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NOTE ON A CONJECTURE OF R. A. SATNOIANU

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Abstract. In this paper, a conjecture posed in [R. A. Satnoianu, The proof of the conjectured inequality from the Mathematical Olympiad, Washington DC 2001, Gazeta Matematica, 106 (2001), 390–393] is studied once again, a generalization of Satnoianu's inequality is established.

1. Introduction

In [1], R. A. Satnoianu studied the inequality

$$\sum_{cyclic} \frac{a}{\sqrt{a^2+\lambda bc}} \geqslant \frac{3}{\sqrt{1+\lambda}} \quad (a,b,c>0,\ \lambda \geqslant 8), \tag{1}$$

and conjectured that the above inequality can be extended to a more general case. Satnoianu then proposed the following inequality as a conjecture.

CONJECTURE.

$$\sum_{i=1}^{n} \left(\frac{x_i^{n-1}}{x_i^{n-1} + \lambda \prod_{k\neq i} x_k} \right)^{\frac{1}{n-1}} \geqslant n(1+\lambda)^{-\frac{1}{n-1}} \tag{2}$$

for all $n \geqslant 2,\ x_i > 0,\ i=1,2,\dots,n$ *and any* $\lambda \geqslant n^{n-1}-1$.

Inequality (2) has attracted interest of mathematicians. In [2], W. Janous proved the validity of inequality (2) by means of Lagrange's method of multipliers. Shortly after the publication of Janous's paper, by using the *GA*-convexity inequality, R. A. Satnoianu [3] established a generalized version of inequality (2) as follows

$$\sum_{i=1}^{n} \left(\frac{x_i^{n-1}}{\alpha x_i^{n-1} + \beta \prod_{k\neq i} x_k} \right)^{\frac{1}{n-1}} \geqslant n(\alpha+\beta)^{-\frac{1}{n-1}}, \tag{3}$$

where $n \geqslant 2,\ x_i > 0,\ i=1,2,\dots,n$, $\alpha,\beta > 0$ and $\beta \geqslant (n^{n-1}-1)\alpha$.

Satnoianu's inequality (3) prompts us to ask a natural question: *Under what condition is the inequality (3) still true if the exponent* $\frac{1}{n-1}$ *is replaced by any real number* $\frac{1}{p}$ *?*

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A satisfying answer to this question is given by the following theorem.

THEOREM 1. *Let* $\alpha,\ \beta,\ x_i\,(i=1,2,\ldots,n)$ *be positive real numbers, and let* q *be a real numbers. Then for* $p<0$*, or* $p>0$ *with* $\beta \geqslant (n^{\max\{p,1\}}-1)\alpha$*, the following inequality holds true*

$$\sum_{i=1}^{n}\left(\frac{x_i^q}{\alpha x_i^q+\beta\prod_{k=1}^{n}x_k^{q/n}}\right)^{1/p} \geqslant n(\alpha+\beta)^{-1/p}. \tag{4}$$

2. A Set of Lemmas

In order to prove Theorem 1, we first introduce the following lemmas.

LEMMA 1. (see Wu and Debnath [4]) *Let* $p_j>0,\ \ a_{ij}>0\ \ (i=1,2,\ldots,n,$ $j=1,2,\ldots,m)$. *Then*

$$\prod_{j=1}^{m}\left(\sum_{i=1}^{n}a_{ij}^{\frac{1}{p_j}}\right)^{p_j} \geqslant n^{\min\{p_1+p_2+\cdots+p_m-1,\ 0\}}\sum_{i=1}^{n}\prod_{j=1}^{m}a_{ij}\,. \tag{5}$$

LEMMA 2. (see Wu [5]) *Let* $x_i>0\ \ (i=1,2,\ldots,n)$ *and* $p\geqslant 2$. *Then*

$$\left(\sum_{i=1}^{n}x_i\right)^{p} \geqslant \sum_{i=1}^{n}x_i^p+(n^p-n)\left(\prod_{i=1}^{n}x_i\right)^{p/n}. \tag{6}$$

LEMMA 3. *Let* $a_i>0,\ b_i>0\ \ (i=1,2,\ldots,n)$ *and* $rs>0$. *Then*

$$\sum_{i=1}^{n}\frac{a_i^r}{b_i^s} \geqslant n^{\min\{s-r+1,\,0\}}\left(\sum_{i=1}^{n}a_i\right)^{r}\Big/\left(\sum_{i=1}^{n}b_i\right)^{s}. \tag{7}$$

Proof. Case (I): When $r>0,\ s>0$. Using Lemma 1, we obtain

$$\left(\sum_{i=1}^{n}b_i\right)^{s/r}\left(\sum_{i=1}^{n}\frac{a_i^r}{b_i^s}\right)^{1/r} \geqslant n^{\min\{\frac{s}{r}+\frac{1}{r}-1,\,0\}}\left(\sum_{i=1}^{n}b_i^{s/r}\cdot\frac{a_i}{b_i^{s/r}}\right),$$

thus

$$\sum_{i=1}^{n}\frac{a_i^r}{b_i^s} \geqslant n^{\min\{s-r+1,\,0\}}\left(\sum_{i=1}^{n}a_i\right)^{r}\Big/\left(\sum_{i=1}^{n}b_i\right)^{s}. \tag{8}$$

Case (II): When $r<0,\ s<0$. It implies that $-r>0,\ -s>0$. By appealing the result of Case (I), we have

$$\begin{aligned}\sum_{i=1}^{n}\frac{a_i^r}{b_i^s} = \sum_{i=1}^{n}\frac{b_i^{-s}}{a_i^{-r}} &\geqslant n^{\min\{-r-(-s)+1,\,0\}}\left(\sum_{i=1}^{n}b_i\right)^{-s}\Big/\left(\sum_{i=1}^{n}a_i\right)^{-r}\\ &= n^{\min\{s-r+1,\,0\}}\left(\sum_{i=1}^{n}a_i\right)^{r}\Big/\left(\sum_{i=1}^{n}b_i\right)^{s}.\end{aligned}$$

Lemma 3 is proved. □

3. Proof of the Main Result (Theorem 1)

In our proof of Theorem 1, we consider the following three cases.

Case (I): When $p \geqslant 1$ and $\beta \geqslant (n^{\max\{p,1\}} - 1)\alpha$, which implies that $p \geqslant 1$ and $\beta \geqslant (n^p - 1)\alpha$.

Applying Lemma 3 and Lemma 2 respectively, it follows that

$$\begin{aligned}
&\sum_{i=1}^{n}\left(\frac{x_i^q}{\alpha x_i^q + \beta\prod_{k=1}^{n} x_k^{q/n}}\right)^{1/p} \\
&\quad = \sum_{i=1}^{n}\frac{\left(x_i^{q/(p+1)}\right)^{(p+1)/p}}{\left(\alpha x_i^q + \beta\prod_{k=1}^{n} x_k^{q/n}\right)^{1/p}} \\
&\quad \geqslant n^{\min\{\frac{1}{p}-\frac{p+1}{p}+1,\,0\}}\left[\frac{\left(\sum_{i=1}^{n} x_i^{q/(p+1)}\right)^{(p+1)/p}}{\left(\sum_{i=1}^{n}\left(\alpha x_i^q + \beta\prod_{k=1}^{n} x_k^{q/n}\right)\right)^{1/p}}\right] \\
&\quad = \left[\frac{\left(\sum_{i=1}^{n} x_i^{q/(p+1)}\right)^{p+1}}{\alpha\sum_{i=1}^{n} x_i^q + n\beta\prod_{i=1}^{n} x_i^{q/n}}\right]^{1/p} \\
&\quad \geqslant \left[\frac{\sum_{i=1}^{n} x_i^q + (n^{p+1}-n)(\prod_{i=1}^{n} x_i^{q/n})}{\alpha\sum_{i=1}^{n} x_i^q + n\beta\prod_{i=1}^{n} x_i^{q/n}}\right]^{1/p} \\
&\quad = \left[\frac{n^p}{\alpha+\beta} + \frac{(\beta-(n^p-1)\alpha)\left(\sum_{i=1}^{n} x_i^q - n\prod_{i=1}^{n} x_i^{q/n}\right)}{(\alpha+\beta)\left(\alpha\sum_{i=1}^{n} x_i^q + n\beta\prod_{i=1}^{n} x_i^{q/n}\right)}\right]^{1/p} \\
&\quad \geqslant n(\alpha+\beta)^{-1/p}.
\end{aligned}$$

The latter inequality follows from the hypothesis $\beta - (n^p - 1)\alpha \geqslant 0$ and the arithmetic-geometric means inequality $\sum_{i=1}^{n} x_i^q - n\prod_{i=1}^{n} x_i^{q/n} \geqslant 0$.

Case (II): When $0 < p < 1$ and $\beta \geqslant (n^{\max\{p,1\}} - 1)\alpha$, which implies that $0 < p < 1$ and $\beta \geqslant (n-1)\alpha$.

Applying Lemma 3 and Lemma 2 respectively, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\left(\frac{x_i^q}{\alpha x_i^q + \beta\prod_{k=1}^{n} x_k^{q/n}}\right)^{1/p} \\
&\quad = \sum_{i=1}^{n}\frac{\left(x_i^{q/2}\right)^{2/p}}{\left(\alpha x_i^q + \beta\prod_{k=1}^{n} x_k^{q/n}\right)^{1/p}}
\end{aligned}$$

$$
\begin{aligned}
&\geqslant n^{\min\{\frac{1}{p}-\frac{2}{p}+1,\,0\}}\left[\frac{\left(\sum_{i=1}^{n} x_i^{q/2}\right)^{2/p}}{\left(\sum_{i=1}^{n}\left(\alpha x_i^q+\beta\prod_{k=1}^{n} x_k^{q/n}\right)\right)^{1/p}}\right]\\
&= n^{1-\frac{1}{p}}\left[\frac{\left(\sum_{i=1}^{n} x_i^{q/2}\right)^{2}}{\alpha\sum_{i=1}^{n} x_i^q+n\beta\prod_{i=1}^{n} x_i^{q/n}}\right]^{1/p}\\
&\geqslant n^{1-\frac{1}{p}}\left[\frac{\sum_{i=1}^{n} x_i^q+(n^2-n)(\prod_{i=1}^{n} x_i^{q/n})}{\alpha\sum_{i=1}^{n} x_i^q+n\beta\prod_{i=1}^{n} x_i^{q/n}}\right]^{1/p}\\
&= n^{1-\frac{1}{p}}\left[\frac{n}{\alpha+\beta}+\frac{(\beta-(n-1)\alpha)\left(\sum_{i=1}^{n} x_i^q-n\prod_{i=1}^{n} x_i^{q/n}\right)}{(\alpha+\beta)\left(\alpha\sum_{i=1}^{n} x_i^q+n\beta\prod_{i=1}^{n} x_i^{q/n}\right)}\right]^{1/p}\\
&\geqslant n(\alpha+\beta)^{-1/p}.
\end{aligned}
$$

The latter inequality follows from the hypothesis $\beta-(n-1)\alpha \geqslant 0$ and the arithmetic-geometric means inequality $\sum_{i=1}^{n} x_i^q - n\prod_{i=1}^{n} x_i^{q/n} \geqslant 0$.

Case (III): When $p<0$ and $\alpha,\beta>0$. Using the arithmetic-geometric means inequality gives

$$
\begin{aligned}
\sum_{i=1}^{n}\left(\frac{x_i^q}{\alpha x_i^q+\beta\prod_{k=1}^{n} x_k^{q/n}}\right)^{1/p} &\geqslant n\left[\prod_{i=1}^{n}\left(\frac{x_i^q}{\alpha x_i^q+\beta\prod_{k=1}^{n} x_k^{q/n}}\right)^{1/p}\right]^{1/n}\\
&= n\left[\prod_{i=1}^{n}\left(\alpha+\beta x_i^{-q}\prod_{k=1}^{n} x_k^{q/n}\right)^{1/n}\right]^{-1/p}.
\end{aligned}
\tag{9}
$$

On the other hand, applying Hölder inequality, one obtains

$$
\begin{aligned}
&\prod_{i=1}^{n}\left(\alpha+\beta x_i^{-q}\prod_{k=1}^{n} x_k^{q/n}\right)^{1/n}\\
&\qquad\geqslant\left(\alpha^{1/n}\right)^n+\left(\beta^{1/n}\right)^n\left(\prod_{i=1}^{n} x_i^{-q/n}\right)\left(\left(\prod_{k=1}^{n} x_k^{q/n}\right)^{1/n}\right)^n\\
&\qquad=\alpha+\beta.
\end{aligned}
$$

Note that $-1/p>0$, combining inequality (9) and the above inequality leads to the desired inequality:

$$
\sum_{i=1}^{n}\left(\frac{x_i^q}{\alpha x_i^q+\beta\prod_{k=1}^{n} x_k^{q/n}}\right)^{1/p} \geqslant n(\alpha+\beta)^{-1/p}.
$$

The proof of Theorem 1 is complete. □

REMARK. Letting in particular $p = n - 1$ and $q = n$ $(n \geqslant 2)$ in Theorem 1 yields the Satnoianu's inequality (3). If we put in the Satnoianu's inequality (3) $\alpha = 1$ and $\beta = \lambda$, we obtain the conjectured inequality (2). Further, the inequality (1) would follow from inequality (2) with $n = 3$. It is worth mentioning that inequality (1) appeared first as an Open problem 10944 in the *American Mathematical Monthly* [6].

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