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OSTROWSKI'S INEQUALITY IN PRE–HILBERT C^*–MODULES

LJILJANA ARAMBAŠIĆ AND RAJNA RAJIĆ

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Abstract. In this paper we show that Ostrowski's inequality can be generalized to the case of pre-Hilbert C^*-modules. We also obtain some results related to Ostrowski's inequality for elements of C^*-algebras.

1. Introduction and preliminaries

The following result is known as Ostrowski type inequality in an inner-product space:

THEOREM 1.1. *Let $(H,(\cdot,\cdot))$ be a real or complex inner-product space and $x,y \in H$ two linearly independent vectors. If $z \in H$ is such that $(x,z)=0$, then*

$$|(z,y)|^2 \leqslant \frac{\|z\|^2}{\|x\|^2}(\|x\|^2\|y\|^2 - |(x,y)|^2). \tag{1}$$

The equality in (1) holds if and only if

$$z = \nu(y - \frac{\overline{(x,y)}}{\|x\|^2}x),$$

where $\nu \in \mathbb{C}$ is such that $|\nu| = \frac{\|x\|\,\|z\|}{\sqrt{\|x\|^2\|y\|^2 - |(x,y)|^2}}$.

This was proved by H. Šikić and T. Šikić in 2001, [8], by the use of an argument based on orthogonal projections in inner-product spaces. Also, in [2] the same was proved by the use of elementary arguments and the Cauchy-Schwarz inequality in inner-product spaces. (Notice that Theorem 1.1 was proved in [8] under the additional assumption $(z,y)=1$, but replacing y with $\frac{y}{(y,z)}$ we get Theorem 1.1.)

In the special case $H=\mathbb{R}^n$, $n \in \mathbb{N}$, the inequality (1) was obtained by Ostrowski (see [4]). In the case of L^2-functions this result was proved by C.E.M. Pearce, J. Pečarić and S. Varošanec in [5].

In this paper we extend Theorem 1.1 to elements of a pre-Hilbert C^*-module. Pre-Hilbert C^*-modules generalize inner-product spaces by allowing the inner product

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to take values in a more general C^*-algebra than the field of complex numbers. Formal definitions of a C^*-algebra and a pre-Hilbert C^*-module are as follows.

A Banach $*$-algebra is a complex Banach algebra $\mathscr{A}$ with a conjugate-linear involution $*:\mathscr{A}\to\mathscr{A}$ such that $a^{**}=a$ and $(ab)^*=b^*a^*$ for all $a,b\in\mathscr{A}$. A *C^*-algebra* is a Banach $*$-algebra with the additional norm condition $\|a^*a\|=\|a\|^2$. It easily follows that involution is an isometry. The simplest examples of C^*-algebras are the algebras of all bounded linear operators $\mathbf{B}(H)$ and all compact operators $\mathbf{K}(H)$ on a Hilbert space H with the usual adjoint operation. General references for the theory of C^*-algebras are [1], [6] or [7].

A *pre-Hilbert C^*-module V over a C^*-algebra $\mathscr{A}$*, or a *pre-Hilbert $\mathscr{A}$-module* is a (right) $\mathscr{A}$-module together with an $\mathscr{A}$-valued inner product $\langle\cdot,\cdot\rangle : V\times V\to\mathscr{A}$ satisfying the conditions:

(1) $\langle x,\alpha y+\beta z\rangle=\alpha\langle x,y\rangle+\beta\langle x,z\rangle$ for $x,y,z\in V,\ \alpha,\beta\in\mathbb{C}$,
(2) $\langle x,ya\rangle=\langle x,y\rangle a$ for $x,y\in V,\ a\in\mathscr{A}$,
(3) $\langle x,y\rangle^*=\langle y,x\rangle$ for $x,y\in V$,
(4) $\langle x,x\rangle\geqslant 0$ for $x\in V$,
(5) $\langle x,x\rangle=0$ if and only if $x=0$.

It is straightforward that a C^*-algebra valued inner product is conjugate-linear in the first variable. We can define a norm on V by $\|x\|=\|\langle x,x\rangle\|^{\frac{1}{2}}$.

For a pre-Hilbert $\mathscr{A}$-module V the following inequality holds:

$$\|xa\|\leqslant\|x\|\|a\|\quad (x\in V,a\in\mathscr{A}).$$

Also, the Cauchy-Schwarz inequality holds, that is,

$$\|\langle x,y\rangle\|\leqslant\|x\|\|y\|\quad (x,y\in V).\tag{2}$$

A pre-Hilbert $\mathscr{A}$-module which is complete with respect to its norm is called a *Hilbert C^*-module over $\mathscr{A}$*, or a *Hilbert $\mathscr{A}$-module*.

Clearly, every inner-product space is a pre-Hilbert $\mathbb{C}$-module (and every Hilbert space is a Hilbert $\mathbb{C}$-module). Also, every C^*-algebra is a Hilbert C^*-module over itself with the inner product $\langle a,b\rangle=a^*b$. The Banach space $\mathbf{B}(H_1,H_2)$ of all bounded linear operators between Hilbert spaces H_1 and H_2 is a Hilbert $\mathbf{B}(H_1)$-module, where the inner product is defined as $\langle T,S\rangle=T^*S$, and T^* denotes the adjoint operator of T. For more details about Hilbert C^*-modules the reader is referred to [3] or [9].

We say that two elements x,y of a pre-Hilbert C^*-module are *orthogonal* if $\langle x,y\rangle=0$.

Throughout this paper, if x is an element of a pre-Hilbert $\mathscr{A}$-module V, $|x|$ refers to the unique positive square root of $\langle x,x\rangle$. In the case of a C^*-algebra we get the usual $|a|=(a^*a)^{\frac{1}{2}}$.

2. The main result

It is well known that in a pre-Hilbert C^*-module V, besides (2), the following stronger version of the Cauchy-Schwarz inequality holds:

$$\langle y,x\rangle\langle x,y\rangle\leqslant\|x\|^2\langle y,y\rangle\quad (x,y\in V).\tag{3}$$

Let us first consider the case of equality in (3).

LEMMA 2.1. *Let V be a pre-Hilbert $\mathscr{A}$-module and $x,y \in V$, $x \neq 0$. Then it holds*

$$\langle y,x\rangle \langle x,y\rangle = \|x\|^2\langle y,y\rangle \Leftrightarrow y = \frac{1}{\|x\|^2}x\langle x,y\rangle .$$

Proof. We may assume that $\|x\| = 1$.

First, let us suppose that $y = x\langle x,y\rangle$. Then $\langle y,x\rangle \langle x,y\rangle = \langle y,x\rangle \langle x,x\langle x,y\rangle \rangle = \langle y,x\rangle \langle x,x\rangle \langle x,y\rangle$ which implies that

$$\begin{aligned} 0 &= \langle x\langle x,y\rangle - y, x\langle x,y\rangle - y\rangle \\ &= \langle x\langle x,y\rangle , x\langle x,y\rangle \rangle - \langle x\langle x,y\rangle , y\rangle - \langle y, x\langle x,y\rangle \rangle + \langle y,y\rangle \\ &= \langle y,x\rangle \langle x,x\rangle \langle x,y\rangle - \langle y,x\rangle \langle x,y\rangle - \langle y,x\rangle \langle x,y\rangle + \langle y,y\rangle \\ &= \langle y,y\rangle - \langle y,x\rangle \langle x,y\rangle . \end{aligned}$$

Hence, $\langle y,x\rangle \langle x,y\rangle = \langle y,y\rangle$.

Conversely, suppose that $\langle y,x\rangle \langle x,y\rangle = \langle y,y\rangle$. Let us prove that $y = x\langle x,y\rangle$. Since $\langle y,x\rangle \langle x,x\rangle \langle x,y\rangle \leqslant \|\langle x,x\rangle\| \langle y,x\rangle \langle x,y\rangle = \langle y,x\rangle \langle x,y\rangle$ (see [6, Proposition 1.3.5]), we have

$$0 \leqslant \langle x\langle x,y\rangle - y, x\langle x,y\rangle - y\rangle \leqslant \langle y,y\rangle - \langle y,x\rangle \langle x,y\rangle = 0.$$

Thus, $\langle x\langle x,y\rangle - y, x\langle x,y\rangle - y\rangle = 0$, from which it follows $y = x\langle x,y\rangle$. □

Now we prove the statement which generalizes Theorem 1.1 for elements of a pre-Hilbert C^*-module.

THEOREM 2.2. *Let $\mathscr{A}$ be a C^*-algebra and V a pre-Hilbert C^*-module over $\mathscr{A}$. Let $x,y \in V$, $x \neq 0$. Let $z \in V$, $z \neq 0$, be such that $\langle x,z\rangle = 0$. Then*

$$|\langle z,y\rangle|^2 \leqslant \frac{\|z\|^2}{\|x\|^2}(\|x\|^2|y|^2 - |\langle x,y\rangle|^2). \tag{4}$$

The equality in (4) holds if and only if $y - \frac{1}{\|x\|^2}x\langle x,y\rangle = \frac{1}{\|z\|^2}z\langle z,y\rangle$.

Proof. Without loss of generality we can assume that $\|x\| = \|z\| = 1$.

Let us put $v = y - x\langle x,y\rangle$. Then it holds

$$\langle v,z\rangle = \langle y - x\langle x,y\rangle , z\rangle = \langle y,z\rangle - \langle y,x\rangle \langle x,z\rangle = \langle y,z\rangle , \tag{5}$$

from which by using (3) we get

$$\langle y,z\rangle \langle z,y\rangle = \langle v,z\rangle \langle z,v\rangle \leqslant \|z\|^2\langle v,v\rangle = \langle v,v\rangle . \tag{6}$$

Since $\langle y,x\rangle \langle x,x\rangle \langle x,y\rangle \leqslant \|\langle x,x\rangle\| \langle y,x\rangle \langle x,y\rangle = \langle y,x\rangle \langle x,y\rangle$, we obtain

$$\begin{aligned} \langle v,v\rangle &= \langle y - x\langle x,y\rangle , y - x\langle x,y\rangle \rangle \\ &= \langle y,y\rangle - 2\langle y,x\rangle \langle x,y\rangle + \langle y,x\rangle \langle x,x\rangle \langle x,y\rangle \\ &\leqslant \langle y,y\rangle - 2\langle y,x\rangle \langle x,y\rangle + \langle y,x\rangle \langle x,y\rangle \\ &= \langle y,y\rangle - \langle y,x\rangle \langle x,y\rangle . \end{aligned}$$

From this and (6) it follows that

$$|\langle z,y\rangle|^2 = \langle y,z\rangle\langle z,y\rangle \leqslant \langle y,y\rangle - \langle y,x\rangle\langle x,y\rangle = |y|^2 - |\langle x,y\rangle|^2$$

which proves (4).

It follows from the above calculation that the equality in (4) holds if and only if the following two conditions are satisfied

$$\langle v,z\rangle\langle z,v\rangle = \langle v,v\rangle \quad \text{and} \quad \langle y,x\rangle\langle x,x\rangle\langle x,y\rangle = \langle y,x\rangle\langle x,y\rangle. \tag{7}$$

According to Lemma 2.1 and (5), the first condition of (7) is equivalent to $v = \frac{1}{\|z\|^2}z\langle z,v\rangle = z\langle z,v\rangle = z\langle z,y\rangle$, that is,

$$y - x\langle x,y\rangle = z\langle z,y\rangle. \tag{8}$$

To complete the proof, it is enough to show that (8) implies the second condition of (7). Indeed, since $\langle x,z\rangle = 0$ by (8) we have

$$\begin{aligned}\langle y,x\rangle\langle x,y\rangle &= \langle y,x\rangle\langle x, x\langle x,y\rangle + z\langle z,y\rangle\rangle \\ &= \langle y,x\rangle\langle x,x\rangle\langle x,y\rangle + \langle y,x\rangle\langle x,z\rangle\langle z,y\rangle \\ &= \langle y,x\rangle\langle x,x\rangle\langle x,y\rangle.\end{aligned}$$

□

Since every C^*-algebra can be recognized as a Hilbert C^*-module over itself with the inner product $\langle a,b\rangle = a^*b$, the following result is an immediate consequence of Theorem 2.2.

COROLLARY 2.3. *Let $\mathscr{A}$ be a C^*-algebra. Let $a,b \in \mathscr{A}$, $a \neq 0$. Let $c \in \mathscr{A}$, $c \neq 0$, be such that $a^*c = 0$. Then*

$$|c^*b|^2 \leqslant \frac{\|c\|^2}{\|a\|^2}(\|a\|^2|b|^2 - |a^*b|^2). \tag{9}$$

The equality in (9) holds if and only if $b - \frac{1}{\|a\|^2}aa^*b = \frac{1}{\|c\|^2}cc^*b$.

In [8] H. Šikić and T. Šikić observed that Ostrowski's inequality in inner-product spaces is actually a statement about projections. From Theorem 2.2 we can see the nature of Ostrowski's inequality in pre-Hilbert C^*-modules. To do this, let us first state some notations and definitions.

By V we denote a pre-Hilbert C^*-module over a C^*-algebra $\mathscr{A}$. Recall that a map $T: V \to V$ is *$\mathscr{A}$-linear* if $T(xa) = T(x)a$ for all $x \in V$, $a \in \mathscr{A}$. We shall say that a bounded $\mathscr{A}$-linear map $T: V \to V$ is *positive* if $\langle Tx,x\rangle \geqslant 0$ for all $x \in V$. If $T: V \to V$ is a positive map, we write $T \geqslant 0$. Also, if $T,S: V \to V$ are two linear maps satisfying $S - T \geqslant 0$, we shall write $T \leqslant S$. Let us mention here that in the case of a Hilbert C^*-module V, the algebra $\mathbf{B}(V)$ of all adjointable operators on V (i.e., the set of all maps $A: V \to V$ for which there is a map $A^*: V \to V$ such that $\langle Ax,y\rangle = \langle x,A^*y\rangle$ for all $x,y \in V$) is a C^*-algebra. Every element $T \in \mathbf{B}(V)$ is a bounded $\mathscr{A}$-linear map [3, p. 8] and T is positive as an element of the C^*-algebra $\mathbf{B}(V)$ if and only if $\langle Tx,x\rangle \geqslant 0$ for all $x \in V$ (see [3, Lemma 4.1]).

Let us now introduce a class of operators analogous to the finite-rank operators on an inner-product space. For $x,y \in V$ we define $\theta_{x,y} : V \to V$ by

$$\theta_{x,y}(z) = x\langle y,z\rangle \quad (z \in V).$$

Obviously, $\theta_{x,y}$ is an $\mathscr{A}$-linear map and $\|\theta_{x,y}\| \leqslant \|x\|\|y\|$ for all $x,y \in V$. It is straightforward that for such maps it holds

$$\theta_{x,y}\theta_{z,v} = \theta_{x\langle y,z\rangle ,v} \quad (x,y,z,v \in V). \tag{10}$$

Observe that $\theta_{x,x}$ is positive for all $x \in V$. Namely,

$$\langle \theta_{x,x}(y),y\rangle = \langle x\langle x,y\rangle ,y\rangle = \langle y,x\rangle \langle x,y\rangle = |\langle x,y\rangle |^2 \geqslant 0 \quad (y \in V). \tag{11}$$

Thus, the inequality (4) from Theorem 2.2 can be expressed in the following form.

COROLLARY 2.4. *Let $\mathscr{A}$ be a C^*-algebra and V a pre-Hilbert C^*-module over $\mathscr{A}$. Let $x,z \in V$ be such that $\|x\| = \|z\| = 1$ and $\langle x,z\rangle = 0$. Then*

$$\theta_{x,x} + \theta_{z,z} \leqslant I,$$

where $I : V \to V$ denotes the identity operator.

Proof. By definition $\theta_{x,x} + \theta_{z,z} \leqslant I$ if and only if

$$\langle \theta_{x,x}(y),y\rangle + \langle \theta_{z,z}(y),y\rangle \leqslant \langle y,y\rangle \quad (y \in V),$$

that is, by (11), if and only if

$$|\langle x,y\rangle |^2 + |\langle z,y\rangle |^2 \leqslant |y|^2 \quad (y \in V).$$

Therefore, our statement follows from (4) of Theorem 2.2. □

REMARK 2.5. (a) Observe that the condition $\langle x,z\rangle = 0$ is equivalent to $\theta_{x,x}\theta_{z,z} = 0$. Indeed, if $\langle x,z\rangle = 0$ then it follows from (10) that $\theta_{x,x}\theta_{z,z} = \theta_{x\langle x,z\rangle ,z} = \theta_{0,z} = 0$. Conversely, if $\theta_{x,x}\theta_{z,z} = 0$ then $\langle z,x\rangle \langle x,z\rangle \langle z,x\rangle = \langle z, \theta_{x,x}\theta_{z,z}(x)\rangle = 0$ from which we get

$$\|\langle x,z\rangle \|^4 = \|\langle z,x\rangle \langle x,z\rangle \|^2 = \|\langle z,x\rangle \langle x,z\rangle \langle z,x\rangle \langle x,z\rangle \| = 0,$$

that is, $\langle x,z\rangle = 0$.

(b) Note that for a (unit) vector $x \in V$, $\theta_{x,x}^2 \neq \theta_{x,x}$ in general. But, if V is an inner-product space and $x \in V$, $\|x\| = 1$, then by (10) it holds $\theta_{x,x}^2 = \theta_{x\langle x,x\rangle ,x} = \theta_{x,x}$, so $\theta_{x,x}$ is a rank one projection on the subspace of V spanned by x. Thus, in the case of an inner-product space V, the inequality (4) from Theorem 2.2 can be stated in terms of rank one projections as follows:

Whenever x,z are two orthogonal unit vectors of an inner-product space V, then $\theta_{x,x}$ and $\theta_{z,z}$ are two orthogonal projections whose product is zero, so $\theta_{x,x} + \theta_{z,z} \leqslant I$.

(c) The condition $\langle x,z\rangle = 0$ from Theorem 2.2 is not a necessary condition on elements x and z for Ostrowski's inequality to hold. As an example, we can take a C^*-algebra $\mathscr{A} = M_3(\mathbb{C})$ of all 3×3 complex matrices, regarded as a Hilbert C^*-module over itself. Then we choose the elements

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

of this algebra. We get $\|A\| = \|B\| = \|C\| = 1$,

$$|C^*B|^2 = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leqslant \begin{bmatrix} \frac{3}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{\|C\|^2}{\|A\|^2}(\|A\|^2|B|^2 - |A^*B|^2),$$

which is (9) from Corollary 2.3, but $A^*C = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0.$

However, if $x,z \in V$, $\|x\| = \|z\| = 1$, are elements of a pre-Hilbert $\mathscr{A}$-module such that $\langle x,x\rangle$ or $\langle z,z\rangle$ is a projection and the inequality (4) holds, then it must be $\langle x,z\rangle = 0$. Namely, if $\langle x,x\rangle$ is a projection, by putting $y := x$ in (4) we get

$$|\langle z,x\rangle|^2 \leqslant |x|^2 - |\langle x,x\rangle|^2 = \langle x,x\rangle - \langle x,x\rangle^2 = 0,$$

so $\langle x,z\rangle = 0$. Similarly, in the case when $\langle z,z\rangle$ is a projection, it is enough to put $y := z$ in (4) to conclude $\langle x,z\rangle = 0$.

In what follows we apply Theorem 2.2 to the case of a particular Hilbert C^*-module.

First, let $\mathscr{A}$ be an arbitrary C^*-algebra. Then we define

$$\ell_2(\mathscr{A}) = \{(a_i)_{i\in\mathbb{N}} : a_i \in \mathscr{A}, \forall i \in \mathbb{N}, \sum_{i\in\mathbb{N}} a_i^*a_i \text{ converges in } \mathscr{A}\}.$$

With the operations $\lambda(a_i)_i + (b_i)_i = (\lambda a_i + b_i)_i$, $(a_i)_i a = (a_i a)_i$ and

$$\langle (a_i)_i, (b_i)_i\rangle = \sum_{i\in\mathbb{N}} a_i^* b_i,$$

$\ell_2(\mathscr{A})$ becomes a Hilbert C^*-module over $\mathscr{A}$. Then the following corollary holds.

COROLLARY 2.6. *Let $\mathscr{A}$ be a C^*-algebra and $(a_i)_i, (b_i)_i \in \ell_2(\mathscr{A})$, $\sum_{i\in\mathbb{N}} a_i^*a_i \neq 0$. Then for every $(c_i)_i \in \ell_2(\mathscr{A})$ such that $\sum_{i\in\mathbb{N}} c_i^*c_i \neq 0$ and $\sum_{i\in\mathbb{N}} c_i^*a_i = 0$ it holds*

$$\sum_{i\in\mathbb{N}} c_i^*b_i \sum_{i\in\mathbb{N}} b_i^*c_i \leqslant \frac{\|\sum_{i\in\mathbb{N}} c_i^*c_i\|}{\|\sum_{i\in\mathbb{N}} a_i^*a_i\|}(\|\sum_{i\in\mathbb{N}} a_i^*a_i\| \sum_{i\in\mathbb{N}} b_i^*b_i - \sum_{i\in\mathbb{N}} b_i^*a_i \sum_{i\in\mathbb{N}} a_i^*b_i),$$

with the equality if and only if

$$b_j = \frac{1}{\|\sum_{i\in\mathbb{N}} a_i^*a_i\|} a_j \sum_{i\in\mathbb{N}} a_i^*b_i + \frac{1}{\|\sum_{i\in\mathbb{N}} c_i^*c_i\|} c_j \sum_{i\in\mathbb{N}} c_i^*b_i,$$

for every $j \in \mathbb{N}$.

Similarly, for Hilbert $\mathscr{A}$ -modules $V_1, \dots, V_n$ we define their direct sum

$$V = V_1 \oplus \cdots \oplus V_n = \{(v_1, \dots, v_n) : v_i \in V_i, \forall i = 1, \dots n\}$$

which is a Hilbert $\mathscr{A}$ -module with $\langle (v_i), (w_i) \rangle = \sum_{i=1}^n \langle v_i, w_i \rangle$ (see [3, p. 5]). In the special case when $V_i = \mathbf{B}(H_1, H_2)$ for all $i = 1, \dots, n$ we get the following corollary.

COROLLARY 2.7. *Let H_1 and H_2 be Hilbert spaces and $n \in \mathbb{N}$. Then for every $A_1, \dots, A_n$, $B_1, \dots, B_n$, $C_1, \dots, C_n \in \mathbf{B}(H_1, H_2)$ such that $\sum_{i=1}^n A_i^* A_i \neq 0$, $\sum_{i=1}^n C_i^* C_i \neq 0$ and $\sum_{i=1}^n C_i^* A_i = 0$ it holds*

$$\sum_{i=1}^n C_i^* B_i \sum_{i=1}^n B_i^* C_i \leqslant \frac{\| \sum_{i=1}^n C_i^* C_i \|}{\| \sum_{i=1}^n A_i^* A_i \|} \Big(\| \sum_{i=1}^n A_i^* A_i \| \sum_{i=1}^n B_i^* B_i - \sum_{i=1}^n B_i^* A_i \sum_{i=1}^n A_i^* B_i \Big),$$

with the equality if and only if

$$B_j = \frac{1}{\| \sum_{i=1}^n A_i^* A_i \|} A_j \sum_{i=1}^n A_i^* B_i + \frac{1}{\| \sum_{i=1}^n C_i^* C_i \|} C_j \sum_{i=1}^n C_i^* B_i,$$

for every $j = 1, \dots, n$.

3. Some applications to C^* -algebras

Our next results are some consequences of Theorem 2.2 in the case of a C^* -algebra regarded as a Hilbert C^* -module over itself. Before stating the results, we prove the following technical lemma that will be useful in the sequel.

LEMMA 3.1. *Let $\mathscr{A}$ be a C^* -algebra and $a, b \in \mathscr{A}$. Then we have:*

$$ab^* = 0 \Leftrightarrow |a||b| = 0.$$

Proof. Let us suppose first that $ab^* = 0$. Then we have $|a|^2|b|^2 = a^*ab^*b = 0$. Therefore, $|a|^2|b|^2 = |b|^2|a|^2 = 0$. Since $|b|^2$ commutes with $|a|^2$, we have $|a|^2|b| = |b||a|^2$ and then similarly $|a||b| = |b||a|$. Now, $|a||b|$ is positive, as the product of two commuting positive elements, and $(|a||b|)^2 = |a|^2|b|^2 = 0$. Thus, we get $|a||b| = 0$.

Conversely, let us suppose that $|a||b| = 0$. Then $|a|^2|b|^2 = 0$, from which it follows that

$$(ba^*)(ba^*)^*(ba^*)(ba^*)^* = ba^*ab^*ba^*ab^* = b|a|^2|b|^2|a|^2b^* = 0.$$

Therefore, $(ba^*)(ba^*)^* = 0$ and then $ba^* = 0$. Hence, $ab^* = 0$. □

In what follows $\sigma(a)$ will stand for the spectrum of an arbitrary element a of a C^* -algebra. By $C(S)$ we denote the algebra of all continuous complex functions on some compact set $S \subset \mathbb{C}$ with pointwise operations and the norm $\|f\| = \sup\{|f(t)| : t \in S\}$.

PROPOSITION 3.2. *Let $\mathscr{A}$ be a C^* -algebra with the unit e and $a, b \in \mathscr{A}$ nonzero normal elements such that $ab^* = 0$. Then, for every $f \in C(\sigma(a))$ and $g \in C(\sigma(b))$ such that $f(0) = g(0) = 0$ and $f(a) \neq 0$, $g(b) \neq 0$ it holds $|f(a)||g(b)| = 0$ and*

$$\frac{|f(a)|}{\|f(a)\|} + \frac{|g(b)|}{\|g(b)\|} \leqslant e,$$

with the equality if and only if there is an orthogonal projection $p \in \mathscr{A}$ *such that*

$$|f(a)| = \|f(a)\|p \quad and \quad |g(b)| = \|g(b)\|(e-p).$$

Proof. Since $f \in C(\sigma(a))$ and $g \in C(\sigma(b))$, by the Weierstrass approximation theorem, there are sequences of polynomials p_n and q_n such that $p_n \to f$ and $q_n \to g$ uniformly on $\sigma(a)$ and $\sigma(b)$, respectively. In particular, $p_n(0) \to f(0) = 0$ and $q_n(0) \to g(0) = 0$, as $0 \in \sigma(a)$ and $0 \in \sigma(b)$. (Namely, from $ab^* = 0$ it follows $b = 0$ if $0 \notin \sigma(a)$, and $a = 0$ if $0 \notin \sigma(b)$.)

Since $ab^* = 0$, it follows that $p_n(a)b^* = p_n(0)b^*$ $(n \in \mathbb{N})$, so from functional calculus we obtain $f(a)b^* = 0$. Thus, $bf(a)^* = 0$, so $q_n(b)f(a)^* = q_n(0)f(a)^*$ $(n \in \mathbb{N})$, from which it follows $g(b)f(a)^* = 0$ and then $f(a)g(b)^* = 0$. Therefore, if we replace a, b and c in Corollary 2.3 with $f(a)^*$, e and $g(b)^*$ respectively, then we get

$$\frac{|f(a)|^2}{\|f(a)\|^2} + \frac{|g(b)|^2}{\|g(b)\|^2} \leqslant e.$$

By Lemma 3.1 we have $|f(a)||g(b)| = 0$, so

$$\left(\frac{|f(a)|}{\|f(a)\|} + \frac{|g(b)|}{\|g(b)\|}\right)^2 = \frac{|f(a)|^2}{\|f(a)\|^2} + \frac{|g(b)|^2}{\|g(b)\|^2} \leqslant e.$$

From this, by using [6, Proposition 1.3.8], it follows that $\frac{|f(a)|}{\|f(a)\|} + \frac{|g(b)|}{\|g(b)\|} \leqslant e$ since $\frac{|f(a)|}{\|f(a)\|} + \frac{|g(b)|}{\|g(b)\|}$ is positive.

Furthermore, if the equality $\frac{|f(a)|}{\|f(a)\|} + \frac{|g(b)|}{\|g(b)\|} = e$ holds, then we have

$$\left(\frac{|f(a)|}{\|f(a)\|}\right)^2 - \frac{|f(a)|}{\|f(a)\|} = \frac{|f(a)|}{\|f(a)\|}\left(\frac{|f(a)|}{\|f(a)\|} - e\right) = -\frac{|f(a)|}{\|f(a)\|}\frac{|g(b)|}{\|g(b)\|} = 0,$$

i.e., $\left(\frac{|f(a)|}{\|f(a)\|}\right)^2 = \frac{|f(a)|}{\|f(a)\|}$ and analogously $\left(\frac{|g(b)|}{\|g(b)\|}\right)^2 = \frac{|g(b)|}{\|g(b)\|}$. Thus, $\frac{|f(a)|}{\|f(a)\|}$ and $\frac{|g(b)|}{\|g(b)\|}$ are two orthogonal projections whose sum is e. The converse is obvious. □

We conclude this paper by applying the previous proposition on some special functions f and g.

COROLLARY 3.3. *Let* $\mathscr{A}$ *be a* C^**-algebra with the unit* e *and* $a, b \in \mathscr{A}$ *such that* $\|a\| = \|b\| = 1$ *and* $ab^* = 0$. *Then, for all* $\alpha, \beta > 0$ *it holds* $|a|^\alpha |b|^\beta = 0$ *and*

$$|a|^\alpha + |b|^\beta \leqslant e$$

with the equality if and only if $|a|^\alpha = p$ *and* $|b|^\beta = e - p$ *for some orthogonal projection* $p \in \mathscr{A}$. *Also,*

$$\frac{|a|^\alpha}{\alpha} + \frac{|b|^\beta}{\beta} \leqslant \max\left\{\frac{1}{\alpha}, \frac{1}{\beta}\right\} e$$

with the equality if and only if $\alpha = \beta$ *and there is an orthogonal projection* $p \in \mathscr{A}$ *such that* $|a|^\alpha = p$ *and* $|b|^\alpha = e - p$.

Proof. Let us define the functions $f,g : [0,\infty) \to \mathbb{R}$ by putting $f(t) = t^\alpha$ and $g(t) = t^\beta$ for some $\alpha,\beta > 0$. Clearly, $f \in C(\sigma(|a|))$, $g \in C(\sigma(|b|))$ and $f(0) = g(0) = 0$. So, $|a|^\alpha = f(|a|)$ and $|b|^\beta = g(|b|)$ are well-defined elements of the C^* -algebras generated by $|a|$ and $|b|$, respectively.

Since $f(t) \geqslant 0,\ t \in \sigma(|a|)$ and $g(t) \geqslant 0,\ t \in \sigma(|b|)$, the elements $|a|^\alpha = f(|a|)$ and $|b|^\beta = g(|b|)$ must be positive.

Furthermore, since $1 = \|a\| = \||a|\| \in \sigma(|a|) \subseteq [0, \||a|\|] = [0,1]$ and $1 = \|b\| = \||b|\| \in \sigma(|b|) \subseteq [0, \||b|\|] = [0,1]$ we obtain

$$\||a|^\alpha\| = \|f(|a|)\| = \sup\{|f(t)| : t \in \sigma(|a|)\} = \sup\{t^\alpha : t \in \sigma(|a|)\} = 1$$

and similarly $\||b|^\beta\| = 1$. By Lemma 3.1 we have $|a||b| = 0$. It remains to apply Proposition 3.2 on the elements $|a|$ and $|b|$ and the functions f and g to get the first statement of this corollary.

Now we have

$$\frac{|a|^\alpha}{\alpha} + \frac{|b|^\beta}{\beta} \leqslant \max\left\{\frac{1}{\alpha}, \frac{1}{\beta}\right\}(|a|^\alpha + |b|^\beta) \leqslant \max\left\{\frac{1}{\alpha}, \frac{1}{\beta}\right\}e.$$

Let us consider the case of equality. So, suppose that $\frac{|a|^\alpha}{\alpha} + \frac{|b|^\beta}{\beta} = \max\{\frac{1}{\alpha}, \frac{1}{\beta}\}e$. We may assume that $\max\{\frac{1}{\alpha}, \frac{1}{\beta}\} = \frac{1}{\alpha}$. Then we get $|b|^\beta = \frac{\beta}{\alpha}(e - |a|^\alpha)$, so

$$0 = |a|^\alpha |b|^\beta = \frac{\beta}{\alpha}(|a|^\alpha - (|a|^\alpha)^2).$$

Thus, $|a|^\alpha = (|a|^\alpha)^2$, i.e., $p := |a|^\alpha$ is an orthogonal projection. Since $e - p$ is a nonzero orthogonal projection, we have $\|e - p\| = 1$ and

$$1 = \||b|^\beta\| = \frac{\beta}{\alpha}\|e - p\| = \frac{\beta}{\alpha}.$$

It follows that $\alpha = \beta$ and then $|b|^\alpha = e - p$. The converse is obvious. □

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Ljiljana Arambašić
Department of Mathematics
University of Zagreb
Bijenička cesta 30
10000 Zagreb
Croatia
e-mail: ljsekul@math.hr

Rajna Rajić
Faculty of Mining, Geology and Petroleum Engineering
University of Zagreb
Pierottijeva 6
10000 Zagreb
Croatia
e-mail: rajna.rajic@zg.t-com.hr