1}^n y_j \right\| + \frac{1}{n} \sum_{j=1}^n |\beta_j - \beta_k| \|y_j\| \right\} \end{aligned} \quad (5.7)$$

that provides simpler upper and lower bounds for the norm of the *unweighted Čebyšev functional* $C_n(\beta, \mathbf{y})$.

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REFERENCES

- [1] S. S. DRAGOMIR, *Bounds for the normalised Jensen functional*. Bull. Austral. Math. Soc., **74**(3) (2006), 471–478.
- [2] S. S. DRAGOMIR, *Another Grüss type inequality for sequences of vectors in normed linear spaces and applications*. J. Comput. Anal. Appl., **4**(2) (2002), 155–172.
- [3] S. S. DRAGOMIR, *A Grüss type inequality for sequences of vectors in normed linear spaces and applications*. Tamsui Oxf. J. Math. Sci., **20**(2) (2004), 143–159.
- [4] S. S. DRAGOMIR, *Grüss type discrete inequalities in normed linear spaces, revisited*. Nonlinear Funct. Anal. Appl., **9**(4) (2004), 577–591.
- [5] S. S. DRAGOMIR, *Bounding the Čebyšev functional for sequences of vectors in normed linear spaces*, Filomat, **18** (2004), 15–26.

- [6] S. S. DRAGOMIR AND G. L. BOOTH, *Grüss-Lupaş type inequality and its applications for the estimation of p -moments of guessing mappings*. Math. Commun., **5**(2) (2000), 117–126.
- [7] C. F. DUNKL AND K. S. WILLIAMS, *A simple norm inequality*. Amer. Math. Monthly, **71** (1964), 53–54.
- [8] M. KATO, K.-S. SAITO AND T. TAMURA, *Sharp triangle inequality and its reverse in Banach spaces*, Math. Inequal. & Appl., **10**(2) (2007), 461–470.
- [9] L. MALIGRANDA, *Simple norm inequalities*, Amer. Math. Monthly, **113** (2006), 256–260.
- [10] J. L. MASSERA AND J. J. SCHÄFFER, *Linear differential equations and functional analysis (I)*, Ann. of Math., **67** (1958), 517–573.
- [11] P. R. MERCER, *The Dunkl-Williams inequality in an inner product space*, Math. Inequal. & Appl., **10**(2) (2007), 447–450.
- [12] D. S. MITRINOVIĆ, J. PEČARIĆ AND A. M. FINK, *Classical and New Inequalities in Analysis*. Mathematics and its Applications (East European Series), 61. Kluwer Academic Publishers Group, Dordrecht, 1993.
- [13] J. PEČARIĆ AND R. RAJIĆ, *The Dunkl-Williams inequality with n elements in normed linear spaces*, Math. Inequal. & Appl., **10**(2) (2007), 461–470.
- [14] J. PEČARIĆ, F. PROSCHAN AND Y. L. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*. Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.

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