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THE WEAK TYPE INEQUALITY FOR THE MAXIMAL OPERATOR OF THE MARCINKIEWICZ–FEJÉR MEANS OF THE TWO–DIMENSIONAL WALSH–KACZMARZ SYSTEM

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Abstract. The main aim of this paper is to prove that the maximal operator $\sigma^{\#}$ of the Marcinkiewicz-Fejér means of the two-dimensional Fourier series with respect to the Walsh-Kaczmarz system is bounded from the martingale Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$ and is not bounded from the martingale Hardy space $H_{1/2}$ to the space $L_{1/2}$ provided that the supremum in the maximal operator is taken over spatial indices.

1. Introduction

In 1939 Marcinkiewicz [4] proved for the two-dimensional trigonometric system that the Marcinkiewicz means of a function converge to the function itself almost evrywhere for all $f \in L\log L([0,2\pi]^2)$. Zhizhiashvili [14] improved this result for $f \in L([0,2\pi]^2)$.

For the two-dimensional Walsh-Fourier series Weisz [10] proved that the maximal operator

$$\sigma^* f := \sup_{n\geqslant 1} \frac{1}{n} \left| \sum_{j=0}^{n-1} S_{j,j}(f) \right|$$

is bounded from the two-dimensional dyadic martingale Hardy space H_p to the space L_p for $p > 2/3$. In [3] the author showed that in theorem of Weisz the assumption $p > 2/3$ is essential.

In 1948 Šneider [9] introduced the Walsh-Kaczmarz system and showed that the inequality

$$\limsup_{n\to\infty} \frac{D_n^{\kappa}(x)}{\log n} \geqslant C > 0$$

holds a.e. In 1974 Schipp [6] and Young [13] proved that the Walsh-Kaczmarz system is a convergence system. Skvorcov [8] in 1981 showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to f for any continuous functions f. Gát [1] proved, for any integrable functions, that the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere to the function. Gát's Theorem

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was extended by Simon [7] to H_p spaces, namely, he proved that the maximal operator of Fejér means σ^* of the one-dimensional Fourier series with respect to the Walsh-Kaczmarz system is bounded from Hardy spaces H_p to the spaces L_p for $p>1/2$. He also showed (H_p,L_p)-boundedness for every $p>0$ if the maximal operator of the Fejér means is considered only of order 2^n. In the endpoint case $p=1/2$ Weisz [12] proved that σ^* is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.

In [2] it is proved that the maximal operator

$$\sigma^{\#} f := \sup_{n\geqslant 1} \frac{1}{2^n}\left|\sum_{j=0}^{2^n-1} S_{j,j}^{k}(f)\right|$$

is bounded from the Hardy space H_p to the space L_p for $p>1/2$. The main aim of this paper is to prove that for the boudedness of the maximal operator $\sigma^{\#} f$ from the Hardy space H_p to the space L_p the assumption $p>1/2$ is essential, In particular, we prove that the maximal operator $\sigma^{\#} f$ is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$ (see Theorem 1). By interpolation it follows that $\sigma^{\#} f$ is not bounded from the Hardy space H_p to the space weak-L_p for $0<p<1/2$. In the endpoint case $p=1/2$ we proved that (see Theorem 2) $\sigma^{\#} f$ is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.

2. Definitions and Notation

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N}:=\mathbf{P}\cup\{0\}$. Denote Z_2 the discrete cyclic group of order 2, that is $Z_2=\{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singletion is 1/2. Let G be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of G are of the form $x=(x_0,x_1,...,x_k,...)$ with $x_k\in\{0,1\}\,(k\in\mathbf{N})$. The group operation on G is the coordinate-wise addition, the measure (denote by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$I_0(x):=G,\ I_n(x):=I_n(x_0,...,x_{n-1}):=\{y\in G:\ y=(x_0,...,x_{n-1},y_n,y_{n+1},...)\},$$

$$(x\in G, n\in\mathbf{N}).$$

These sets are called the dyadic intervals. Let $0=(0:i\in\mathbf{N})\in G$ denote the null element of G, $I_n:=I_n(0)\ (n\in\mathbf{N})$. Set $e_n:=(0,...,0,1,0,...)\in G$ the n th coordinate of which is 1 and the rest are zeros $(n\in\mathbf{N})$. Denote $\overline{I}_n:=G\setminus I_n$.

For $k\in\mathbf{N}$ and $x\in G$ denote

$$r_k(x):=(-1)^{x_k}$$

the k-th Rademacher function. If $n\in\mathbf{N}$, then $n=\sum_{i=0}^{\infty} n_i 2^i$, where $n_i\in\{0,1\}\ (i\in\mathbf{N})$, i. e. n is expressed in the number system of base 2. Denote $|n|:=\max\{j\in\mathbf{N}: n_j\neq 0\}$, that is, $2^{|n|}\leqslant n<2^{|n|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in G, n \in \mathbf{P}).$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 := 1$ and for $n \geqslant 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k}.$$

For $A \in \mathbf{N}$ define the transformation $\tau_A : G \to G$ by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \ldots, x_0, x_A, x_{A+1}, \ldots).$$

By the definition of τ_A (see [8]), we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbf{N}, x \in G).$$

The Dirichlet kernels are defined by

$$D_n^{\alpha}(x) := \sum_{k=0}^{n-1} \alpha_k(x),$$

where $\alpha_k = w_k$ or κ_k. Recall that

$$D_{2^n}(x) := D_{2^n}^{w}(x) = D_{2^n}^{\kappa}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \in \overline{I}_n. \end{cases} \tag{1}$$

The Fourier coefficients (if f is an integrable function), the partial sums of Fourier series, the Fejér means and the Fejér kernels are defined as follows:

$$\hat{f}^{\alpha}(n) := \int_G f \alpha_n, \quad S_n^{\alpha}(f,x) := \sum_{k=0}^{n-1} \hat{f}^{\alpha}(k) \alpha_k(x)$$

$$\sigma_n^{\alpha}(f,x) := \frac{1}{n} \sum_{k=0}^{n-1} S_k^{\alpha}(f,x), \quad K_n^{\alpha}(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k^{\alpha}(x).$$

Next, we introduce some notation with respect to the theory of two-dimensional Walsh system. The group $G \times G$ is called the two-dimensional Walsh group. The Kroneker product $(\alpha_{m,n} : m, n \in \mathbf{N})$ of the two Walsh(Kaczmarz) system is said to be the two-dimensional Walsh(Kaczmarz) system. Thus

$$\alpha_{m,n}\left(x^1, x^2\right) := \alpha_m\left(x^1\right) \alpha_n\left(x^2\right).$$

The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, the Marcinkiewicz means, the Dirichlet kernels and the Matcinkiewicz kernels are defined as follows

$$\hat{f}^{\alpha}(n_1,n_2) := \int_{G\times G} f\alpha_{n_1,n_2}, \;\; S^{\alpha}_{n_1,n_2}f(x^1,x^2) := \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\hat{f}^{\alpha}(k_1,k_2)\alpha_{k_1,k_2}(x^1,x^2),$$

$$\sigma^{\alpha}_n f(x^1,x^2) := \frac{1}{n}\sum_{k=0}^{n-1} S^{\alpha}_{k,k}f(x^1,x^2),$$

$$D^{\alpha}_{k,l}(x^1,x^2) := D^{\alpha}_k(x^1)D^{\alpha}_l(x^2), \;\; K^{\alpha}_n(x^1,x^2) := \frac{1}{n}\sum_{k=0}^{n-1} D^{\alpha}_{k,k}(x^1,x^2).$$

The σ-algebra generated by the dyadic 2-dimensional $I_k \times I_k$ cube of measure $2^{-k}\times 2^{-k}$ will be denoted by $F_k\,(k\in\mathbf{N})$.

Denote by $f = \left(f^{(n)}, n\in\mathbf{N}\right)$ one-parameter martingale with respect to $(F_n, n\in\mathbf{N})$ (for details see, e. g. [11] The maximal function of a martingale f is defined by

$$f^* = \sup_{n\in\mathbf{N}}\left|f^{(n)}\right|.$$

In case $f\in L_1(G\times G)$, the maximal function can also be given by

$$f^*\left(x^1,x^2\right) = \sup_{n\in\mathbf{N}}\frac{1}{\mu\left(I_n(x^1)\times I_n(x^2)\right)}\left|\int_{I_n(x^1)\times I_n(x^2)} f\left(u^1,u^2\right)d\mu\left(u^1,u^2\right)\right|,$$

$$\left(x^1,x^2\right)\in G\times G.$$

For $0<p<\infty$ the martingale Hardy space $H_p(G\times G)$ consists all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f\in L_1(G\times G)$ then it is easy to show that the sequence $(S_{2^n,2^n}(f) : n\in\mathbf{N})$ is a martingale. If f is a martingale, that is $f = (f^{(0)}, f^{(1)}, \ldots)$ then the Walsh-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}^{\alpha}(i,j) = \lim_{k\to\infty}\int_{G\times G} f^{(k)}\left(x^1,x^2\right)\alpha_i\left(x^1\right)\alpha_j\left(x^2\right)d\mu\left(x^1,x^2\right).$$

The Walsh-Fourier coefficients of $f\in L_1(G\times G)$ are the same as the ones of the martingale $(S_{2^n,2^n}(f) : n\in\mathbf{N})$ obtained from f.

For the martingale f we consider the maximal operator

$$\sigma^{\#}f(x^1,x^2) = \sup_A|\sigma^k_{2^A}f(x^1,x^2)|.$$

A bounded measurable function a is a p-atom, if there exists a dyadic 2-dimensional cube I^2, such that

a) $\int\limits_{I^2} a d\mu = 0$;
b) $\|a\|_\infty \leqslant \mu(I^2)^{-1/p}$;
c) supp $a \subset I^2$.

3. Formulation of Main Results

THEOREM 1. *The maximal operator $\sigma^\#$ is not bounded from the Hardy space $H_{1/2}(G\times G)$ to the space $L_{1/2}(G\times G)$.*

THEOREM 2. *The maximal operator $\sigma^\#$ is bounded from the Hardy space $H_{1/2}(G\times G)$ to the space weak-$L_{1/2}(G\times G)$.*

4. Auxiliary Propositions

We shall need the following lemmas (see [2, 5, 11]).

LEMMA 1. (Weisz) *Suppose that an operator V is sublinear and, for some $0 < p < 1$*

$$\sup_{\rho>0} \rho^p \mu\{x \in (G\times G)\setminus(I\times I) : |Va(x)| > \rho\} \leqslant c_p < \infty,$$

for every p-atom a, where I denote the support of the atom. If V is bounded from L_{p_1} to L_{p_1} for a fixed $1 < p_1 \leqslant \infty$, then

$$\|Vf\|_{weak\text{-}L_p(G\times G)} \leqslant c_p \|f\|_{H_p}.$$

LEMMA 2. (Nagy) *Let $A,s,l \in \mathbf{N}, s \leqslant l < A, \left(x^1,x^2\right) \in (I_s\setminus I_{s+1})\times(I_l\setminus I_{l+1})$. Then*

$$K_{2^A}^w\left(x^1,x^2\right) = \begin{cases} 0 & \text{if} \quad \exists\, i \in B_1,\ x_i^1 \neq x_i^2, \\ 0 & \text{if} \quad \forall\, i \in B_1,\ x_i^1 = x_i^2,\ \exists\, m \in B_2,\ x^1 - e_s - e_m \notin I_{l+1},\ x_m^1 = 1, \\ 2^{s+m-2} & \text{if} \quad \forall\, i \in B_1,\ x_i^1 = x_i^2,\ \exists\, m \in B_2,\ x^1 - e_s - e_m \in I_{l+1},\ x_m^1 = 1, \\ 2^{2s-1} & \text{if} \quad x^1 - e_s \in I_{l+1}\left(\forall\, i \in B_1,\ x_i^1 = x_i^2\right), \end{cases}$$

where $B_1 = \{l+1,...,A-1\}$, $B_2 = \{s+1,...,l\}$.

LEMMA 3. (Nagy) *Let $A \in \mathbf{N}, \left(x^1,x^2\right) \in G\times G$. Then*

$$\begin{aligned} 2^A K_{2^A}^\kappa\left(x^1,x^2\right) &= 1 + \sum_{j=0}^{A-1} 2^j D_{2^j,2^j}\left(x^1,x^2\right) + \sum_{j=0}^{A-1} 2^j D_{2^j}\left(x^1\right) r_j\left(x^2\right) K_{2^j}^w\left(\tau_j\left(x^2\right)\right) \\ &+ \sum_{j=0}^{A-1} 2^j D_{2^j}\left(x^2\right) r_j\left(x^1\right) K_{2^j}^w\left(\tau_j\left(x^1\right)\right) \\ &+ \sum_{j=0}^{A-1} 2^j r_j\left(x^1+x^2\right) K_{2^j}^w\left(\tau_j\left(x^1\right), \tau_j\left(x^2\right)\right). \end{aligned}$$

LEMMA 4. (Gát, Goginava, Nagy) *Let* $x\in I_N(x_0,...,x_{l-1},x_l=1,0,...,0)$ *and* $j>N$. *Then*

$$\int_{I_N} K_{2^j}^w(\tau_j(x+t))\,d\mu(t)\leqslant \frac{c}{2^l}\mathbf{1}_{I_N(0,...,0,x_l=1,0,...,0)}(x).$$

LEMMA 5. (Gát, Goginava, Nagy) *Let*

$$x^1\in I_N\left(x_0^1,...,x_{s-1}^1,x_s^1=1,0,...,0\right),$$
$$x^2\in I_N\left(x_0^2,...,x_{l-1}^2,x_l^2=1,0,...,0\right),$$

$$0\leqslant s\leqslant l<N.$$

Then for $j>N$ we have

$$\int_{I_N\times I_N} K_{2^j}^w\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)d\mu\left(t^1,t^2\right)$$
$$\leqslant c\sum_{m=s}^{l}2^{-l-m}\mathbf{1}_{I_N\left(x_0^1,...,x_{s-1}^1,x_s^1=1,0,...,0,x_m^2=1,0,...,0,x_l^2=1,0,...,0\right)}\left(x^2\right).$$

LEMMA 6. (Gát, Goginava, Nagy) *Let* $\left(x^1,x^2\right)\in I_N\times I_N\left(x_0^2,...,x_{l-1}^2,x_l^2=1,0,...,0\right)$, $l=0,...,N-1$. *Then for* $j>N$ *we have*

$$\int_{I_N\times I_N} K_{2^j}^w\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)d\mu\left(t^1,t^2\right)$$
$$\leqslant c\sum_{s=l}^{N-1}2^{-l-s}\mathbf{1}_{I_N\left(0,...,0,x_l^2=1,0,...,0,x_s^2=1,0,...,0\right)}\left(x^2\right).$$

5. Proofs of Main Results

Proof of Theorem 1. Let $A\in\mathbf{P}$ and

$$f_A\left(x^1,x^2\right):=\left(D_{2^{A+1}}\left(x^1\right)-D_{2^A}\left(x^1\right)\right)\left(D_{2^{A+1}}\left(x^2\right)-D_{2^A}\left(x^2\right)\right).$$

It is evident that

$$\widehat{f}_A^k(i,j)=\begin{cases}1, \text{ if } i,j=2^A,...,2^{A+1}-1,\\ 0, \text{ otherwise.}\end{cases}$$

Then we can write that

$$S_{j,j}^k f_A\left(x^1,x^2\right)=\begin{cases}0, \text{ if } j=0,...,2^A,\\ \left(D_j\left(x^1\right)-D_{2^A}\left(x^1\right)\right)\left(D_j\left(x^2\right)-D_{2^A}\left(x^2\right)\right), \text{ if } j=2^A+1,...,2^{A+1}-1\\ f_A\left(x^1,x^2\right),\ j\geqslant 2^{A+1}.\end{cases} \tag{2}$$

We have

$$f_A^*\left(x^1,x^2\right) = \sup_j \left|S_{2^j,2^j} f_A\left(x^1,x^2\right)\right| = \left|f_A\left(x^1,x^2\right)\right|, \tag{3}$$

$$\|f_A\|_{H_p} = \|f_A^*\|_p = \|D_{2^A}\|_p^2 = 2^{2A(1-1/p)}.$$

Since

$$D_{j+2^A}^k(x) - D_{2^A}^k(x) = w_{2^A}(x) D_j^w(\tau_A(x)), \ \ j=1,2,...,2^A,$$

from (2) we obtain

$$\begin{aligned}
\sigma^{\#} f_A\left(x^1,x^2\right) &= \sup_n |\sigma_{2^n}^k f_A(x^1,x^2)|. \geqslant \left|\sigma_{2^{A+1}} f_A(x^1,x^2)\right| \qquad (4)\\
&= \frac{1}{2^{A+1}}\left|\sum_{j=0}^{2^{A+1}-1} S_{j,j}^k\left(f_A;x^1,x^2\right)\right| \\
&= \frac{1}{2^{A+1}}\left|\sum_{j=2^A+1}^{2^{A+1}-1}\left(D_j^k\left(x^1\right)-D_{2^A}\left(x^1\right)\right)\left(D_j^k\left(x^2\right)-D_{2^A}\left(x^2\right)\right)\right| \\
&= \frac{1}{2^{A+1}}\left|\sum_{j=1}^{2^A-1}\left(D_{j+2^A}^k\left(x^1\right)-D_{2^A}\left(x^1\right)\right)\left(D_{j+2^A}^k\left(x^2\right)-D_{2^A}\left(x^2\right)\right)\right| \\
&= \frac{1}{2^{A+1}}\left|\sum_{j=1}^{2^A-1} D_j^w\left(\tau_A\left(x^1\right)\right) D_j^w\left(\tau_A\left(x^2\right)\right)\right| \\
&= \frac{1}{2}\left|K_{2^A}^w\left(\tau_A\left(x^1\right),\tau_A\left(x^2\right)\right)\right|.
\end{aligned}$$

Then from Lemma 2 we obtain

$$\begin{aligned}
&\int_{G\times G} \left|K_{2^A}^w\left(\tau_A\left(x^1\right),\tau_A\left(x^2\right)\right)\right|^{1/2} d\mu\left(x^1,x^2\right) \qquad (5)\\
&= \int_{G\times G} \left|K_{2^A}^w\left(x^1,x^2\right)\right|^{1/2} d\mu\left(x^1,x^2\right) \\
&\geqslant \sum_{m^1=0}^{A-1}\sum_{x^1_{m^1+1}=0}^{1}\cdots\sum_{x^1_{A-1}=0}^{1} \int_{\substack{I_A\left(0,\ldots,0,x^1_{m^1}=1,x^1_{m^1+1},\ldots,x^1_{A-1}\right)\\ \times I_A\left(0,\ldots,0,x^1_{m^1}=1,x^1_{m^1+1},\ldots,x^1_{A-1}\right)}} \left|K_{2^A}\left(x^1,x^2\right)\right|^{1/2} d\mu\left(x^1,x^2\right) \\
&\geqslant c\sum_{m^1=0}^{A-1} 2^{m^1} \sum_{x^1_{m^1+1}=0}^{1}\cdots\sum_{x^1_{A-1}=0}^{1} \int_{G\times G} 1_{I_A\left(0,\ldots,0,x^1_{m^1}=1,x^1_{m^1+1},\ldots,x^1_{A-1}\right)}\left(x^1\right) \\
&\quad \times 1_{I_A\left(0,\ldots,0,x^1_{m^1}=1,x^1_{m^1+1},\ldots,x^1_{A-1}\right)}\left(x^2\right) d\mu\left(x^1,x^2\right) \\
&\geqslant c\sum_{m^1=0}^{A-1} 2^{m^1} 2^{A-m^1}\frac{1}{2^{2A}} \geqslant \frac{cA}{2^A},
\end{aligned}$$

Combining (3), (4) and (5) we have

$$\frac{\left\|\sigma^{\#} f_A\right\|_{1/2}}{\left\|f_A\right\|_{H_{1/2}}} \geqslant \frac{cA^2}{2^{2A}2^{2A(1-2)}} \geqslant cA^2 \to \infty \text{ as } A \to \infty.$$

Theorem 1 is proved. □

Proof of Theorem 2. We shall apply Lemma 1, we may suppose that $a \in L_\infty$ is a 1/2-atom with support I_N. Since $\sigma_{2^A} a\left(x^1,x^2\right)$ if $A \leqslant N$ we may assume that $A > N$.

Suppose that $\rho = c2^\lambda$ for some $\lambda \in \mathbf{N}$.

It is evident that

$$\begin{aligned}
&\mu\left\{\left(x^1,x^2\right) \in \overline{I_N \times I_N} : \left|\sigma^{\#} a\left(x^1,x^2\right)\right| > c2^\lambda\right\} \qquad (6)\\
&= \mu\left\{\left(x^1,x^2\right) \in \overline{I}_N \times \overline{I}_N : \left|\sigma^{\#} a\left(x^1,x^2\right)\right| > c2^\lambda\right\}\\
&\quad + \mu\left\{\left(x^1,x^2\right) \in I_N \times \overline{I}_N : \left|\sigma^{\#} a\left(x^1,x^2\right)\right| > c2^\lambda\right\}\\
&\quad + \mu\left\{\left(x^1,x^2\right) \in \overline{I}_N \times I_N : \left|\sigma^{\#} a\left(x^1,x^2\right)\right| > c2^\lambda\right\}\\
&= A + B + C.
\end{aligned}$$

Using Lemma 3 for $\left(x^1,x^2\right) \in \overline{I_N \times I_N}$ we write

$$\begin{aligned}
&\sigma_{2^A} a\left(x^1,x^2\right)\\
&= \frac{1}{2^A} \int_{I_N \times I_N} a\left(t^1,t^2\right) \left(1 + \sum_{j=0}^{A-1} 2^j D_{2^j,2^j}\left(x^1+t^1,x^2+t^2\right)\right.\\
&\quad + \sum_{j=0}^{A-1} 2^j D_{2^j}\left(x^1+t^1\right) r_j\left(x^2+t^2\right) K_{2^j}^w\left(\tau_j\left(x^2+t^2\right)\right)\\
&\quad + \sum_{j=0}^{A-1} 2^j D_{2^j}\left(x^2+t^2\right) r_j\left(x^1+t^1\right) K_{2^j}^w\left(\tau_j\left(x^1+t^1\right)\right)\\
&\quad \left. + \sum_{j=0}^{A-1} 2^j r_j\left(x^1+t^1+x^2+t^2\right) K_{2^j}^w\left(\tau_j\left(x^1+t^1\right), \tau_j\left(x^2+t^2\right)\right)\right) dt^1 dt^2\\
&= \frac{1}{2^A} \int_{I_N \times I_N} a\left(t^1,t^2\right) \sum_{j=N+1}^{A-1} 2^j D_{2^j}\left(x^1+t^1\right) r_j\left(x^2+t^2\right) K_{2^j}^w\left(\tau_j\left(x^2+t^2\right)\right) dt^1 dt^2\\
&\quad + \frac{1}{2^A} \int_{I_N \times I_N} a\left(t^1,t^2\right) \sum_{j=N+1}^{A-1} 2^j D_{2^j}\left(x^2+t^2\right) r_j\left(x^1+t^1\right) K_{2^j}^w\left(\tau_j\left(x^1+t^1\right)\right) dt^1 dt^2\\
&\quad + \frac{1}{2^A} \int_{I_N \times I_N} a\left(t^1,t^2\right) \sum_{j=N+1}^{A-1} 2^j r_j\left(x^1+t^1+x^2+t^2\right) K_{2^j}^w\left(\tau_j\left(x^1+t^1\right), \tau_j\left(x^2+t^2\right)\right) dt^1 dt^2\\
&= \sigma_{2^A}^{(1)} a\left(x^1,x^2\right) + \sigma_{2^A}^{(2)} a\left(x^1,x^2\right) + \sigma_{2^A}^{(3)} a\left(x^1,x^2\right). \qquad (7)
\end{aligned}$$

Let $\left(x^1,x^2\right)\in I_N\left(x_0^1,...,x_{s-1}^1,x_s^1=1,0,...,0\right)\times I_N\left(x_0^2,...,x_{l-1}^2,x_l^2=1,0,...,0\right)$, $0\leqslant s\leqslant l<N$. Using (1), Lemma 5 and the fact that $|a|\leqslant c2^{4N}$ we have

$$\sigma_{2^A}^{(1)}a\left(x^1,x^2\right)=0, \tag{8}$$

$$\sigma_{2^A}^{(2)}a\left(x^1,x^2\right)=0, \tag{9}$$

$$\begin{aligned}
&\left|\sigma_{2^A}^{(3)}a\left(x^1,x^2\right)\right| \\
&\leqslant \frac{1}{2^A}\sum_{j=N+1}^{A-1}2^j\int\limits_{I_N\times I_N}\left|a\left(t^1,t^2\right)\right|\left|K_{2^j}^w\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)\right|d\mu\left(t^1,t^2\right)\\
&\leqslant c\frac{2^{4N}}{2^A}\sum_{j=N+1}^{A-1}2^j\sum_{m=s}^{l}2^{-l-m}\mathbf{1}_{I_N\left(x_0^1,...,x_{s-1}^1,x_s^1=1,0,...,0,x_m^2=1,0,...,0,x_l^2=1,0,...,0\right)}\left(x^2\right) \qquad (10)\\
&\leqslant c\frac{2^{4N}}{2^l}\sum_{m=s}^{l}2^{-m}\mathbf{1}_{I_N\left(x_0^1,...,x_{s-1}^1,x_s^1=1,0,...,0,x_m^2=1,0,...,0,x_l^2=1,0,...,0\right)}\left(x^2\right).
\end{aligned}$$

Let $\lambda>4N$. Then it is evident that

$$\mu\left\{\left(x^1,x^2\right)\in\overline{I}_N\times\overline{I}_N:\sup_{A>N}\left|\sigma_{2^A}^{(3)}a\left(x^1,x^2\right)\right|\geqslant c2^\lambda\right\}=0.$$

Hence we can suppose that $\lambda\leqslant 4N$.

Let $2N<\lambda\leqslant 4N$. Then by (10) and from the simple calculation we can write

$$\begin{aligned}
&\mu\left\{\left(x^1,x^2\right)\in\overline{I}_N\times\overline{I}_N:\sup_{A>N}\left|\sigma_{2^A}^{(3)}a\left(x^1,x^2\right)\right|\geqslant c2^\lambda\right\} \qquad (11)\\
&\leqslant\sum_{s=0}^{2N-[\lambda/2]}\sum_{m=s}^{2N-[\lambda/2]}\sum_{l=m}^{4N-[\lambda]-m}\frac{2^s}{2^{2N}}+\sum_{s=0}^{3N-[\lambda]}\sum_{l=s}^{2N-[\lambda/2]}\sum_{m=s}^{l}\frac{2^s}{2^{2N}}\\
&\quad+\sum_{s=0}^{3N-[\lambda]}\sum_{l=2N-[\lambda/2]}^{N}\sum_{m=s}^{4N-[\lambda]-l}\frac{2^s}{2^{2N}}+\sum_{s=3N-[\lambda]}^{2N-[\lambda/2]}\sum_{l=s}^{2N-[\lambda/2]}\sum_{m=s}^{l}\frac{2^s}{2^{2N}}\\
&\quad+\sum_{s=3N-[\lambda]}^{2N-[\lambda/2]}\sum_{l=2N-[\lambda/2]}^{4N-[\lambda]-s}\sum_{m=s}^{4N-[\lambda]-l}\frac{2^s}{2^{2N}}\\
&\leqslant c\sum_{s=0}^{2N-[\lambda/2]}\frac{\left(2N-\lambda/2-s+1\right)^2}{2^{2N-s}}\\
&\leqslant\frac{c}{2^{\lambda/2}}.
\end{aligned}$$

Let $0<\lambda\leqslant 2N$. Then we have

$$\begin{aligned}
&\mu\left\{\left(x^1,x^2\right)\in\overline{I}_N\times\overline{I}_N:\sup_{A>N}\left|\sigma_{2^A}^{(3)}a\left(x^1,x^2\right)\right|\geqslant c2^\lambda\right\} \qquad (12)\\
&\leqslant c\sum_{s=0}^{N}\sum_{l=s}^{N}\sum_{m=s}^{l}\frac{2^s}{2^{2N}}\leqslant\frac{c}{2^N}\leqslant\frac{c}{2^{\lambda/2}}.
\end{aligned}$$

Combining (8)-(12) we obtain that

$$A \leqslant \frac{c}{2^{\lambda/2}}. \tag{13}$$

Let $\left(x^1,x^2\right) \in I_N \times I_N\left(x_0^2,...,x_{l-1}^2,x_l^2 = 1,0,...,0\right),\ 0 \leqslant l < N$. Then we can write

$$\sigma_{2^A} a\left(x^1,x^2\right) = \sigma_{2^A}^{(1)} a\left(x^1,x^2\right) + \sigma_{2^A}^{(3)} a\left(x^1,x^2\right) \tag{14}$$

Using (1), Lemma 4 and the fact that $|a| \leqslant c2^{4N}$ we have

$$\begin{aligned}\left|\sigma_{2^A}^{(1)} a\left(x^1,x^2\right)\right| &\leqslant c\frac{2^{4N}}{2^A}\sum_{j=N+1}^{A-1} 2^j \int\limits_{I_N \times I_N} D_{2^j}\left(x^1+t^1\right) K_{2^j}^w\left(\tau_j\left(x^2+t^2\right)\right) d\mu\left(t^1,t^2\right)\\ &\leqslant c\frac{2^{4N}}{2^A}\sum_{j=N+1}^{A-1} 2^j \int\limits_{I_N} K_{2^j}^w\left(\tau_j\left(x^2+t^2\right)\right) d\mu\left(t^2\right).\\ &\leqslant c\frac{2^{4N}}{2^l}\mathbf{1}_{I_N\left(0,...,0,x_l^2=1,0,...,0\right)}\left(x^2\right).\end{aligned}$$

Then from the simple calculation we can write

$$\begin{aligned}&\mu\left\{\left(x^1,x^2\right) \in I_N \times \overline{I}_N : \sup_{A>N}\left|\sigma_{2^A}^{(1)} a\left(x^1,x^2\right)\right| \geqslant c2^{\lambda}\right\} \qquad (15)\\ &\leqslant c\sum_{l=0}^{4N-[\lambda]}\frac{1}{2^{2N}} \leqslant c\frac{4N-\lambda}{2^{2N}} \leqslant \frac{c}{2^{\lambda/2}}.\end{aligned}$$

For $\sigma_{2^A}^{(3)} a\left(x^1,x^2\right)$ we have (see Lemma 6)

$$\begin{aligned}\left|\sigma_{2^A}^{(3)} a\left(x^1,x^2\right)\right| &\leqslant \frac{1}{2^A}\sum_{j=N+1}^{A-1} 2^j \int\limits_{I_N \times I_N}\left|a\left(t^1,t^2\right)\right| K_{2^j}^w\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right) dt^1 dt^2\\ &\leqslant \frac{c2^{4N}}{2^A}\sum_{j=N+1}^{A-1} 2^j \sum_{s=l}^{N-1} 2^{-l-s}\mathbf{1}_{I_N\left(0,...,0,x_l^2=1,0,...,0,x_s^2=1,0,...,0\right)}\left(x^2\right)\\ &\leqslant \frac{c2^{4N}}{2^l}\sum_{s=l}^{N-1} 2^{-s}\mathbf{1}_{I_N\left(0,...,0,x_l^2=1,0,...,0,x_s^2=1,0,...,0\right)}\left(x^2\right).\end{aligned}$$

Let $2N < \lambda \leqslant 4N$. Then from the simple calculation we can write

$$\begin{aligned}&\mu\left\{\left(x^1,x^2\right) \in I_N \times \overline{I}_N : \sup_{A>N}\left|\sigma_{2^A}^{(3)} a\left(x^1,x^2\right)\right| \geqslant c2^{\lambda}\right\} \qquad (16)\\ &\leqslant c\left\{\sum_{l=0}^{2N-[\lambda/2]}\sum_{s=l}^{4N-[\lambda]-l}\frac{1}{2^{2N}} + \sum_{s=0}^{2N-[\lambda/2]}\sum_{l=0}^{s}\frac{1}{2^{2N}} + \sum_{s=2N-[\lambda/2]}^{N}\sum_{l=0}^{4N-[\lambda]-s}\frac{1}{2^{2N}}\right\}\\ &\leqslant \frac{c\left(2N-\lambda/2\right)^2}{2^{2N}} \leqslant \frac{c}{2^{\lambda/2}}.\end{aligned}$$

Let $0<\lambda\leqslant 2N$. Then we have

$$\mu\left\{\left(x^1,x^2\right)\in I_N\times\overline{I}_N:\sup_{A>N}\left|\sigma_{2^A}^{(3)}a\left(x^1,x^2\right)\right|\geqslant c2^{\lambda}\right\} \tag{17}$$
$$\leqslant c\sum_{s=0}^{N}\sum_{l=0}^{s}\frac{1}{2^{2N}}\leqslant\frac{cN^2}{2^{2N}}\leqslant\frac{c}{2^{\lambda/2}}.$$

Combining (14)-(17) we obtain that

$$B\leqslant\frac{c}{2^{\lambda/2}}. \tag{18}$$

Analogously, we can prove that

$$C\leqslant\frac{c}{2^{\lambda/2}}. \tag{19}$$

Combining (6), (13), (18) and (19) we obtain that

$$2^{\lambda/2}\mathrm{mes}\left\{\left(x^1,x^2\right)\in\overline{I_N\times I_N}:\left|\sigma^{\#}a\left(x^1,x^2\right)\right|>c2^{\lambda}\right\}\leqslant c<\infty.$$

Theorem 2 is proved. □

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