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DIFFERENTIAL INEQUALITY CONDITIONS FOR DOMINANCE BETWEEN CONTINUOUS ARCHIMEDEAN T–NORMS

SUSANNE SAMINGER-PLATZ, BERNARD DE BAETS AND HANS DE MEYER

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Abstract. Dominance between triangular norms (t-norms) is a versatile relationship. For continuous Archimedean t-norms, dominance can be verified by checking one of many sufficient conditions derived from a generalization of the Mulholland inequality. These conditions pertain to various convexity properties of compositions of additive generators and their inverses. In this paper, assuming differentiability of these additive generators, we propose equivalent sufficient conditions that can be expressed as inequalities involving derivatives of the additive generators, avoiding the need of composing them. We demonstrate the powerfulness of the results by the straightforward rediscovery of dominance relationships in the Schweizer-Sklar t-norm family, as well as by unveiling some formerly unknown dominance relationships in the Sugeno-Weber t-norm family. Finally, we illustrate that the results can also be applied to members of different parametric families of t-norm.

1. Introduction

The dominance relation was originally introduced in the framework of probabilistic metric spaces [23] and was soon abstracted to operations on a partially ordered set [21]. The dominance relation, in particular between t-norms, plays a profound role in various topics, such as the construction of Cartesian products of probabilistic metric and normed spaces [5, 21, 23], the construction of many-valued equivalence relations [2, 3, 26] and many-valued order relations [1], and in the preservation of various properties during (dis-)aggregation processes in flexible querying, preference modelling and computer-assisted assessment [2, 4, 14, 17]. These applications instigated the study of the dominance relation in the broader context of aggregation functions [12, 14, 17].

Additional to these application aspects, the dominance relation is an interesting mathematical notion *per se*. Because of the common neutral element, dominance constitutes a reflexive and antisymmetric relation on the class of t-norms. Since counterexamples for its transitivity were not readily found, it remained an intriguing open problem [8, 19, 21, 22, 25] for more than 20 years whether or not it was an order relation. Only recently the question was answered to the negative [18, 20]. However, due to its relevance in applications, it is still of interest to determine subclasses of t-norms on which the dominance relation establishes an order relation. Of particular importance are continuous Archimedean t-norms, as they are the basic elements of which

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all continuous t-norms are composed. Moreover, they can be represented by means of continuous additive generators.

It was shown in [16], see also [13, 24, 25] for earlier results dealing with strict t-norms only, that dominance between continuous Archimedean t-norms can be equivalently expressed as a functional inequality involving compositions of the additive generators (and their inverses) of the corresponding t-norms. This inequality, being a generalization of the *Minkowski inequality*, is often referred to as the *Mulholland inequality*. Although sufficient and necessary conditions for its fulfilment are already known, see [13, 16, 24, 25], and can be visualized easily for two t-norms, they have hardly ever been used for proving resp. disproving dominance between two arbitrary members of a family or families of t-norms. The aim of the present contribution is to establish easy-to-check conditions that involve directly the additive generators and their derivatives (provided they exist).

After a short introduction on t-norms, we summarize the known sufficient and necessary conditions for dominance. Subsequently, we derive new differential conditions for dominance between continuous Archimedean t-norms and demonstrate their strength by applying them to some parametric families of triangular norms leading to new results on dominance between two continuous Archimedean t-norms.

2. Triangular norms and the dominance relation

We briefly summarize some basic properties of t-norms for a thorough understanding of this paper (for further details see, e.g., [7, 8, 9, 10, 11, 15, 17, 18]).

DEFINITION 1. A *t-norm* $T : [0,1]^2 \to [0,1]$ is a binary operation on the unit interval that is commutative, associative, increasing and has 1 as neutral element.

Well-known examples of t-norms are the *minimum* $T_{\mathbf{M}}$, the *product* $T_{\mathbf{P}}$, the *Łukasiewicz t-norm* $T_{\mathbf{L}}$ and the *drastic product* $T_{\mathbf{D}}$, defined by $T_{\mathbf{M}}(u,v) = \min(u,v)$, $T_{\mathbf{P}}(u,v) = u \cdot v$, $T_{\mathbf{L}}(u,v) = \max(u+v-1,0)$, and

$$T_{\mathbf{D}}(u,v) = \begin{cases} \min(u,v), & \text{if } \max(u,v) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since t-norms are just functions from the unit square to the unit interval, their comparison is done pointwisely: $T_1 \leqslant T_2$ if $T_1(u,v) \leqslant T_2(u,v)$ for all $u,v \in [0,1]$, expressed as "T_1 is *weaker* than T_2" or "T_2 is *stronger* than T_1". The minimum $T_{\mathbf{M}}$ is the strongest of all t-norms, the drastic product $T_{\mathbf{D}}$ is the weakest of all t-norms. Furthermore, it holds that $T_{\mathbf{P}} \geqslant T_{\mathbf{L}}$.

Just as triangular norms, the dominance relation finds its origin in the field of probabilistic metric spaces [21, 23]. It was originally introduced for associative operations (with common neutral element) on a partially ordered set [21], and has been further investigated for t-norms [15, 19, 20, 25] and aggregation functions [14, 17]. We state the definition for t-norms only.

DEFINITION 2. Consider two t-norms T_1 and T_2. We say that T_1 *dominates* T_2 (or T_2 is dominated by T_1), denoted by $T_1 \gg T_2$, if for all $x,y,u,v \in [0,1]$ it holds that

$$T_1(T_2(x,y),T_2(u,v)) \geqslant T_2(T_1(x,u),T_1(y,v)). \tag{1}$$

Note that every t-norm is dominated by itself and by $T_{\mathbf{M}}$; moreover, it dominates $T_{\mathbf{D}}$. Since all t-norms have neutral element 1, dominance between two t-norms implies their comparability: $T_1 \gg T_2$ implies $T_1 \geqslant T_2$. The converse does not hold, not even for strict t-norms [8]. Due to the induced comparability it also follows that dominance is an antisymmetric relation on the class of t-norms.

DEFINITION 3. A t-norm T is called *Archimedean* if for all $u,v \in]0,1[$ there exists an $n \in \mathbb{N}$ such that

$$T(\underbrace{u,\ldots,u}_{n \text{ times}}) < v.$$

DEFINITION 4.

(i) A t-norm T is called *strict* if it is continuous and strictly monotone, i.e., for all $u,v,w \in [0,1]$ it holds that

$$T(u,v) < T(u,w) \quad \text{whenever} \quad u > 0 \text{ and } v < w.$$

(ii) A t-norm T is called *nilpotent* if it is continuous and if each $u \in]0,1[$ is a nilpotent element of T, i.e., there exists some $n \in \mathbb{N}$ such that

$$T(\underbrace{u,\ldots,u}_{n \text{ times}}) = 0.$$

A continuous t-norm T is Archimedean if and only if for all $u \in]0,1[$ it holds that $T(u,u) < u$. The class of continuous Archimedean t-norms can be partitioned into two disjoint subclasses: the class of strict t-norms and the class of nilpotent t-norms. The product $T_{\mathbf{P}}$ is strict, whereas the Łukasiewicz t-norm $T_{\mathbf{L}}$ is nilpotent.

Note that for a strict t-norm T it holds that $T(u,v) > 0$ for all $u,v \in]0,1]$, while for a nilpotent t-norm T it holds that for every $u \in]0,1[$ there exists some $v \in]0,1[$ such that $T(u,v) = 0$ (each $u \in]0,1[$ is a so-called *zero divisor*). Therefore, for a nilpotent t-norm T_1 and a strict t-norm T_2 it can never hold that $T_1 \geqslant T_2$ and, as a consequence, never that $T_1 \gg T_2$.

Of particular interest in the discussion of continuous Archimedean t-norms and dominance between them is the notion of an *additive generator*.

DEFINITION 5. An *additive generator* of a t-norm T is a strictly decreasing function $t\colon [0,1] \to [0,\infty]$ which is right-continuous in 0 and satisfies $t(1) = 0$ such that for all $u,v \in [0,1]$ it holds that

$$T(u,v) = t^{(-1)}(t(u)+t(v)),$$

with

$$t^{(-1)}(u) = t^{-1}(\min(t(0),u))$$

the pseudo-inverse of the decreasing function t.

An additive generator is uniquely determined up to a positive multiplicative constant. A t-norm T with additive generator t is continuous if and only if t is continuous. Continuous Archimedean t-norms are exactly those t-norms with a continuous additive generator. Any additive generator of a strict t-norm satisfies $t(0)=\infty$, while that of a nilpotent t-norm satisfies $t(0)<\infty$. In the case of strict t-norms, the pseudo-inverse $t^{(-1)}$ of an additive generator t coincides with its standard inverse t^{-1}. In any case, the following relationships hold between an additive generator t and its pseudo-inverse $t^{(-1)}$

$$t\circ t^{(-1)}|_{\mathrm{Ran}(t)}=\mathrm{id}_{\mathrm{Ran}(t)} \quad \text{and} \quad t^{(-1)}\circ t=\mathrm{id}_{[0,1]}\,.$$

3. The generalized Mulholland inequality and related conditions

The dominance relation between two continuous Archimedean t-norms can be expressed in terms of their generators. This was shown for strict t-norms in [25] and was generalized as follows in [16].

PROPOSITION 1. *Consider two continuous Archimedean t-norms T_1 and T_2 with additive generators t_1 and t_2. Then T_1 dominates T_2 if and only if the function $h=t_1\circ t_2^{(-1)}\colon\ [0,\infty]\to[0,\infty]$ fulfills for all $a,b,c,d\in[0,t_2(0)]$ the inequality*

$$h^{(-1)}(h(a)+h(c))+h^{(-1)}(h(b)+h(d))\geqslant h^{(-1)}(h(a+b)+h(c+d))\,, \tag{2}$$

with $h^{(-1)}=t_2\circ t_1^{(-1)}\colon\ [0,\infty]\to[0,\infty]$ the pseudo-inverse of the increasing function h.

Since (1) is trivially fulfilled for arbitrary t-norms T_1 and T_2 as soon as 0 appears among the arguments, it suffices to prove that (2) holds for all $a,b,c,d\in\,]0,t_2(0)[$ in order to show dominance between the continuous Archimedean t-norms considered.

In case some function $f\colon\ [0,\infty]\to[0,\infty]$ fulfills (2) for all $a,b,c,d\in[0,\infty]$, we say that it fulfills the *generalized Mulholland inequality*. In [16] (see also [6, 13, 25]), sufficient and necessary conditions for the generalized Mulholland inequality to hold for a function $f\colon\ [0,\infty]\to[0,\infty]$, which is continuous and strictly increasing on some subdomain $[0,t]$, with $t\in[0,\infty[$, and for which $f(0)=0$ holds, have been investigated. Properties such as the convexity, the geometric convexity, and the logarithmic convexity of a function showed up to be most relevant.

DEFINITION 6. A function $f\colon\ [0,\infty[\,\to[0,\infty[$ is called *geometric convex* (*geo-convex* for short) on $]0,t[$, with $t\in\,]0,\infty[$, if for all $x,y\in\,]0,t[$ it holds that

$$f(\sqrt{xy})\leqslant\sqrt{f(x)f(y)}\,.$$

It is called *logarithmic convex* (*log-convex* for short) on $]0,t[$ if the function $\log\circ f\colon\ [0,\infty[\,\to[-\infty,\infty[$ is convex on $]0,t[$.

For a continuous function f such that $f(]0,\infty[)\subseteq\,]0,\infty[$, its geo-convexity on $]0,t[$ is equivalent to the convexity of the function $\log\circ f\circ\exp$ on $]-\infty,\log(t)[$. Clearly, if $f(0)=0$, then the geo-convexity holds also on $[0,t[$. Further, if f is increasing, then its log-convexity implies its geo-convexity. Moreover, the relationship between the geo-convexity of a function and that of its derivative can be expressed in the following way.

LEMMA 2. [6, 16] *Consider a function* $f\colon\ ]0,\infty[\ \to\ ]0,\infty[$ *with* $\lim_{x\to 0} f(x) = 0$ *and such that* f *is differentiable on* $[0,t[$, *with* $t \in\ ]0,\infty[$, *and* $f'(x) > 0$ *for all* $x \in\ ]0,t[$. *If* f' *is geo-convex on* $]0,t[$, *then so is* f.

Applying these relationships and the results obtained in [16] to the dominance relation between continuous Archimedean t-norms we can state the following:

PROPOSITION 3. [16] *Consider two continuous Archimedean t-norms* T_1 *and* T_2 *with additive generators* t_1 *and* t_2. *If* T_1 *dominates* T_2, *then the function* $h = t_1 \circ t_2^{(-1)}\colon$ $[0,\infty] \to [0,\infty]$ *is convex on* $]0,t_2(0)[$.

PROPOSITION 4. [16] *Consider two continuous Archimedean t-norms* T_1 *and* T_2 *with additive generators* t_1 *and* t_2. *If the function* $h = t_1 \circ t_2^{(-1)}\colon\ [0,\infty] \to [0,\infty]$ *is convex on* $]0,t_2(0)[$ *and log- or geo-convex on* $]0,t_2(0)[$, *then* h *fulfills* (2) *for all* $a,b,c,d \in [0,t_2(0)]$, *i.e.,* T_1 *dominates* T_2.

PROPOSITION 5. [16] *Consider two continuous Archimedean t-norms* T_1 *and* T_2 *with additive generators* t_1 *and* t_2. *If the function* $h = t_1 \circ t_2^{(-1)}\colon\ [0,\infty] \to [0,\infty]$ *is differentiable and convex on* $]0,t_2(0)[$, *and* h' *is log- or geo-convex on* $]0,t_2(0)[$, *then* h *fulfills* (2) *for all* $a,b,c,d \in [0,t_2(0)]$, *i.e.,* T_1 *dominates* T_2.

The relationships between the above sufficient conditions for dominance are summarized in Fig. 1.

Consider two continuous Archimedean t-norms T_1 and T_2 with additive generators t_1 and t_2. If the function

$$h = t_1 \circ t_2^{(-1)}\colon\ [0,\infty] \to [0,\infty]$$

is convex on $]0,t_2(0)[$ and ...

h' exists and h' is log-convex on $]0,t_2(0)[$	$\Rightarrow$	h' exists and h' is geo-convex on $]0,t_2(0)[$
		$\Downarrow$
h is log-convex on $]0,t_2(0)[$	$\Rightarrow$	h is geo-convex on $]0,t_2(0)[$
		$\Downarrow$
		$T_1 \gg T_2$

Figure 1. Sufficient conditions for dominance between two continuous Archimedean t-norms T_1 *and* T_2

Corresponding conditions for the subclass of strict t-norms have already been discussed in [25]. Although these sufficient conditions can be visualized easily, concrete proofs might become cumbersome, in particular for two members of a parametric family, because h is a compound function. In fact, the conditions mentioned above have never been used for (dis-)proving dominance apart from one particular case: for proving dominance between members of a family of t-norms whose additive generators are powers of some basic additive generator. In this case the generalized Mulholland inequality turns into the Minkowski inequality whose solution is well known (see [8] for further details).

However, if the additive generators have derivatives of sufficiently high order, the sufficient conditions expressed as properties of h can be reformulated as equivalent (differential) conditions on the corresponding additive generators. As such we can provide localized conditions that are equivalent to the global ones and allow to (dis-)prove dominance between two continuous Archimedean t-norms.

4. Differential inequality conditions

Throughout this section, T_1 and T_2 are two continuous Archimedean t-norms with continuous additive generators t_1 and t_2. Then the function

$$h = t_1 \circ t_2^{(-1)}\colon\ [0,\infty] \to [0,\infty]$$

is continuous and strictly increasing on $]0,t_2(0)[$, $h(0)=0$ and $h(]0,t_2(0)[) \subseteq]0,t_1(0)[$. Further, we assume that t_1 and t_2 are sufficiently often (i.e., once, twice or three times) differentiable. It then holds in particular that $t_1'(u)<0$ and $t_2'(u)<0$ for all $u \in]0,1[$. For every $x \in]0,t_2(0)[$, there exists a unique $u \in]0,1[$ such that $x = t_2(u)$ and $t_2^{-1}(x) = u$. The identity

$$\frac{d}{dx}x = \frac{d}{dx}t_2(t_2^{-1}(x)) = \left.\frac{dt_2(u)}{du}\right|_{u=t_2^{-1}(x)} \cdot \frac{dt_2^{-1}(x)}{dx} = 1$$

allows to express the derivatives of h at x in terms of the derivatives of t_1 and t_2 at $u = t_2^{-1}(x)$. Explicitly,

$$h(x) = t_1(t_2^{-1}(x)) = t_1(u)|_{u=t_2^{-1}(x)}, \tag{3}$$

$$\begin{aligned} h'(x) &= \frac{d}{dx}h(x) = \frac{d}{dx}(t_1(t_2^{-1}(x))) = \left.\frac{dt_1(u)}{du}\right|_{u=t_2^{-1}(x)} \cdot \frac{dt_2^{-1}(x)}{dx} \\ &= \left.\frac{dt_1(u)}{du}\cdot\frac{1}{\frac{dt_2(u)}{du}}\right|_{u=t_2^{-1}(x)} = \left.\frac{t_1'(u)}{t_2'(u)}\right|_{u=t_2^{-1}(x)}, \end{aligned} \tag{4}$$

$$\begin{aligned} h''(x) &= \frac{d}{dx}h'(x) = \left.\frac{d}{dx}\frac{t_1'(u)}{t_2'(u)}\right|_{u=t_2^{-1}(x)} = \left.\frac{d}{du}\frac{t_1'(u)}{t_2'(u)}\right|_{u=t_2^{-1}(x)} \cdot \frac{dt_2^{-1}(x)}{dx} \\ &= \left.\frac{t_1''(u)t_2'(u) - t_2''(u)t_1'(u)}{t_2'^2(u)}\cdot\frac{1}{\frac{dt_2(u)}{du}}\right|_{u=t_2^{-1}(x)} = \left.\frac{t_1''(u)t_2'(u) - t_2''(u)t_1'(u)}{t_2'^3(u)}\right|_{u=t_2^{-1}(x)}. \end{aligned} \tag{5}$$

Let us now turn to the convexity, the logarithmic and the geometric convexity of h and its derivative.

PROPOSITION 6. *The function h is convex on $]0,t_2(0)[$, i.e.,*

$$h''(x) \geqslant 0 \tag{6}$$

for all $x \in]0,t_2(0)[$, if and only if

$$t_1'(u)t_2''(u) - t_1''(u)t_2'(u) \geqslant 0 \tag{7}$$

for all $u \in]0,1[$.

Proof. Since $h''(x)$ can be expressed by (5) and $t_2'(u) < 0$ for all $u \in]0,1[$, it follows immediately that

$$\forall x \in]0,t_2(0)[: h''(x) \geqslant 0 \quad \Leftrightarrow \quad \forall u \in]0,1[: t_1'(u)t_2''(u) - t_1''(u)t_2'(u) \geqslant 0\,.$$

PROPOSITION 7. *The function h is log-convex on $]0,t_2(0)[$, i.e.,*

$$h(x)h''(x) - h'^2(x) \geqslant 0 \tag{8}$$

for all $x \in]0,t_2(0)[$, if and only if

$$t_1'^2(u)t_2'(u) + t_1(u)\big(t_1'(u)t_2''(u) - t_1''(u)t_2'(u)\big) \geqslant 0 \tag{9}$$

for all $u \in]0,1[$.

Proof. The function h is log-convex on $]0,t_2(0)[$ if and only if

$$(\log \circ h)''(x) = \frac{h(x)h''(x) - h'^2(x)}{h^2(x)} \geqslant 0$$

for all $x \in]0,t_2(0)[$. Since $h(x) > 0$ for all $x > 0$, we can write equivalently

$$h(x)h''(x) - h'^2(x) \geqslant 0$$

for all $x \in]0,t_2(0)[$. Using (3)–(5), the latter turns out to be equivalent to

$$t_1(u)\frac{t_1''(u)t_2'(u) - t_2''(u)t_1'(u)}{t_2'^3(u)} - \frac{t_1'^2(u)}{t_2'^2(u)} \geqslant 0,$$

or also

$$t_1'^2(u)t_2'(u) + t_1(u)\left(t_1'(u)t_2''(u) - t_1''(u)t_2'(u)\right) \geqslant 0$$

for all $u \in]0,1[$.

PROPOSITION 8. *The function h is geo-convex on $]0,t_2(0)[$, i.e.,*

$$h(x)h'(x) + x\left(h(x)h''(x) - h'^2(x)\right) \geqslant 0 \tag{10}$$

for all $x \in]0,t_2(0)[$, if and only if

$$\frac{t_1'^2(u) - t_1(u)t_1''(u)}{t_1(u)t_1'(u)} \geqslant \frac{t_2'^2(u) - t_2(u)t_2''(u)}{t_2(u)t_2'(u)} \tag{11}$$

for all $u \in]0,1[$.

Proof. First, we show that the geometric convexity of h on $]0,t_2(0)[$ is equivalent to Eq. (10) for all $x \in]0,t_2(0)[$. The geometric convexity of h on $]0,t_2(0)[$ is equivalent to the convexity of the function $\chi = \log \circ h \circ \exp\colon [-\infty, \log(t_2(0))] \to [-\infty, \log(t_1(0))]$ on $]-\infty, \log(t_2(0))[$. Since h is twice differentiable, also χ is twice differentiable and

$$\begin{aligned}
\chi'(v) &= \tfrac{h'(e^v)}{h(e^v)} e^v;\\
\chi''(v) &= \left(\frac{h'(e^v)}{h(e^v)} + e^v \frac{h(e^v)h''(e^v) - h'(e^v)^2}{h(e^v)^2}\right) e^v\\
&= \frac{e^v}{h(e^v)^2}\left(h'(e^v)h(e^v) + e^v(h(e^v)h''(e^v) - h'(e^v)^2)\right).
\end{aligned}$$

Since always $\frac{e^v}{h(e^v)^2} > 0$, $\chi''(v) \geqslant 0$ is equivalent to $h'(e^v)h(e^v) + e^v(h(e^v)h''(e^v) - h'(e^v)^2) \geqslant 0$, or, replacing e^v by x, to

$$h(x)h'(x) + x\left(h(x)h''(x) - h'^2(x)\right) \geqslant 0.$$

Using Eqs. (3)–(5), the validity of Eq. (10) for all $x \in]0,t_2(0)[$ turns out to be equivalent to

$$t_1(u) \cdot \tfrac{t_1'(u)}{t_2'(u)} + t_2(u)\left(t_1(u) \cdot \tfrac{t_1''(u)t_2'(u) - t_2''(u)t_1'(u)}{t_2'^3(u)} - \tfrac{t_1'^2(u)}{t_2'^2(u)}\right) \geqslant 0,$$

or also

$$\frac{t_1'^2(u) - t_1(u)t_1''(u)}{t_1(u)t_1'(u)} \geqslant \frac{t_2'^2(u) - t_2(u)t_2''(u)}{t_2(u)t_2'(u)}$$

for all $u \in]0,1[$.

REMARK 9. Investigating the differential formulations of the convexity, log-convexity and geo-convexity of h, it becomes evident that the log-convexity of h implies its convexity as well as its geo-convexity. Indeed, if h is log-convex, i.e.

$$t_1'^2(u)t_2'(u) + t_1(u)\left(t_1'(u)t_2''(u) - t_1''(u)t_2'(u)\right) \geqslant 0,$$

it follows that

$$t_1(u)\big(t_1'(u)t_2''(u)-t_1''(u)t_2'(u)\big)\geqslant -t_1'^2(u)t_2'(u)\geqslant 0\,,$$

since $t_2'(u)<0$ for all $u\in\,]0,1[$. As $t_1(u)>0$ for all $u\in\,]0,1[$, it must hold that $t_1'(u)t_2''(u)-t_1''(u)t_2'(u)\geqslant 0$ for all $u\in\,]0,1[$, i.e., h is convex.

Again assume that h is log-convex, i.e. $h(x)h''(x)-h'^2(x)\geqslant 0$ for all $x\in\,]0,t_2(0)[$. Since for any such x it holds that x, $h(x)$ and $h'(x)$ are positive, it also holds that

$$h(x)h'(x)+x\left(h(x)h''(x)-h'^2(x)\right)\geqslant 0$$

for all $x\in\,]0,t_2(0)[$, i.e. h is geo-convex on $]0,t_2(0)[$.

Similarly as for Eqs. (3)–(5), for all $x\in\,]0,t_2(0)[$, the third derivative of h at x can be expressed as

$$\begin{aligned}
h'''(x) &= \tfrac{d}{dx}h''(x) = \tfrac{d}{du}\left(\tfrac{t_1''(u)t_2'(u)-t_2''(u)t_1'(u)}{t_2'^3(u)}\right)\Bigg|_{u=t_2^{-1}(x)}\cdot\frac{dt_2^{-1}(x)}{dx}\\
&= \frac{t_2'^3(u)(t_1'''(u)t_2'(u)-t_2'''(u)t_1'(u))-3t_2'^2(u)t_2''(u)(t_1''(u)t_2'(u)-t_2''(u)t_1'(u))}{t_2'^6(u)}\cdot\frac{1}{\frac{dt_2(u)}{du}}\Bigg|_{u=t_2^{-1}(x)}\\
&= \tfrac{1}{t_2'^5(u)}\left(3t_1'(u)t_2''^2(u)-t_1'(u)t_2'(u)t_2'''(u)-3t_1''(u)t_2'(u)t_2''(u)+t_1'''(u)t_2'^2(u)\right)\Bigg|_{u=t_2^{-1}(x)}.
\end{aligned}$$

Substitution in the corresponding formulas and reshuffling the inequalities leads to the following corollaries which we state without their easy but tedious and cumbersome proofs.

COROLLARY 10. *The function h' is log-convex on $]0,t_2(0)[$, i.e.,*

$$h'(x)h'''(x)-h''^2(x)\geqslant 0 \tag{12}$$

for all $x\in\,]0,t_2(0)[$, if and only if

$$\begin{aligned}
&t_1'^2(u)\left(2t_2''^2(u)-t_2'(u)t_2'''(u)\right)\\
&\qquad\geqslant t_2'^2(u)\left(t_1''^2(u)-t_1'(u)t_1'''(u)\right)+t_1'(u)t_1''(u)t_2'(u)t_2''(u)
\end{aligned} \tag{13}$$

for all $u\in\,]0,1[$.

COROLLARY 11. *The function h' is geo-convex on $]0,t_2(0)[$, i.e.,*

$$h'(x)h''(x)+x\left(h'(x)h'''(x)-h''^2(x)\right)\geqslant 0 \tag{14}$$

for all $x\in\,]0,t_2(0)[$, if and only if

$$\begin{aligned}
&t_2(u)\Big(t_1'(u)t_2'(u)\big(t_1'''(u)t_2'(u)-t_2'''(u)t_1'(u)\big)\\
&\qquad -\big(t_1''(u)t_2'(u)-t_2''(u)t_1'(u)\big)\big(2t_1'(u)t_2''(u)+t_1''(u)t_2'(u)\big)\Big)\\
&\qquad\qquad \geqslant t_1'(u)t_2'^2(u)\left(t_1'(u)t_2''(u)-t_1''(u)t_2'(u)\right)
\end{aligned} \tag{15}$$

for all $u\in\,]0,1[$.

5. Dominance within a single parametric family of t-norms

Although the differential inequality conditions look cumbersome at first sight, they often reduce to easy-to-check inequalities when applied to members of parametric families of t-norms, as we will demonstrate in this and the following section. First, we consider the family of Schweizer-Sklar t-norms. Although it is known [22] that dominance within this family is in accordance with the ordering of the parameters, we provide an alternative (and easier) proof based on the new differential inequality conditions in order to illustrate their strength. Second, we examine dominance within the family of Sugeno-Weber t-norms, leading to relationships not yet established so far, since most of its members are nilpotent t-norms. We tackle these problems by following the scheme of sufficient conditions displayed in Fig. 3. We will provide the differential inequality for the necessary convexity of h as well as the differential inequality corresponding to the strongest sufficient condition leading to the discovery of a dominance relationship.

5.1. The family of Schweizer-Sklar t-norms

The family of Schweizer-Sklar t-norms $(T_\lambda^{\mathbf{SS}})_{\lambda\in[-\infty,\infty]}$ is given by

$$T_\lambda^{\mathbf{SS}}(u,v) = \begin{cases} T_{\mathbf{M}}(u,v), & \text{if } \lambda = -\infty, \\ T_{\mathbf{P}}(u,v), & \text{if } \lambda = 0, \\ T_{\mathbf{D}}(u,v), & \text{if } \lambda = \infty, \\ \max(u^\lambda + v^\lambda - 1, 0)^{1/\lambda}, & \text{if } \lambda \in\,]-\infty,0[\,\cup\,]0,\infty[. \end{cases}$$

For $\lambda \in]-\infty,\infty[$, $T_\lambda^{\mathbf{SS}}$ is a continuous Archimedean t-norm with additive generator

$$t_\lambda^{\mathbf{SS}}(u) = \tfrac{1-u^\lambda}{\lambda}, \text{ if } \lambda \in\,]-\infty,0[\,\cup\,]0,\infty[, \quad \text{and} \quad t_0^{\mathbf{SS}}(u) = -\log u, \text{ if } \lambda = 0$$

for all $u \in [0,1]$; parameters $\lambda \in]-\infty,0]$ lead to strict t-norms, while parameters $\lambda \in]0,\infty[$ lead to nilpotent t-norms.

In the sequel of this section, we omit the superscript indicating the family when discussing properties of additive generators. Since we deal with Schweizer-Sklar t-norms only, no ambiguity can occur.

Clearly, the derivatives of the additive generators exist and are given, for all $\lambda \in]-\infty,\infty[$ and all $u \in]0,1[$, by:

$$\begin{aligned} t'_\lambda(u) &= -u^{\lambda-1}, \\ t''_\lambda(u) &= -(\lambda-1)u^{\lambda-2}, \\ t'''_\lambda(u) &= -(\lambda-1)(\lambda-2)u^{\lambda-3}. \end{aligned}$$

The family of Schweizer-Sklar t-norms is ordered according to its parameter: $T_\lambda^{\mathbf{SS}} \geqslant T_\mu^{\mathbf{SS}}$ if and only if $\lambda \leqslant \mu$. Moreover, since $T_{\mathbf{M}}$ dominates every t-norm, and every t-norm dominates itself as well as $T_{\mathbf{D}}$, it suffices to investigate dominance between two Schweizer-Sklar t-norms $T_\lambda^{\mathbf{SS}}$ and $T_\mu^{\mathbf{SS}}$ with parameters $-\infty < \lambda < \mu < \infty$.

Note that the function $h = t_\lambda \circ t_\mu^{(-1)}\colon [0,\infty] \to [0,\infty]$ is continuous, strictly increasing and differentiable on $\left]0, t_\mu(0)\right[$ and fulfills $h(0)=0$. If h is convex on $\left]0, t_\mu(0)\right[$ and if either h or h' is log- or geo-convex on $\left]0, t_\mu(0)\right[$, then $T_\lambda^{\mathbf{SS}}$ dominates $T_\mu^{\mathbf{SS}}$.

Convexity of h. The function h is convex on $\left]0, t_\mu(0)\right[$ if and only if, for all $u \in \left]0,1\right[$,

$$\begin{aligned} t'_\lambda(u)t''_\mu(u) - t''_\lambda(u)t'_\mu(u) \geqslant 0 \quad &\Leftrightarrow \\ (\mu-1)u^{\lambda+\mu-3} - (\lambda-1)u^{\lambda+\mu-3} \geqslant 0 \quad &\Leftrightarrow \\ (\mu-\lambda)u^{\lambda+\mu-3} \geqslant 0 \quad &\Leftrightarrow \\ \mu \geqslant \lambda\,. \end{aligned}$$

Geo-convexity of h'. Substituting the expressions for the derivatives of the additive generators in (15) shows that the function h' is geo-convex on $\left]0, t_\mu(0)\right[$ if and only if, for all $u \in \left]0,1\right[$,

$$\begin{aligned} t_\mu(u)\Big(u^{\lambda+\mu-2}\big((\lambda-1)(\lambda-2)u^{\lambda+\mu-4} - (\mu-1)(\mu-2)u^{\lambda+\mu-4}\big) \\ -\big((\lambda-1)u^{\lambda+\mu-3} - (\mu-1)u^{\lambda+\mu-3}\big)\big(2(\mu-1)u^{\lambda+\mu-3} + (\lambda-1)u^{\lambda+\mu-3}\big)\Big) \\ \geqslant -u^{\lambda+2\mu-3}\Big((\mu-1)u^{\lambda+\mu-3} - (\lambda-1)u^{\lambda+\mu-3}\Big), \end{aligned}$$

with rearrangements and simple calculations leading to

$$t_\mu(u)\mu(\mu-\lambda) \geqslant -u^\mu(\mu-\lambda)\,.$$

In case $\mu = 0$, the latter condition reduces to $0 \geqslant \lambda$, or, equivalently, $\mu \geqslant \lambda$. In case $\mu \neq 0$, the condition reads explicitly

$$\begin{aligned} \left(\tfrac{1-u^\mu}{\mu}\right)\mu(\mu-\lambda) \geqslant -u^\mu(\mu-\lambda) \quad &\Leftrightarrow \\ (\mu-\lambda)(1-u^\mu+u^\mu) \geqslant 0 \quad &\Leftrightarrow \\ \mu \geqslant \lambda\,. \end{aligned}$$

Hence, neither the convexity of h nor the geo-convexity of h' imposes further restrictions on λ and μ.

COROLLARY 12. *Consider the family of Schweizer-Sklar t-norms* $(T_\lambda^{\mathbf{SS}})_{\lambda\in[-\infty,\infty]}$. *For all* $\lambda, \mu \in [-\infty,\infty]$ *it holds that* $T_\lambda^{\mathbf{SS}}$ *dominates* $T_\mu^{\mathbf{SS}}$ *if and only if* $\lambda \geqslant \mu$.

We stress that this result is obtained here much more economically than in [22].

5.2. The family of Sugeno-Weber t-norms

The second family we consider is the family of Sugeno-Weber t-norms. Two arguments support its consideration: first, dominance relationships between two members of this family have not yet been laid bare; second, it is of particular interest as all but two of its menbers are nilpotent t-norms.

The family of Sugeno-Weber t-norms $(T_\lambda^{\mathbf{SW}})_{\lambda\in[0,\infty]}$ is given by

$$T_\lambda^{\mathbf{SW}}(u,v)=\begin{cases}T_{\mathbf{P}}(u,v), & \text{if } \lambda=0,\\ T_{\mathbf{D}}(u,v), & \text{if } \lambda=\infty,\\ \max(0,(1-\lambda)uv+\lambda(u+v-1)), & \text{if } \lambda\in\,]0,\infty[.\end{cases}$$

For $\lambda\in[0,\infty[$, $T_\lambda^{\mathbf{SW}}$ is a continuous Archimedean t-norm with additive generator

$$t_\lambda^{\mathbf{SW}}(u)=\begin{cases}-(1-\lambda)\log(\lambda+(1-\lambda)u), & \text{if } \lambda\in[0,\infty[\setminus\{1\},\\ 1-u, & \text{if } \lambda=1\end{cases}$$

for all $u\in[0,1]$; parameters $\lambda\in\,]0,\infty[$ lead to nilpotent t-norms (with $T_1^{\mathbf{SW}}=T_{\mathbf{L}}$ as special case), while $T_0^{\mathbf{SW}}=T_{\mathbf{P}}$ is the only strict member. Note that, for better readability, we again omit the superscript indicating the family when discussing properties of additive generators.

Clearly, the derivatives of the additive generators exist and are, for all $\lambda\in[0,\infty[\setminus\{1\}$ and all $u\in\,]0,1[$, given by:

$$\begin{aligned}t'_\lambda(u)&=-\tfrac{(1-\lambda)^2}{\lambda+(1-\lambda)u},\\ t''_\lambda(u)&=\tfrac{(1-\lambda)^3}{(\lambda+(1-\lambda)u)^2},\\ t'''_\lambda(u)&=-\tfrac{2(1-\lambda)^4}{(\lambda+(1-\lambda)u)^3}.\end{aligned}$$

In case $\lambda=1$, it holds that $t'_1(u)=-1$ and $t''_1(u)=t'''_1(u)=0$ for all $u\in\,]0,1[$.

The family of Sugeno-Weber t-norms is ordered according to its parameter: $T_\lambda^{\mathbf{SW}}\geqslant T_\mu^{\mathbf{SW}}$ if and only if $\lambda\leqslant\mu$. Moreover, since every t-norm dominates itself as well as $T_{\mathbf{D}}$, it suffices to investigate dominance between two Sugeno-Weber t-norms $T_\lambda^{\mathbf{SW}}$ and $T_\mu^{\mathbf{SW}}$ with parameters $0\leqslant\lambda<\mu<\infty$.

Note that the function $h=t_\lambda\circ t_\mu^{(-1)}\colon [0,\infty]\to[0,\infty]$ is continuous, strictly increasing and differentiable on $\left]0,t_\mu(0)\right[$ and fulfills $h(0)=0$. If h is convex on $\left]0,t_\mu(0)\right[$ and if either h or h' is log- or geo-convex on $\left]0,t_\mu(0)\right[$, then $T_\lambda^{\mathbf{SW}}$ dominates $T_\mu^{\mathbf{SW}}$.

Convexity of h. The function h is convex on $\left]0,t_\mu(0)\right[$ if and only if, for all $u\in\,]0,1[$,

$$t'_\lambda(u)t''_\mu(u)-t''_\lambda(u)t'_\mu(u)\geqslant 0.$$

In case $\lambda\neq 1\neq\mu$, the latter inequality is equivalent to

$$\begin{aligned}\frac{(1-\lambda)^3}{(\lambda+(1-\lambda)u)^2}\frac{(1-\mu)^2}{\mu+(1-\mu)u}&\geqslant\frac{(1-\lambda)^2}{\lambda+(1-\lambda)u}\frac{(1-\mu)^3}{(\mu+(1-\mu)u)^2}\quad\Leftrightarrow\\ \frac{1-\lambda}{\lambda+(1-\lambda)u}&\geqslant\tfrac{1-\mu}{\mu+(1-\mu)u}\quad\Leftrightarrow\end{aligned}$$

$$\begin{aligned}(1-\lambda)(\mu+(1-\mu)u) \geqslant (1-\mu)(\lambda+(1-\lambda)u) \qquad &\Leftrightarrow \\ \mu \geqslant \lambda\,. &\end{aligned}$$

In case $\lambda = 1$, the condition reduces to $-t''_\mu(u) \geqslant 0$ being equivalent to $\mu \geqslant 1 = \lambda$. In case $\mu = 1$, the condition becomes $t''_\lambda(u) \geqslant 0$, i.e., $\lambda \leqslant 1 = \mu$. Summarizing, in all cases h is convex if and only if $\mu \geqslant \lambda$.

Log-convexity of h'. Substituting the expressions for the derivatives of the additive generators in (13) and applying basic transformations shows that for all $\lambda \neq 1 \neq \mu$ the function h' is log-convex on $\left]0, t_\mu(0)\right[$ if and only if, for all $u \in]0,1[$,

$$\frac{(1-\lambda)^5(1-\mu)^4(\mu-\lambda)}{(\lambda+(1-\lambda)u)^4(\mu+(1-\mu)u)^3} \geqslant 0 \quad \Leftrightarrow \quad (\mu-\lambda)(1-\lambda) \geqslant 0\,.$$

This inequality clearly holds whenever $\lambda < 1$ and $\mu > \lambda$. In case $\lambda = 1 < \mu$, we obtain in a similar way the condition

$$2t''^2_\mu(u) - t'_\mu(u)t'''_\mu(u) = 2\frac{(1-\mu)^6}{(\mu+(1-\mu)u)^4} - \frac{2(1-\mu)^6}{(\mu+(1-\mu)u)^4} \geqslant 0\,,$$

which trivially holds. Finally, in case $\lambda < \mu = 1$, we end up with the following equivalent inequality

$$t''^2_\lambda(u) - t'_\lambda(u)t'''_\lambda(u) = -\frac{(1-\lambda)^6}{(\lambda+(1-\lambda)u)^4} \leqslant 0\,,$$

which is also obviously fulfilled.

The above results can be summarized as follows.

COROLLARY 13. *Consider the family of Sugeno-Weber t-norms $(T^{\mathbf{SW}}_\lambda)_{\lambda\in[0,\infty]}$. For all $\lambda,\mu \in [0,\infty]$ such that*

$$\lambda \leqslant \min(1,\mu)$$

it holds that $T^{\mathbf{SW}}_\lambda \gg T^{\mathbf{SW}}_\mu$.

This means in particular that any Sugeno-Weber t-norm greater than or equal to the Łukasiewicz t-norm dominates any other, but smaller Sugeno-Weber t-norm. Naturally, this raises the question whether dominance is also in accordance with the ordering of the parameters when both t-norms are smaller than the Łukasiewicz t-norm, i.e, when $1 < \lambda < \mu$. However, in general this need not be the case as the following example demonstrates.

EXAMPLE 1. Consider the Sugeno-Weber t-norms $T^{\mathbf{SW}}_{51}$ and $T^{\mathbf{SW}}_{101}$ and let $x = y = u = v = \frac{975}{1000}$. Then $T^{\mathbf{SW}}_{51}(x,x) = \frac{147}{160}$ and $T^{\mathbf{SW}}_{101}(x,x) = \frac{142}{160}$ such that

$$T^{\mathbf{SW}}_{51}(T^{\mathbf{SW}}_{101}(x,x), T^{\mathbf{SW}}_{101}(x,x)) = \tfrac{182}{1280} < \tfrac{227}{1280} = T^{\mathbf{SW}}_{101}(T^{\mathbf{SW}}_{51}(x,x), T^{\mathbf{SW}}_{51}(x,x))\,,$$

showing that $T^{\mathbf{SW}}_{51}$ does not dominate $T^{\mathbf{SW}}_{101}$, although $\lambda = 51 \leqslant 101 = \mu$.

So far, we have only exploited the log-convexity of h'. Of course, the remaining sufficient conditions can still be applied. We provide them in two forms: first, after substituting the expressions for the derivatives of the additive generators, and second, in their simplest form after applying several transformations. Further, we discuss the case $1 < \lambda < \mu$ only in order to gain additional insight into dominance between two Sugeno-Weber t-norms.

Geo-convexity of h'. The function h' is geo-convex on $\left]0, t_\mu(0)\right[$ if and only if, for all $u \in \left]0,1\right[$,

$$-\frac{(1-\lambda)^5(1-\mu)^5(\mu-\lambda)\log(\mu+(1-\mu)u)}{(\lambda+(1-\lambda)u)^4(\mu+(1-\mu)u)^3} \geqslant -\frac{(1-\lambda)^4(1-\mu)^6(\mu-\lambda)}{(\lambda+(1-\lambda)u)^3(\mu+(1-\mu)u)^4} \quad \Leftrightarrow$$

$$(\mu-\lambda)(1-\lambda)\left(\frac{\mu+(1-\mu)u}{1-\mu}\log(\mu+(1-\mu)u) - \frac{\lambda+(1-\lambda)u}{1-\lambda}\right) \leqslant 0.$$

In case $1 < \lambda < \mu$, we need to show that, for all $u \in \left]0,1\right[$,

$$\frac{\mu+(1-\mu)u}{\mu-1}\log(\mu+(1-\mu)u) \leqslant \frac{\lambda+(1-\lambda)u}{\lambda-1}. \tag{16}$$

Note that for all $u \in [0,1]$, it holds that the function $f_u\colon \left]1,\infty\right[\to \left]0,\infty\right[$, $f_u(t) = \frac{t+(1-t)u}{t-1}$ is decreasing, since $\frac{df_u}{dt}(t) = -\frac{1}{(t-1)^2} < 0$. Therefore, for $\mu \geqslant \lambda$ it holds that

$$\frac{\mu+(1-\mu)u}{\mu-1} \leqslant \frac{\lambda+(1-\lambda)u}{\lambda-1}.$$

Hence, as long as the factor $\log(\mu+(1-\mu)u)$, which is always positive for $\mu > 1$, is upper bounded by 1, also (16) follows. This requires that $\mu+(1-\mu)u \leqslant e$ for all $u \in \left]0,1\right[$, i.e. $\mu \leqslant e$. We conclude that h' is geo-convex at least when $1 < \lambda < \mu \leqslant e$.

Log-convexity of h. The function h is log-convex on $\left]0, t_\mu(0)\right[$ if and only if, for all $u \in \left]0,1\right[$,

$$-\frac{(1-\lambda)^4(1-\mu)^2}{(\lambda+(1-\lambda)u)^2(\mu+(1-\mu)u)} \geqslant \frac{(1-\lambda)^3(\mu-\lambda)(1-\mu)^2\log(\lambda+(1-\lambda)u)}{(\lambda+(1-\lambda)u)^2(\mu+(1-\mu)u)^2} \quad \Leftrightarrow$$

$$\mu+(1-\mu)u+\frac{\mu-\lambda}{1-\lambda}\log(\lambda+(1-\lambda)u) \leqslant 0.$$

As u approaches 1, the left-hand side approaches 1 as well. Therefore, h can never be log-convex.

Geo-convexity of h. The function h is geo-convex on $\left]0, t_\mu(0)\right[$ if and only if, for all $u \in \left]0,1\right[$,

$$\frac{(\lambda-1)(\log(\lambda+(1-\lambda)u)+1)}{(\lambda+(1-\lambda)u)\log(\lambda+(1-\lambda)u)} \leqslant \frac{(\mu-1)(\log(\mu+(1-\mu)u)+1)}{(\mu+(1-\mu)u)\log(\mu+(1-\mu)u)}. \tag{17}$$

In case $1 < \lambda < \mu$, we consider the function $g_u\colon\ ]1,\infty[\ \to\ ]0,\infty[$,

$$g_u(t) = \frac{(t-1)(\log(t+(1-t)u)+1)}{(t+(1-t)u)\log(t+(1-t)u)},$$

which is increasing whenever

$$\frac{dg_u}{dt}(t) = \frac{\log^2(t+(1-t)u)+\log(t+(1-t)u)+(t-1)(u-1)}{\big((t+(1-t)u)\log(t+(1-t)u)\big)^2}$$

is positive for all $t \in\ ]1,\infty[$. Note that for $t > 1$ it holds that $\log(t+(1-t)u) > 0$ for all $u \in\ ]0,1[$, and hence, $\frac{dg_u}{dt}(t)$ is positive whenever

$$p(t) = \log^2(t) + \log(t) - t + 1 \geqslant 0\,.$$

Numerical investigations (using Maple) show that this is the case for $1 \leqslant t \leqslant 6.00914$ (with 6.00914 denoting the second root of $p(t) = 0$). Therefore, h is geo-convex at least when $1 < \lambda < \mu \leqslant 6.00914$.

Of course, this does not contradict the findings on the geo-convexity of h'. Interestingly, the geo-convexity investigation allows us to extend Corollary 13.

COROLLARY 14. *Consider the family of Sugeno-Weber t-norms* $(T_\lambda^{\mathbf{SW}})_{\lambda\in[0,\infty]}$. *For all* $\lambda,\mu \in [0,\infty]$ *such that*

(i) either $\lambda \leqslant \min(1,\mu)$,

(ii) or $1 < \lambda \leqslant \mu \leqslant 6.00914$,

it holds that $T_\lambda^{\mathbf{SW}} \gg T_\mu^{\mathbf{SW}}$.

Having in mind the geo-convexity study of h, it is intuitively clear that as λ approaches 1 from the right in (17), even larger values of μ will do (knowing that for $\lambda = 1$, μ can be arbitrarily large). However, this problem becomes numerically intractable.

6. Dominance between two parametric families of t-norms

Finally, we turn to the investigation of dominance between a member of the family of Dombi t-norms and a member of the family of Yager t-norms. Since (apart from the limit cases) the Dombi t-norms are strict and the Yager t-norms are nilpotent, it suffices to investigate when a Dombi t-norm dominates a Yager t-norm. The investigation of such a mixed case (strict versus nilpotent) is possible thanks to the new conditions applicable to all continuous Archimedean t-norms.

The family of Dombi t-norms $(T_\lambda^{\mathbf{D}})_{\lambda\in[0,\infty]}$ is given by

$$T_\lambda^{\mathbf{D}}(u,v) = \begin{cases} T_{\mathbf{D}}(u,v), & \text{if } \lambda = 0,\\ T_{\mathbf{M}}(u,v), & \text{if } \lambda = \infty,\\ \dfrac{1}{1+\left(\left(\frac{1-u}{u}\right)^\lambda + \left(\frac{1-v}{v}\right)^\lambda\right)^{1/\lambda}}, & \text{if } \lambda \in\ ]0,\infty[. \end{cases}$$

For $\lambda \in]0,\infty[$, $T_\lambda^{\mathbf{D}}$ is a strict t-norm with generator $t_\lambda^{\mathbf{D}}(u) = (\frac{1-u}{u})^\lambda$ for all $u \in [0,1]$. The derivatives of the additive generators are, for all $\lambda \in]0,\infty[$ and all $u \in [0,1]$, given by:

$$(t_\lambda^{\mathbf{D}})'(u) = -\frac{\lambda(1-u)^{\lambda-1}}{u^{\lambda+1}},$$
$$(t_\lambda^{\mathbf{D}})''(u) = (\lambda+1-2u)\frac{\lambda(1-u)^{\lambda-2}}{u^{\lambda+2}}.$$

Similarly, the family of Yager t-norms $(T_\mu^{\mathbf{Y}})_{\mu\in[0,\infty]}$ is defined by

$$T_\mu^{\mathbf{Y}}(u,v) = \begin{cases} T_{\mathbf{D}}(u,v), & \text{if } \mu = 0,\\ T_{\mathbf{M}}(u,v), & \text{if } \mu = \infty,\\ \max(1-((1-u)^\mu + (1-v)^\mu)^{1/\mu}, 0), & \text{if } \mu \in]0,\infty[. \end{cases}$$

For $\mu \in]0,\infty[$, $T_\mu^{\mathbf{Y}}$ is a nilpotent t-norm with additive generator $t_\mu^{\mathbf{Y}}(u) = (1-u)^\mu$ for all $u \in [0,1]$. The derivatives of the additive generators are, for all $\mu \in]0,\infty[$ and all $u \in]0,1[$, given by:

$$(t_\mu^{\mathbf{Y}})'(u) = -\mu(1-u)^{\mu-1},$$
$$(t_\mu^{\mathbf{Y}})''(u) = \mu(\mu-1)(1-u)^{\mu-2}.$$

Note that for both families it holds that the additive generators of the continuous Archimedean members are powers of the basic additive generators $t_1^{\mathbf{D}}(u) = \frac{1-u}{u}$ and $t_1^{\mathbf{Y}}(u) = 1-u$. Investigating dominance within each of these families then turns the generalized Mulholland inequality into the Minkowski inequality and dominance within each family is in accordance with the ordering of the parameters (see also [8]), i.e.,

$$T_{\lambda_1}^{\mathbf{D}} \gg T_{\lambda_2}^{\mathbf{D}} \quad \Leftrightarrow \quad \lambda_1 \geqslant \lambda_2 \qquad \text{and} \qquad T_{\mu_1}^{\mathbf{Y}} \gg T_{\mu_2}^{\mathbf{Y}} \quad \Leftrightarrow \quad \mu_1 \geqslant \mu_2 .$$

We will now investigate for which λ and μ it holds that the Dombi t-norm $T_\lambda^{\mathbf{D}}$ dominates the Yager t-norm $T_\mu^{\mathbf{Y}}$. Since for both families the limiting members are $T_{\mathbf{D}}$ and $T_{\mathbf{M}}$, it suffices to consider $\lambda,\mu \in]0,\infty[$ only. Note that the function $h = t_\lambda^{\mathbf{D}} \circ (t_\mu^{\mathbf{Y}})^{(-1)}\colon [0,\infty] \to [0,\infty]$ is continuous, strictly increasing and differentiable on $\left]0, t_\mu^{\mathbf{Y}}(0)\right[$ and fulfills $h(0) = 0$. If h is convex on $\left]0, t_\mu^{\mathbf{Y}}(0)\right[$ and if either h or h' is log- or geo-convex on $\left]0, t_\mu^{\mathbf{Y}}(0)\right[$, then $T_\lambda^{\mathbf{D}}$ dominates $T_\mu^{\mathbf{Y}}$. For the sake of brevity we will further omit the indication of the families; λ and μ therefore not only indicate the specific parameter but also the corresponding family.

Convexity of h. The function h is convex on $\left]0, t_\mu(0)\right[$ if and only if, for all $u \in]0,1[$,

$$\begin{aligned} t'_\lambda(u)t''_\mu(u) - t''_\lambda(u)t'_\mu(u) \geqslant 0 \quad &\Leftrightarrow \\ -\lambda\mu(\mu-1)\frac{(1-u)^{\lambda+\mu-3}}{u^{\lambda+1}} + (\lambda+1-2u)\lambda\mu\frac{(1-u)^{\lambda+\mu-3}}{u^{\lambda+2}} \geqslant 0, \quad &\Leftrightarrow \\ \lambda\mu\frac{(1-u)^{\lambda+\mu-3}}{u^{\lambda+2}}(\lambda+1-u(\mu+1)) \geqslant 0, \quad &\Leftrightarrow \\ \lambda+1 \geqslant u(\mu+1). \end{aligned}$$

This inequality is fulfilled for all $u \in]0,1[$ if and only if $\lambda \geqslant \mu$.

Geo-convexity of h. The function h is geo-convex on $\left]0, t_\mu(0)\right[$ if and only if, for all $u \in]0,1[$,

$$\frac{{t'_\lambda}^2(u) - t_\lambda(u)t''_\lambda(u)}{t_\lambda(u)t'_\lambda(u)} \geqslant \frac{{t'_\mu}^2(u) - t_\mu(u)t''_\mu(u)}{t_\mu(u)t'_\mu(u)}$$

being equivalent to

$$\begin{aligned} (\lambda - (\lambda + 1 - 2u))\tfrac{1}{u(u-1)} \geqslant \tfrac{(\mu-(\mu-1))}{(u-1)} \quad &\Leftrightarrow \\ \tfrac{2u-1}{u(u-1)} \geqslant \tfrac{1}{(u-1)} \quad &\Leftrightarrow \\ u \leqslant 1, \end{aligned}$$

which obviously is fulfilled for all $u \in]0,1[$.

COROLLARY 15. *Consider the families of Dombi t-norms $(T^{\mathbf{D}}_\lambda)_{\lambda \in [0,\infty]}$ and of Yager t-norms $(T^{\mathbf{Y}}_\mu)_{\mu \in [0,\infty]}$. For all $\lambda, \mu \in [0,\infty]$ it holds that $T^{\mathbf{D}}_\lambda$ dominates $T^{\mathbf{Y}}_\mu$ if and only if $\lambda \geqslant \mu$.*

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Susanne Saminger-Platz
Department of Knowledge-Based Mathematical Systems
Johannes Kepler University Linz
Altenbergerstrasse 69
A-4040 Linz, Austria
e-mail: susanne.saminger-platz@jku.at

Bernard De Baets
Department of Applied Mathematics, Biometrics and Process Control
Ghent University
Coupure links 653
B-9000 Gent, Belgium
e-mail: bernard.debaets@ugent.be

Hans De Meyer
Department of Applied Mathematics and Computer Science
Ghent University
Krijgslaan 281 S9
B-9000 Gent, Belgium
e-mail: hans.demeyer@ugent.be