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A STUDY ON ABSOLUTE SUMMABILITY FACTORS FOR A TRIANGULAR MATRIX

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Abstract. In this paper we obtain an absolute summability factor theorem for lower triangular matrices.

Sulaiman [2] obtained sufficient conditions for $\sum a_n\lambda_n$ to be $|\overline{N}, p_n|_k$ summable, $k \in \mathbb{N}$. Unfortunately he used an incorrect definition of absolute summability (see, e.g., [1]). In this paper we obtain the corresponding summability factor theorem for a lower triangular matrix, and obtain the correct form of [2] as a special case. Let A be a lower triangular matrix, $\{s_n\}$ a sequence. Then we put

$$A_n := \sum_{v=0}^{n} a_{nv}s_v.$$

A series $\sum a_n$ is said to be $|A|_k$ summable, $k \in \mathbb{N}$ if

$$\sum_{n=1}^{\infty} n^{k-1}|A_n - A_{n-1}|^k < \infty. \tag{1}$$

We may associate with A two lower triangular matrices $\overline{A}$ and $\hat{A}$ defined as follows:

$$\bar{a}_{nv} = \sum_{r=v}^{n} a_{nr}, \qquad n, v \in \mathbb{N}_0$$

and

$$\hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \qquad n = 1, 2, 3, \ldots.$$

Also we shall define

$$y_n := \sum_{i=0}^{n} a_{ni}s_i = \sum_{i=0}^{n} a_{ni} \sum_{v=0}^{i} \lambda_v a_v$$
$$= \sum_{v=0}^{n} \lambda_v a_v \sum_{i=v}^{n} a_{ni} = \sum_{v=0}^{n} \bar{a}_{nv}\lambda_v a_v$$

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and

$$Y_n := y_n - y_{n-1} = \sum_{\nu=0}^{n} (\bar{a}_{n\nu} - \bar{a}_{n-1,\nu})\lambda_\nu a_\nu = \sum_{\nu=0}^{n} \hat{a}_{n\nu}\lambda_\nu a_\nu. \tag{2}$$

Given any sequence $\{x_n\}$, the notation $x_n \asymp O(1)$ means $x_n = O(1)$ and $1/x_n = O(1)$. For any matrix entry $a_{n\nu}$, $\Delta_\nu a_{n\nu} := a_{n\nu} - a_{n\nu+1}$.

THEOREM 1. *Let A be a lower triangular matrix with nonnegative entries such that*

(i) $\bar{a}_{n0} = 1, n = 0, 1, \ldots,$
(ii) $a_{n-1,\nu} \geqslant a_{n\nu}$ for $\mathrm{n} \geqslant \nu + 1$,
(iii) $na_{nn} \asymp O(1)$, *and*
(iv) $\sum_{\nu=1}^{n-1} a_{\nu\nu}|\hat{a}_{n,\nu+1}| = O(a_{nn})$.

Let t_n^1 denote the nth $(C,1)$ mean of $\{na_n\}$. If

(v) $\displaystyle\sum_{\nu=1}^{\infty} a_{\nu\nu}|\lambda_\nu|^k|t_\nu^1|^k = O(1)$,
(vi) $\displaystyle\sum_{\nu=1}^{\infty} |a_{\nu\nu}|^{1-k}|\Delta\lambda_\nu|^k|t_\nu^1|^k = O(1)$,

then the series $\sum a_n\lambda_n$ is $|A|_k$ summable, $k \in \mathbb{N}$.

Proof. From (i) it follows that $\hat{a}_{n,0} = 0$.
Using (2) we may write

$$\begin{aligned}
Y_n &= \sum_{\nu=1}^{n} \left(\frac{\hat{a}_{n\nu}\lambda_\nu}{\nu}\right)\nu a_\nu \\
&= \sum_{\nu=1}^{n} \left(\frac{\hat{a}_{n\nu}\lambda_\nu}{\nu}\right)\Big[\sum_{r=1}^{\nu} ra_r - \sum_{r=1}^{\nu-1} ra_r\Big] \\
&= \sum_{\nu=1}^{n-1} \Delta_\nu\Big(\frac{\hat{a}_{n\nu}\lambda_\nu}{\nu}\Big)\sum_{r=1}^{\nu} ra_r + \frac{\hat{a}_{nn}\lambda_n}{n}\sum_{\nu=1}^{n} \nu a_\nu \\
&= \sum_{\nu=1}^{n-1} (\Delta_\nu\hat{a}_{n\nu})\lambda_\nu \frac{\nu+1}{\nu} t_\nu^1 + \sum_{\nu=1}^{n-1} \hat{a}_{n,\nu+1}(\Delta\lambda_\nu)\frac{\nu+1}{\nu} t_\nu^1 \\
&\quad + \sum_{\nu=1}^{n-1} \hat{a}_{n,\nu+1}\lambda_{\nu+1}\frac{1}{\nu} t_\nu^1 + \frac{(n+1)a_{nn}\lambda_n t_n^1}{n} \\
&= T_{n1} + T_{n2} + T_{n3} + T_{n4}, \quad \text{say.}
\end{aligned}$$

To prove the theorem it will be sufficient, by Minkowski's inequality, to show that

$$\sum_{n=1}^{\infty} n^{k-1}|T_{nr}|^k < \infty, \quad \text{for} \quad \mathrm{r} = 1, 2, 3, 4.$$

Using Hölder's inequality and (iii) and (v),

$$
\begin{aligned}
I_1 :&= \sum_{n=1}^{m} n^{k-1}|T_{n1}|^k = \sum_{n=1}^{m} n^{k-1}\Big|\sum_{\nu=1}^{n-1}\Delta_\nu \hat{a}_{n\nu}\lambda_\nu \frac{\nu+1}{\nu} t_\nu^1\Big|^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1}\Big(\sum_{\nu=1}^{n-1}|\Delta_\nu \hat{a}_{n\nu}||\lambda_\nu||t_\nu^1|\Big)^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1}\Big(\sum_{\nu=1}^{n-1}|\Delta_\nu \hat{a}_{n\nu}||\lambda_\nu|^k|t_\nu^1|^k\Big)\times\Big(\sum_{\nu=1}^{n-1}|\Delta_\nu \hat{a}_{n\nu}|\Big)^{k-1}.
\end{aligned}
$$

From (ii)

$$
\begin{aligned}
\Delta_\nu \hat{a}_{n\nu} &= \hat{a}_{n\nu} - \hat{a}_{n,\nu+1} \\
&= \bar{a}_{n\nu} - \bar{a}_{n-1,\nu} - \bar{a}_{n,\nu+1} + \bar{a}_{n-1,\nu+1} \\
&= a_{n\nu} - a_{n-1,\nu} \leqslant 0.
\end{aligned}
$$

Thus from(i)

$$
\sum_{\nu=0}^{n-1}|\Delta_\nu \hat{a}_{n\nu}| = \sum_{\nu=0}^{n-1}(a_{n-1,\nu} - a_{n\nu}) = 1 - 1 + a_{nn} = a_{nn}.
$$

Using (iii) and (v)

$$
\begin{aligned}
I_1 :&= \sum_{n=1}^{m} n^{k-1}|T_{n1}|^k = O(1)\sum_{n=1}^{m+1}(na_{nn})^{k-1}\sum_{\nu=1}^{n-1}|\lambda_\nu|^k|t_\nu^1|^k|\Delta_\nu \hat{a}_{n\nu}| \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_\nu|^k|t_\nu^1|^k\sum_{n=\nu+1}^{m+1}(na_{nn})^{k-1}|\Delta_\nu \hat{a}_{n\nu}| \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_\nu|^k|t_\nu^1|^k\sum_{n=\nu+1}^{m+1}|\Delta_\nu \hat{a}_{n\nu}| \\
&= O(1)\sum_{\nu=1}^{m}a_{\nu\nu}|\lambda_\nu|^k|t_\nu^1|^k \\
&= O(1).
\end{aligned}
$$

Using Hölder's inequality, (iii) and (iv),

$$
\begin{aligned}
I_2 :&= \sum_{n=2}^{m+1} n^{k-1}|T_{n2}|^k = \sum_{n=2}^{m+1} n^{k-1}\Big|\sum_{\nu=1}^{n-1}\hat{a}_{n,\nu+1}(\Delta\lambda_\nu)\frac{\nu+1}{\nu}t_\nu^1\Big|^k \\
&\leqslant \sum_{n=2}^{m+1} n^{k-1}\Big[\sum_{\nu=1}^{n-1}\hat{a}_{n,\nu+1}|\Delta\lambda_\nu|\frac{\nu+1}{\nu}|t_\nu^1|\Big]^k
\end{aligned}
$$

$$= O(1)\sum_{n=2}^{m+1} n^{k-1}\Big[\sum_{v=1}^{n-1}|\Delta\lambda_v||t_v^1||\hat{a}_{n,v+1}|\Big]^k$$
$$= O(1)\sum_{n=2}^{m+1} n^{k-1}\Big[\sum_{v=1}^{n-1}|\Delta\lambda_v|^k|t_v^1|^k|a_{vv}|^{1-k}|\hat{a}_{n,v+1}|\Big]\times\Big[\sum_{v=1}^{n-1}|a_{vv}||\hat{a}_{n,v+1}|\Big]^{k-1}$$
$$= O(1)\sum_{n=2}^{m+1}(na_{nn})^{k-1}\sum_{v=1}^{n-1}|a_{vv}|^{1-k}||\Delta\lambda_v|^k|t_v^1|^k|\hat{a}_{n,v+1}|.$$

Using (iii)

$$I_2 = O(1)\sum_{v=1}^{m}|a_{vv}|^{1-k}|\Delta\lambda_v|^k|t_v^1|^k\sum_{n=v+1}^{m+1}(na_{nn})^{k-1}\hat{a}_{n,v+1}$$
$$= O(1)\sum_{v=1}^{m}|a_{vv}|^{1-k}|\Delta\lambda_v|^k|t_v^1|^k\sum_{n=v+1}^{m+1}\hat{a}_{n,v+1}.$$

From the definition of $\hat{A}$ and using (i) and (ii);

$$\begin{aligned}\hat{a}_{n,v+1} &= \bar{a}_{n,v+1}-\bar{a}_{n-1,v+1}\\ &= \sum_{i=v+1}^{n}a_{ni}-\sum_{i=v+1}^{n-1}a_{n-1,i}\\ &= 1-\sum_{i=0}^{v}a_{ni}-1+\sum_{i=0}^{v}a_{n-1,i}\\ &= \sum_{i=0}^{v}\Big(a_{n-1,i}-a_{n,i}\Big)\geqslant 0. \qquad (3)\end{aligned}$$

From (3)

$$\begin{aligned}\sum_{n=v+1}^{m+1}\Big(\sum_{i=0}^{v}(a_{n-1,i}-a_{ni})\Big) &= \sum_{i=0}^{v}\sum_{n=v+1}^{m+1}(a_{n-1,i}-a_{ni})\\ &= \sum_{i=0}^{v}(a_{v,i}-a_{m+1,i})\\ &\leqslant \sum_{i=0}^{v}a_{v,i}=1. \qquad (4)\end{aligned}$$

Therefore, using (vi), $I_2 = O(1)$.

Using Hölder's inequality, (iii), (iv) and (v)

$$\sum_{n=2}^{m+1}n^{k-1}|T_{n3}|^k = \sum_{n=2}^{m+1}n^{k-1}\Big|\sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}\lambda_{v+1}}{v}t_v^1\Big|^k$$
$$= O(1)\sum_{n=2}^{m+1}n^{k-1}\Big[\sum_{v=1}^{n-1}|a_{vv}||\lambda_{v+1}||\hat{a}_{n,v+1}||t_v^1|\Big]^k$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m+1} n^{k-1}\Big[\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}|^k|a_{\nu\nu}||t_\nu^1|^k|\hat{a}_{n,\nu+1}|\Big]\times\Big[\sum_{\nu=1}^{n-1}|a_{\nu\nu}||\hat{a}_{n,\nu+1}|\Big]^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}(na_{nn})^{k-1}\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}|^k|a_{\nu\nu}||t_\nu^1|^k|\hat{a}_{n,\nu+1}|\\
&= O(1)\sum_{\nu=1}^{m}|\lambda_{\nu+1}|^k|a_{\nu\nu}||t_\nu^1|^k\sum_{n=\nu+1}^{m+1}\hat{a}_{n,\nu+1}\\
&= O(1)\sum_{\nu=1}^{m}a_{\nu\nu}|\lambda_{\nu+1}|^k|t_\nu^1|^k\\
&= O(1).
\end{aligned}
$$

Finally using (iii) and (v)

$$
\begin{aligned}
\sum_{n=1}^{m} n^{k-1}|T_{n4}|^k &= \sum_{n=1}^{m} n^{k-1}\left|\frac{(n+1)a_{nn}\lambda_n t_n^1}{n}\right|^k\\
&= O(1)\sum_{n=1}^{m} n^{k-1}|a_{nn}|^k|\lambda_n|^k|t_n^1|^k\\
&= O(1)\sum_{n=1}^{m}(na_{nn})^{k-1}a_{nn}|\lambda_n|^k|t_n^1|^k\\
&= O(1)\sum_{n=1}^{m}a_{nn}|\lambda_n|^k|t_n^1|^k\\
&= O(1).
\end{aligned}
$$

□

COROLLARY 1. *Let* $\{p_n\}$ *be a positive sequence such that* $P_n := \sum_{k=0}^{n} p_k \to \infty$, *and satisfies*

(i) $np_n \asymp O(P_n)$.

Let t_n^1 *denote the nth* $(C,1)$ *mean of* $\{na_n\}$. *If*

(ii) $\sum_{\nu=1}^{\infty} \nu^{-1}|\lambda_\nu|^k|t_\nu^1|^k = O(1)$,

(iii) $\sum_{\nu=1}^{\infty} \nu^{k-1}|\Delta\lambda_\nu|^k|t_\nu^1|^k = O(1)$, *then the series* $\sum a_n\lambda_n$ *is* $|\overline{N},p_n|_k$ *summable,* $k \in \mathbb{N}$.

Proof. Conditions (i), (ii) and (iv) of Theorem 1 are automatically satisfied for any weighted mean method. Condition (iii) of Theorem 1 becomes condition (i) of Corollary 1. Conditions (v) and (vi) of Theorem 1 become conditions (ii) and (iii)) of Corollary 1. □

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