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GENERALIZED INTEGRAL OPERATORS RELATED WITH p–VALENT ANALYTIC FUNCTIONS

KHALIDA INAYAT NOOR AND MUHAMMAD ARIF

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Abstract. Let $\mathscr{A}(p), p \in N,$ be the class of functions $f : f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$, analytic in the unit disc E. For $n \in N_0,\ n > -p,$ an integral operator $I_{n+p-1} : \mathscr{A}(p) \longrightarrow \mathscr{A}(p)$ is defined as $I_{n+p-1}f = f_{n+p-1}^{(-1)} \star f$ such that $\left(f_{n+p-1}^{(-1)} \star f_{n+p-1}\right)(z) = \frac{z^p}{(1-z)^{p+1}}$ where $f_{n+p-1}(z) = \frac{z^p}{(1-z)^{n+p}}$ and $\star$ denotes convolution. Using this integral operator, some new classes $H_{n,p}(k,\alpha,\beta,\mu,\lambda)$ of $\mathscr{A}(p)$ are introduced and certain interesting properties of these classes are studied. A radius problem is also discussed.

1. Introduction

Let $\mathscr{A}(p)$ denote the class of functions

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \ (p \in N = \{1, 2, \ldots, \}), \tag{1.1}$$

which are analytic and p-valent in the unit disk $E = \{z : |z| < 1\}$. The class $\mathscr{A}(p)$ is closed under the Hadamard product or convolution $\star$ as:

$$(f_1 \star f_2)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k,1} a_{p+k,2} z^{p+k},$$

where

$$f_j(z) = z^p + \sum_{k=1}^{\infty} a_{p+k,j} z^{p+k}, \ (j = 1, 2).$$

Let

$$f_{n+p-1}(z) = \frac{z^p}{(1-z)^{n+p}} \ (n > -p)$$

and let $f_{n+p-1}^{(-1)}(z)$ be defined such that

$$\left(f_{n+p-1} \star f_{n+p-1}^{(-1)}\right)(z) = \frac{z^p}{(1-z)^{p+1}}. \tag{1.2}$$

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We define the integral operator $I_{n+p-1} : \mathscr{A}(p) \longrightarrow \mathscr{A}(p)$ as follows:

$$\begin{aligned} I_{n+p-1}f(z) &= \left(f_{n+p-1}^{(-1)} \star f\right)(z) \\ &= \left[\frac{z^p}{(1-z)^{n+p}}\right]^{(-1)} \star f(z), \ (n > -p). \end{aligned} \tag{1.3}$$

We note that

$$I_0 f = zf' \text{ and } I_1 f = f.$$

From (1.2) and (1.3), we obtain the following identity for the operator I_{n+p-1} :

$$z(I_{n+p}f)' = (n+1)I_{n+p-1}f - (n-p+1)I_{n+p}f. \tag{1.4}$$

It is clear, from (1.4), that

$$\frac{z(I_{n+p}f)'}{I_{n+p}f} = (n+1)\frac{I_{n+p-1}f}{I_{n+p}f} - (n+p-1),$$

and

$$\mathrm{Re}\left\{\frac{z(I_{n+p}f)'}{I_{n+p}f}\right\} > 0 \text{ and } \mathrm{Re}\left\{\frac{I_{n+p-1}f}{I_{n+p}f}\right\} > \frac{n-p+1}{n+1}.$$

For $p = 1$, the integral operator I_n was first introduced and studied by Noor in [5], which is known as Noor Integral operator. For more details, see [2, 6-8].

Let $P_k(\alpha)$ be the class of functions $h(z)$ analytic in the unit disk E satisfying the properties $h(0) = 1$ and

$$\int_0^{2\pi} \left|\frac{\mathrm{Re}\, h(z) - \alpha}{1-\alpha}\right| d\theta \leqslant k\pi, \tag{1.5}$$

where $z = re^{i\theta}$, $k \geqslant 2$ and $0 \leqslant \alpha < 1$. For $\alpha = 0$, we denote $P_k(0)$ as P_k and for $\alpha = 0, k = 2$, we have the class P of functions with positive real part. The case $k = 2$ gives us the class $P(\alpha)$ of functions with positive real part greater than α.

Also we can write, for $h(z) \in P_k(\alpha)$ as

$$h(z) = \frac{1}{2}\int_0^{2\pi} \frac{1 + (1-2\alpha)ze{-it}}{1 - ze^{-it}} d\mu(t), \tag{1.6}$$

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} d\mu(t) = 2 \text{ and } \int_0^{2\pi} |d\mu(t)| \leqslant k. \tag{1.7}$$

From (1.6) and (1.7), we can write, for $h \in P_k(\alpha)$,

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right)h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)h_2(z), \ h_1, h_2 \in P(\alpha). \tag{1.8}$$

We now define the following.

DEFINITION 1.1. Let $n > -p$, $\mu > 0$, $k \geqslant 2$, $0 \leqslant \alpha < 1$ and let $f \in \mathscr{A}(p)$. Then $f \in H_{n,p}(k,\alpha,\beta,\mu,\lambda)$ for $z \in E$, if it satisfies

$$\left\{(1-\lambda)\left(\frac{I_{n+p}f}{I_{n+p}g}\right)^{\mu} + \lambda\left(\frac{I_{n+p-1}f}{I_{n+p-1}g}\right)\left(\frac{I_{n+p}f}{I_{n+p}g}\right)^{\mu-1}\right\} \in P_k(\alpha),$$

where $g \in \mathscr{A}(p)$ satisfies the condition

$$\left\{\frac{I_{n+p-1}g}{I_{n+p}g}\right\} \in P(\beta),\ z \in E,\ \text{with}\ 0 \leqslant \beta = \frac{n-p+1}{n+1} < 1. \tag{1.9}$$

We note that, with $p = 1$, $n = 0$, g is starlike univalent in E.

2. Preliminary Results

To establish our main results, we need the following results.

LEMMA 2.1. [3]. *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let $\Psi(u,v)$ be a complex-valued function satisfying the conditions:*

(i) $\Psi(u,v)$ *is continuous in a domain* $\mathscr{D} \subset \mathscr{C}^2$,

(ii) $(1,0) \in \mathscr{D}$ *and* $\Psi(1,0) > 0$.

(iii) $\operatorname{Re}\Psi(iu_2, v_1) \leqslant 0$ *whenever* $(iu_2, v_1) \in \mathscr{D}$ *and* $v_1 \leqslant -\frac{1}{2}(1+u_2^2)$.

If $h(z) = 1 + \sum_{m=1}^{\infty} c_m z^m$ *is a function, analytic in* E, *such that* $(h(z), zh'(z)) \in \mathscr{D}$ *and* $\operatorname{Re}(h(z), zh'(z)) > 0$ *for* $z \in E$, *then* $\operatorname{Re} h(z) > 0$.

LEMMA 2.2. *Let $q(z)$ be analytic in E with $q(0) = 1$ and $\operatorname{Re} q(z) > 0$, $z \in E$. Then, for $|z| = r$, $z \in E$,*

(i)
$$\frac{1-r}{1+r} \leqslant \operatorname{Re} q(z) \leqslant |q(z)| \leqslant \frac{1+r}{1-r},$$

(ii)
$$|q'(z)| \leqslant \frac{2\operatorname{Re} q(z)}{1-r^2}.$$

This result is well-known, see [1].

LEMMA 2.3. [9]. *If $h(z)$ is analytic in E with $h(0) = 1$, and if λ is a complex number satisfying $\operatorname{Re}(\lambda) \geqslant 0$ $(\lambda \neq 0)$, then $\operatorname{Re}\{h(z) + \lambda z h'(z)\} > \alpha$ $(0 \leqslant \alpha < 1)$ implies*

$$\operatorname{Re} h(z) > \alpha + (1-\alpha)(2\gamma_1 - 1),$$

where γ_1 is given by

$$\gamma_1 = \int_0^1 (1 + t^{\operatorname{Re}\lambda})^{-1} dt,$$

which is an increasing function of $\operatorname{Re}\lambda$ and $\frac{1}{2} \leqslant \gamma_1 < 1$. This estimate is sharp in the sense that the bound cannot be improved.

3. Main Results

THEOREM 3.1. *Let* $f \in H_{n,p}(k,\alpha,\beta,\mu,\lambda)$, $\lambda \geqslant 0$. *Then* $\left(\frac{I_{n+p}f}{I_{n+p}g}\right)^{\mu} \in P_k(\gamma)$, *where*

$$\gamma = \frac{2\mu(n+1)\alpha + \lambda\delta}{2\mu(n+1)+\lambda\delta}, \tag{3.1}$$

and $g \in \mathscr{A}(p)$ *satisfies the condition (1.9) and*

$$\delta = \frac{\operatorname{Re} h_0(z)}{|h_0(z)|^2},\ h_0(z) = \frac{I_{n+p-1}g}{I_{n+p}g}.$$

Proof. Set

$$\left(\frac{I_{n+p}f}{I_{n+p}g}\right)^{\mu} = \left[(1-\gamma)h+\gamma\right], \tag{3.2}$$

$h(0)=1,$ and $h(z)$ is analytic in E and we write

$$h(z) = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z). \tag{3.3}$$

Simple calculations yield

$$\begin{aligned}(1-\lambda)\left(\frac{I_{n+p}f}{I_{n+p}g}\right)^{\mu} &+ \lambda\left(\frac{I_{n+p}f}{I_{n+p}g}\right)^{\mu-1}\frac{I_{n+p-1}f}{I_{n+p}g} - \alpha \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{(1-\gamma)h_1(z)+\gamma-\alpha+\frac{\lambda(1-\gamma)zh_1'}{\mu(n+1)h_0}\right\} \\ &\quad -\left(\frac{k}{4}-\frac{1}{2}\right)\left\{(1-\gamma)h_2(z)+\gamma-\alpha+\frac{\lambda(1-\gamma)zh_2'}{\mu(n+1)h_0}\right\}\end{aligned} \tag{3.4}$$

Now we form the functional $\Psi(u,v)$ by choosing $u = h_i(z) = u_1+iu_2$ and $v = zh_i'(z) = v_1+iv_2$. Thus

$$\Psi(u,v) = (1-\gamma)u + (\gamma-\alpha) + \frac{\lambda(1-\gamma)v}{u(n+1)h_0}.$$

The first two conditions of Lemma 2.1 are clearly satisfied. We verify the condition (iii) as follows.

$$\begin{aligned}\operatorname{Re}\psi(iu_2,v_1) &= \gamma-\alpha+\frac{\lambda(1-\gamma)v_1\operatorname{Re}h_0(z)}{\mu(n+1)|h_0(z)|^2} \\ &= (\gamma-\alpha)+\frac{\lambda(1-\gamma)\delta}{\mu(n+1)}v_1, \qquad \text{where } \delta = \frac{\operatorname{Re}h_0}{|h_0|^2}.\end{aligned}$$

Now, for $v_1 \leqslant -\frac{1}{2}(1+u_2^2)$, we have

$$\begin{aligned}
\operatorname{Re}\Psi(iu_2, v_1) &\leqslant (\gamma-\alpha) - \frac{\lambda(1-\gamma)(1+u_2^2)\delta}{2\mu(n+1)} \\
&= \frac{2\mu(n+1)(\gamma-\alpha) - \lambda\delta(1-\gamma) - \lambda\delta(1-\gamma)u_2^2}{2\mu(n+1)} \\
&= \frac{A+Bu_2^2}{2C}, \qquad C>0,
\end{aligned}$$

$$\begin{aligned}
A &= 2\mu(n+1)(\gamma-\alpha) - \lambda\delta(1-\gamma), \\
B &= -\lambda\delta(1-\gamma) \leqslant 0.
\end{aligned}$$

Now $\operatorname{Re}\Psi(iu_2, v_1) \leqslant 0$ if $A \leqslant 0$ and this gives us γ as defined by (3.1). We now apply Lemma 2.1 to conclude that $h_i \in P, z \in E$ and thus $h \in P_k$ which gives us the required result. □

We note that $\gamma = \alpha$ when $\beta = 0$.

THEOREM 3.2. *For* $\lambda \geqslant 1$, *let* $f \in H_{n,p}(k, \alpha, 0, 1, \lambda)$. *Then*

$$\frac{I_{n+p-1}f}{I_{n+p-1}g} \in P_k(\alpha), \quad \textit{for } z \in E.$$

Proof. We can write, for $\lambda \geqslant 1$,

$$\lambda \frac{I_{n+p-1}f}{I_{n+p-1}g} = \left[(1-\lambda)\frac{I_{n+p}f}{I_{n+p}g} + \lambda \frac{I_{n+p-1}f}{I_{n+p-1}g}\right] + (\lambda-1)\frac{I_{n+p}f}{I_{n+p}g}.$$

This implies that

$$\begin{aligned}
\frac{I_{n+p-1}f}{I_{n+p-1}g} &= \frac{1}{\lambda}\left[(1-\lambda)\frac{I_{n+p}f}{I_{n+p}g} + \lambda \frac{I_{n+p-1}f}{I_{n+p-1}g}\right] + (1-\frac{1}{\lambda})\frac{I_{n+p}f}{I_{n+p}g} \\
&= \frac{1}{\lambda}H_1 + (1-\frac{1}{\lambda})H_2.
\end{aligned}$$

Since $H_1, H_2 \in P_k(\alpha)$, by Theorem 3.1, Definition 1.1 and $P_k(\alpha)$ is a convex set, see [4], we obtain the required result. □

THEOREM 3.3. *Let* λ *be a complex number satisfying* $\operatorname{Re}\lambda > 0$. *Let* $f \in \mathscr{A}(p)$ *and satisfy the condition*

$$\left\{(1-\lambda)\left(\frac{I_{n+p}f}{z^p}\right)^{\mu} + \lambda \frac{I_{n+p-1}f}{z^p}\left(\frac{I_{n+p}f}{z^p}\right)^{\mu-1}\right\} \in P_k(\alpha), \quad \textit{for } \mu > 0.$$

Then, for $z \in E$, $\left(\frac{I_{n+p}f}{z^p}\right)^{\mu} \in P_k(\sigma)$, *where*

$$\sigma = \alpha + (1-\alpha)(2\rho-1) \quad \textit{with } \rho = \int_0^1 \left(1 + t^{\frac{\lambda}{\mu(n+1)}}\right)^{-1} dt.$$

The value of σ is best possible and cannot be improved.

Proof. We set

$$\left(\frac{I_{n+p}f}{z^p}\right)^{\mu} = h(z) = \Big(\frac{k}{4}+\frac{1}{2}\Big)h_1(z) - \Big(\frac{k}{4}-\frac{1}{2}\Big)h_2(z),$$

where $h(0) = 1$ and h is analytic in E. Then

$$\left\{(1-\lambda)\left(\frac{I_{n+p}f}{z^p}\right)^{\mu} + \lambda\frac{I_{n+p-1}f}{z^p}\left(\frac{I_{n+p}f}{z^p}\right)^{\mu-1}\right\} = \left[h(z)+\frac{\lambda z h'(z)}{\mu(n+1)}\right] \in P_k(\alpha), \quad z \in E.$$

Using Lemma 2.3, we note that $h_i \in P(\sigma)$,

$$\begin{aligned} \sigma &= \alpha + (1-\alpha)(2\rho-1), \\ \rho &= \int_0^1 \left(1+t^{\frac{\lambda}{\mu(n+1)}}\right)^{-1} dt \end{aligned} \tag{3.5}$$

and consequently $h \in P_k(\sigma)$ and this gives the required result. □

We remark that ρ given by (3.5) can be expressed in terms of hypergeometric function as

$$\begin{aligned} \rho &= \int_0^1 \left(1+t^{\frac{\lambda}{\mu(n+1)}}\right)^{-1} dt \\ &= \frac{\mu(n+1)}{\lambda_1}\int_0^1 u^{\left\{\frac{\mu(n+1)}{\lambda_1}\right\}-1}(1+u)^{-1}du, \ (\lambda_1 = \operatorname{Re}\lambda > 0) \\ &= {}_2F_1\Big(1, \frac{\mu(n+1)}{\lambda_1}; 1+\frac{\mu(n+1)}{\lambda_1}; -1\Big) \\ &= \frac{1}{2}\left[{}_2F_1\Big(1,1;1+\frac{\mu(n+1)}{\lambda_1};\frac{1}{2}\Big)\right]. \end{aligned}$$

THEOREM 3.4. *For* $0 \leqslant \lambda_2 < \lambda_1$, $H_{n,p}(k,\alpha,0,\mu,\lambda_1) \subset H_{n,p}(k,\alpha,0,\mu,\lambda_2)$.

Proof. If $\lambda_2 = 0$, then the proof is immediate from Theorem 3.1. We let $\lambda_2 > 0$ and $f \in H_{n,p}(k,\alpha,0,\mu,\lambda_1)$. There exist two functions $H_1, H_2 \in P_k(\alpha)$ such that

$$\left\{(1-\lambda_1)\left(\frac{I_{n+p}f}{I_{n+p}g}\right)^{\mu} + \lambda_1\frac{I_{n+p-1}f}{I_{n+p}g}\left(\frac{I_{n+p}f}{I_{n+p}g}\right)^{\mu-1}\right\} = H_1(z)$$

and

$$\left(\frac{I_{n+p}f}{I_{n+p}g}\right)^{\mu} = H_2(z).$$

Then

$$\left\{(1-\lambda_2)\left(\frac{I_{n+p}f}{I_{n+p}g}\right)^{\mu} + \lambda_2\frac{I_{n+p-1}f}{I_{n+p-1}g}\left(\frac{I_{n+p}f}{I_{n+p}g}\right)^{\mu-1}\right\} = \frac{\lambda_2}{\lambda_1}H_1(z)+\Big(1-\frac{\lambda_2}{\lambda_1}\Big)H_2(z) \tag{3.6}$$

and since $P_k(\alpha)$ is a convex set, see [4], it follows that the right hand side of (3.6) belongs to $P_k(\alpha)$ and this completes the proof. □

We now consider the converse case of Theorem 3.1. as follows.

THEOREM 3.5. *Let* $\left(\frac{I_{n+p}f}{I_{n+p}g}\right)^{\mu} \in P_k(\alpha)$ *with* $\left\{\frac{I_{n+p-1}g}{I_{n+p}g}\right\} \in P(\beta)$, *for* $z \in E$. *Then* $f \in H_{n,p}(k,\alpha,\beta,\mu,\lambda)$ *for* $|z| < r$, *where* r *is given as*

$$r = \frac{\mu(n+1)}{\{(1-\beta)\mu(n+1)+\lambda\}+\sqrt{(\beta\mu(n+1))^2+\lambda^2+2\lambda(1-\beta)\mu(n+1)}}. \tag{3.7}$$

Proof. Let

$$\begin{aligned} H &= \left(\frac{I_{n+p}f}{I_{n+p}g}\right)^{\mu}, \\ H_0 &= \frac{I_{n+p-1}g}{I_{n+p}g}, \end{aligned}$$

then $H \in P_k(\alpha)$, $H_0 \in P(\beta)$. Proceeding as in Theorem 3.1, for $n > -p$, $\mu > 0$, $k \geqslant 2$, $\operatorname{Re}\lambda \geqslant 0$, $0 \leqslant \alpha, \beta < 1$, and

$$\begin{aligned} H &= (1-\alpha)h + \alpha, \\ H_0 &= (1-\beta)h_0 + \beta, \quad \text{with } h \in P_k,\ h_0 \in P, \end{aligned}$$

we have

$$\begin{aligned} \frac{1}{1-\alpha}\left\{(1-\lambda)\left(\frac{I_{n+p}f}{I_{n+p}g}\right)^{\mu} \right. &+ \left. \lambda\frac{I_{n+p-1}f}{I_{n+p}g}\left(\frac{I_{n+p}f}{I_{n+p}g}\right)^{\mu-1}\right\} - \alpha \\ &= \left\{h(z) + \frac{\lambda}{\mu(n+1)}\frac{zh'(z)}{(1-\beta)h_0(z)+\beta}\right\} \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left[h_1(z) + \frac{\lambda}{\mu(n+1)}\frac{zh_1'(z)}{\{(1-\beta)h_0(z)+\beta}\right] \\ &\quad -\left(\frac{k}{4}-\frac{1}{2}\right)\left[h_2(z) + \frac{\lambda}{\mu(n+1)}\frac{zh_2'(z)}{\{(1-\beta)h_0(z)+\beta\}}\right]. \end{aligned}$$

Using well known results, see [1], for $h_i \in P$,

$$\begin{aligned} |zh_i'(z)| &\leqslant \frac{2r\operatorname{Re}h_i(z)}{1-r^2}, \\ \frac{1-r}{1+r} &\leqslant |h_i(z)| \leqslant \frac{1+r}{1-r}, \end{aligned}$$

we have

$$\begin{aligned}
\operatorname{Re}\left[h_i(z)+\frac{\lambda}{\mu(n+1)}\frac{zh_i'(z)}{[(1-\beta)h_0(z)+\beta]}\right]
&\geqslant \operatorname{Re}h_i(z)\left\{1-\frac{2\lambda r}{\mu(n+1)}\frac{1}{1-r^2}\left(\frac{1+r}{(1-(1-2\beta)r)}\right)\right\}\\
&= \operatorname{Re}h_i(z)\left\{1-\frac{2\lambda r}{\mu(n+1)(1-r)(1-(1-2\beta)r)}\right\}\\
&= \operatorname{Re}h_i(z)\left\{\frac{\mu(n+1)[1-r-(1-2\beta)r+(1-2\beta)r^2]-2\lambda r}{\mu(n+1)(1-r)(1-(1-(1-2\beta)r)}\right\}\\
&\geqslant \operatorname{Re}h_i(z)\left[\frac{\mu(n+1)(1-2\beta)r^2-2[(1-\beta)\mu(n+1)+\lambda]r+\mu(n+1)}{\mu(n+1)(1-r)\{1-(1-2\beta)r\}}\right]. \qquad (3.8)
\end{aligned}$$

Right hand of (3.8) is positive for $|z|<r,$ where r is given by (3.7). $\square$

We note that, for $p=1,\ n=0,\ \mu=1,$ and $\lambda=1,\ \beta=0,\ \left[\frac{f}{g}\right]\in P_k(\alpha),$ for $z\in E$ implies $\left[\frac{f'}{g'}\right]\in P_k(\alpha)$ for $|z|<R=\frac{1}{2+\sqrt{3}}$.

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REFERENCES

[1] A. W. GOODMAN, *Univalent Functions, Vol. I, II*, Polygonal Publishing House, Washington, N. J., 1983.
[2] J. LIU AND K. I. NOOR, *Some properties of Noor integral operator*, J. Nat. Geometry, **21**(2002), 81-90.
[3] S. S. MILLER, *Differential inequalities and Caratheodory functions*, Bull. Amer. Math. Soc., **81**(1975), 79-81.
[4] K. I. NOOR, *On subclasses of close-to-convex functions of higher order*, Inter. J. Math. Math. Sci., 15(1992), 279-290.
[5] K. I. NOOR, *On new classes of integral operators*, J. Nat. Geometry, **16**(1999), 71-80.
[6] K. I. NOOR, *Some classes of p-valent analytic functions defined by certain integral operators*, Appl. Math. Computation, **157**(2004), 835-840.
[7] K. I. NOOR, *Generalized integral operator and multivalent functions*, J. Inequal. Pure Appl. Math., 6(2005), 1-7, article xx.
[8] K. I. NOOR AND M. A. NOOR, *On integral operators*, J. Math. Anal. Appl., **238**(1999), 341-352.
[9] S. PONNUSAMY, *Differential subordination and Bazilevic functions*, Proc. Ind. Acad. Sci., **105**(1995), 169-186.

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Khalida Inayat Noor
Mathematics Department
COMSATS Institute of Information Technology
Islamabad
Pakistan
e-mail: khalidainayat@comsats.edu.pk

Mohammad Arif
Mathematics Department
COMSATS Institute of Information Technology
Islamabad
Pakistan