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INEQUALITIES FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We consider some differential inequalities involving fractional derivatives in the sense of Riemann-Liouville. Bounds for theses fractional differential inequalities are found using desingularization techniques combined with some generalizations of Bihari-type inequalities. Some applications illustrating the usefulness of our results are also provided.

1. Introduction

In this work we establish some boundedness results for fractional differential inequalities of the form

(a) $D^{\alpha}u(t) \leqslant a(t) + \sum_{i=1}^{k} b_i(t)\, D^{\beta_i}u(t)$

(b) $D^{\alpha}u(t) \leqslant a(t) + \sum_{i=1}^{k} b_i(t)\, \left[D^{\beta_i}u(t))\right]^{n}$

(c) $D^{\alpha}u(t) \leqslant a(t) + \sum_{i=1}^{k} b_i(t)\, \left[D^{\beta_i}u(t)\right]^{n_i}$ with $0 < \alpha < 1$, $0 \leqslant \beta_i < \alpha < 1$, $i = 1, \ldots, k$.

It is well known that boundedness results are very useful to understand the behavior of solutions to differential equations. They can be used to prove global existence in case we have a local solution and in presence of either behavior: the solution exists for all time $t \geqslant 0$ or it blows up in finite time in a certain norm. They can also be used to determine the asymptotic behavior of solutions in case we can not find explicit solutions as is the case in most nonlinear problems.

In [6] we considered similar fractional differential inequalities to (a)–(c) but with integrals in the right hand sides. Several results and applications were established. Those results may be seen as generalizations and extensions of analogous results from the integer order case (may be found in [3, 11]) to the fractional order case. This is also what we intend to do in this paper. The situations here are, however, less favorable and the techniques used in [6] are not valid in the present cases.

For convenience, we shall consider here only the case where the order of the fractional differential inequalities α is between 0 and 1. Similar results may be proved for $\alpha > 1$.

Our results may be applied, for instance, to differential equations of the form

$$D^{\alpha}u(t) = f\left(t, u, \{D^{\beta_i}u(t)\}_{i=1}^{k}\right)$$

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where α and β_i are not necessarily integers and f could be a nonlinear function, namely, when f takes the form of one of the right hand sides in (a)–(c). For well posedness of such (Cauchy) problems we refer the reader to [1, 2, 9, 10]. The special case $D^{\alpha}u(t) = f(t,u)$ has been studied by many investigators (see the survey papers [9, 10] and the references therein and also the recent works [4, 5, 7]).

We mention here that an upper bound has been found in the case $\alpha > 1$ and

$$\left|f\left(t, \{D^{\beta_i}u(t)\}_{i=1}^{k+1}\right)\right| \leqslant \sum_{i=1}^{k+1} b_i(t)\left|D^{\beta_i}u(t)\right|$$

with $\beta_{k+1} = \alpha - 1 > \beta_i + \frac{1}{2}$, $i = 1,\dots,k$, by Anastassiou in [1] and Anastassiou et al. in [2]. The technique there is different from ours. It is based on a generalization (to the fractional case) of Opial inequality. However, their method can not be applied to the case $0 < \alpha < 1$.

The rest of the paper is organized as follows. In Section 2 we introduce some definitions, lemmas, and propositions needed in our proofs. Section 3 contains the main results. The last section, Section 4, is devoted to some applications.

2. Preliminaries

In this section we introduce some notations, definitions and lemmas which will be needed later. For more details concerning fractional derivatives, we refer the reader to [8, 12, 13].

We denote by L_p, $1 \leqslant p \leqslant \infty$, the usual Lebesgue spaces, and by $AC([a,b])$ the space of all absolutely continuous functions on $[a,b]$.

DEFINITION 1. Let $f(t) \in L_1(a,b)$, the integral

$$(I^{\alpha}f)(t) := \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(s)}{(t-s)^{1-\alpha}}ds, \qquad t > a,$$

where $\alpha > 0$, is called the Riemann-Liouville fractional integral of order α of the function f.

We also use the notation f_{α} to denote $I^{\alpha}f$.

DEFINITION 2. The expression

$$(D^{\alpha}f)(t) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{f(s)}{(t-s)^{\alpha}}ds,$$

where $0 < \alpha < 1$, is called the Riemann-Liouville fractional derivative of order α of f provided the right-hand side is pointwise defined on (a,b).

Note that $D^{\alpha}f(t) = \frac{d}{dt}I^{1-\alpha}f(t)$. For convenience, we use the notation $I^{-\alpha}$ to denote D^{α} for $\alpha \geqslant 0$.

DEFINITION 3. Let $0 < \alpha < 1$. A function $f(t) \in L_1(a,b)$ is said to have a summable fractional derivative $D^{\alpha}f$ on (a,b) if $f_{1-\alpha} \in AC([a,b])$.

DEFINITION 4. We define the space $I^{\alpha}(L_p(a,b))$, $\alpha > 0$, $1 \leqslant p < \infty$, to be the space of all functions f such that $f = I^{\alpha}\varphi$ for some $\varphi \in L_p(a,b)$.

For convenience we introduce the function

$$R_{\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \qquad t > 0,\ \alpha > 0.$$

PROPOSITION 5. *If $f(t)$ has a summable fractional derivative $D^{\beta}f$, $0 \leqslant \beta < 1$, on (a,b), then for $\alpha \geqslant 0$,*

$$I^{\alpha}D^{\beta}f(t) = f_{\alpha-\beta}(t) - f_{1-\beta}(a)R_{\alpha}(t-a).$$

See ([13], p. 48).

COROLLARY 6. *If $f(t)$ has a summable fractional derivative $D^{\alpha}f$, $0 \leqslant \alpha < 1$, on (a,b), then for $0 \leqslant \beta \leqslant \alpha < 1$ we have*

$$D^{\beta}f(t) = I^{\alpha-\beta}D^{\alpha}f(t) + f_{1-\alpha}(a)R_{\alpha-\beta}(t-a).$$

Proof. In Proposition 5, replace α by $\alpha - \beta$, and replace β by α. □

The following is a Bihari-type inequality (Corollary 5.3, [3], Theorem 2.3.3, [11]).

LEMMA 7. *Let u, ϕ, ψ and k be non-negative continuous functions in $[a,b]$. Let w be a continuous, non-negative and non-decreasing function in $[0,\infty)$, with $w(0) = 0$ and $w(u) > 0$ for $u > 0$, and let $\Phi(t) := \max_{0\leqslant s\leqslant t}\phi(s)$ and $\Psi(t) := \max_{0\leqslant s\leqslant t}\psi(s)$. Assume that*

$$u(t) \leqslant \phi(t) + \psi(t)\int_a^t k(s)\,w(u(s))ds, \qquad t \in [a,b].$$

Then

$$u(t) \leqslant W^{-1}\left[W(\Phi(t)) + \Psi(t)\int_a^t k(s)ds\right], \qquad t \in [a,T),$$

where $W(u) := \int_{u_0}^{u}\frac{d\tau}{w(\tau)}$, $u_0, u > 0$, W^{-1} is the inverse of W and $T > a$ is such that $W(\Phi(t)) + \Psi(t)\int_a^t k(s)ds \in D(W^{-1})$ for all $t \in [a,T)$.

When $w(u) = u^n$, $n > 1$ is a positive integer, then

$$W(u) = \frac{1}{1-n}\left(u^{1-n} - u_0^{1-n}\right), \qquad W^{-1}(u) = \left((1-n)u + u_0^{1-n}\right)^{\frac{1}{1-n}},$$

for some $u_0 > 0$. We obtain the special version:

LEMMA 8. *Let u, ϕ, ψ be non-negative continuous functions in $[a,b]$. Let*

$$\Phi(t) := \max_{0\leqslant s\leqslant t}\phi(s), \qquad \Psi(t) := \max_{0\leqslant s\leqslant t}\psi(s).$$

Assume that

$$u(t) \leqslant \phi(t) + \psi(t)\int_a^t u^n(s)ds, \qquad t \in [a,b],$$

where n is a positive integer > 1. Then

$$u(t) \leqslant \left[\Phi^{1-n}(t) - (n-1)(t-a)\Psi(t)\right]^{\frac{1}{1-n}}, \qquad t \in [a,T], \tag{1}$$

where $T > a$ is such that

$$\Phi^{1-n}(t) - (n-1)(t-a)\Psi(t) > 0,$$

for all $t \in [a,T)$.

Let $I \subset \mathbf{R}$, and let $g_1, g_2 : I \to \mathbf{R} \setminus \{0\}$. We write $g_1 \propto g_2$ if g_2/g_1 is nondecreasing in I.

We will use the following Gronwall-type inequality (Theorem 10.3, [3]).

LEMMA 9. *Let $\phi(t)$ be a positive continuous function in $[a,b]$, $k_i(t,s)$, $i = 1,\ldots,m$, are nonnegative continuous functions for $a \leqslant s \leqslant t < b$ which are nondecreasing in t for any fixed s, $g_i(u)$, $i = 1,\ldots,m$, are nondecreasing continuous functions in $[0,\infty)$, with $g_i(u) > 0$ for $u > 0$ and $u(t)$ is a nonnegative continuous functions in $[a,b]$. If $g_1 \propto g_2 \propto \ldots \propto g_m$ in $(0,\infty)$, then the inequality*

$$u(t) \leqslant \phi(t) + \sum_{i=1}^{m} \int_a^t k_i(t,s)\, g_i(u(s))ds, \qquad t \in [a,b],$$

implies that

$$u(t) \leqslant c_m(t), \qquad a \leqslant t < T,$$

where $c_0(t) := \max_{0 \leqslant s \leqslant t} \phi(s)$,

$$c_i(t) := G_i^{-1}\left[G_i\left(c_{i-1}(t)\right) + \int_a^t k_i(t,s)ds\right], \qquad i = 1,\ldots,m,$$

$$G_i(u) := \int_{u_i}^{u} \frac{dx}{g_i(x)}, \qquad u > 0, \qquad u_i > 0,$$

and T is chosen so that the functions $c_i(t)$, $i = 1,\ldots,m$, are defined for $a \leqslant t < T$.

In particular,

LEMMA 10. *Let $\phi(t)$ be a positive continuous function in $[a,b]$, and $\psi_i(t)$, $i = 1,\ldots,m$, be nonnegative nondecreasing continuous functions in $[a,b]$, then for positive integers $n_i > 1$, the inequality*

$$u(t) \leqslant \phi(t) + \sum_{i=1}^{m} \psi_i(t) \int_a^t u^{n_i}(s)\, ds, \qquad t \in [a,b],$$

implies that

$$u(t) \leqslant c_m(t), \qquad a \leqslant t < T,$$

where

$$c_0(t) := \max_{0 \leqslant s \leqslant t} \phi(s),$$

$$c_i(t) = \left[c_{i-1}^{1-n_i}(t) - (n_i - 1)(t-a)\,\psi_i(t)\right]^{\frac{1}{1-n_i}}, \qquad i = 1,\ldots,m, \tag{2}$$

and T is chosen so that the functions $c_i(t)$, $i = 1,\ldots,m$, are defined for $a \leqslant t < T$.

To study the asymptotic behavior of the functions in (2) we need the following lemma.

LEMMA 11. *Let f and g be positive functions such that*

$$f(t) < F, \qquad g(t) < G\, t^{-q},$$

for some positive constant F, G, and q. Let

$$h(t) = \frac{f(t)}{1 - f(t)\; g(t)}.$$

Then, for any $0 < \epsilon < 1$, and $t > \left(\frac{FG}{1-\epsilon}\right)^{\frac{1}{q}}$,

$$h(t) < F/\epsilon.$$

Now, from this lemma we have the following corollary.

COROLLARY 12. *In Lemma (10), if $\phi(t)$ is uniformly bounded, and $t\,\psi_i(t)$, $i = 1,\dots,m$, has a power-type decay as $t \to \infty$, then $c_i(t)$, $i = 1,\dots,m$, are uniformly bounded for sufficiently large t and*

$$u(t) < c, \qquad c > 0.$$

Proof. Use Lemma 11 with $f(t) = c_{i-1}^{n_i-1}(t)$ and $h(t) = (n_i - 1)\,(t-a)\,\psi_i(t)$. □

We present below some well-known and easy to derive results that we use in our proofs.

LEMMA 13. (Generalized Young's Inequality) *For nonnegative a_i, $i = 1,\dots,k$,*

$$\left(\sum_{i=1}^{k} a_i\right)^n \leqslant k^{n-1} \sum_{i=1}^{k} a_i^n, \tag{3}$$

where k and n are positive integers.

LEMMA 14. (Young's Inequality) *For $f, g \geqslant 0$, $p > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$,*

$$f\,g \leqslant \frac{f^p}{p} + \frac{g^q}{q}. \tag{4}$$

In particular

LEMMA 15. *for nonnegative f and g, and $0 < m < 1$,*

$$f(t)\; g^m(t) \leqslant (1-m)\, f^{\frac{1}{1-m}}(t) + m\, g(t). \tag{5}$$

Proof. In (4), let $q = \frac{1}{m}$ and $p = \frac{1}{1-m}$. □

LEMMA 16. *for positive f and g, $t > a$, and $0 < m < 1$,*

$$f(t)\left[\int_a^t g(s)ds\right]^m \leqslant (1-m)f^{\frac{1}{1-m}}(t) + m\int_a^t g(s)ds. \tag{6}$$

LEMMA 17. *For all $\alpha > 0$ and $\mu < 1$, we have*

$$\int_0^t (t-s)^{\alpha-1}s^{-\mu}ds = \Gamma(\alpha)\ \Gamma(1-\mu)\ R_{\alpha-\mu+1}(t), \qquad t \geqslant 0.$$

For μ replaced by $p\mu$ and α replaced by $p(\alpha-1)+1$ we get

COROLLARY 18. *For $0 < \alpha < 1$, $p < \frac{1}{1-\alpha}$, and $\mu < \frac{1}{p}$ we have*

$$\int_0^t (t-s)^{p(\alpha-1)}s^{-p\mu}ds = \Gamma(p(\alpha-1)+1)\ \Gamma(1-p\mu)\ R_{p(\alpha-\mu-1)+2}(t), \quad t \geqslant 0. \tag{7}$$

In particular when $\mu = 0$, the relation (7) reduces to

$$\int_0^t (t-s)^{p(\alpha-1)}ds = \Gamma(p(\alpha-1)+1)\ R_{p(\alpha-1)+2}(t) = \frac{t^{p(\alpha-1)+1}}{p(\alpha-1)+1}, \quad t \geqslant 0, \tag{8}$$

which can also be found by direct integration.

REMARK 1. Note that the condition $p < \frac{1}{1-\alpha}$ is equivalent to $q > \frac{1}{\alpha}$ for $\frac{1}{p}+\frac{1}{q} = 1$.

LEMMA 19. *For $\phi \in L_1(a,b)$, $0 < \alpha < 1$, and $q > \frac{1}{\alpha}$,*

$$[I^\alpha\phi(t)]^n \leqslant \begin{cases} S_\alpha(t,q,\mu)\ \left(\int_a^t s^{q\mu}\ |\phi(s)|^q\,ds\right)^{\frac{1}{q}}, & n=1,\ \mu < 1-\frac{1}{q} \\ S_\alpha^n(t,n,\mu)\ \int_a^t s^{n\mu}\ |\phi(s)|^n\,ds, & n > \frac{1}{\alpha},\ \mu < 1-\frac{1}{n} \\ \left(1-\frac{n}{q}\right) S_\alpha^{\frac{nq}{q-n}}(t,q,\mu)+\frac{n}{q}\int_a^t s^{q\mu}|\phi(s)|^q\,ds, & 0 < n \leqslant \frac{1}{\alpha},\ \mu < 1-\frac{1}{q}. \end{cases} \tag{9}$$

where

$$S_\alpha(t,q,\mu) = A_{\alpha,q,\mu}(t-a)^{\alpha-\mu-\frac{1}{q}},$$

$$A_{\alpha,q,\mu} = \frac{1}{\Gamma(\alpha)}\left[\frac{\Gamma(1-p\mu)\ \Gamma(p(\alpha-1)+1)}{\Gamma(p(\alpha-\mu-1)+2)}\right]^{\frac{1}{p}},$$

and $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. Because of the condition $p(1-\alpha) < 1$ in Corollary 18 we need to distinguish between the two cases: $n > \frac{1}{\alpha}$ and $n \leqslant \frac{1}{\alpha}$.

For $n = 1$, using Hölder inequality, the result follows from Corollary 18. For $n > \frac{1}{\alpha}$, raise the first inequality in (9) to power n and let q be n. For $0 < n \leqslant \frac{1}{\alpha}$, we use the first inequality raised to power n and Lemma 16 with $m = \frac{n}{q}$ □

The next lemma is crucial to our results. It contains three reference inequalities which will be used repeatedly.

LEMMA 20. *Suppose $u(t)$ is a nonnegative function having a summable nonnegative fractional derivative $D^{\alpha}u$, $0<\alpha<1$. For positive integer n, $0<\beta<\alpha<1$, $t>a$, and $q>\frac{1}{\alpha-\beta}$,*

$$\left[D^{\beta}u(t)\right]^{n} \leqslant \begin{cases} \tilde{R}_{\alpha-\beta}(t,u,1)+S_{\alpha-\beta}(t,q,\mu)\left(\int_a^t [s^{\mu}D^{\alpha}u(s)]^q\,ds\right)^{\frac{1}{q}}, & n=1,\ \mu<1-\frac{1}{q} \\ \tilde{R}_{\alpha-\beta}(t,u,n)+2^{n-1}S^{n}_{\alpha-\beta}(t,n,\mu)\int_a^t [s^{\mu}D^{\alpha}u(s)]^n\,ds, & n>\frac{1}{\alpha-\beta},\ \mu<1-\frac{1}{n} \\ \tilde{S}_{\alpha-\beta}(t,u,n,q,\mu)+\frac{n2^{n-1}}{q}\int_a^t [s^{\mu}D^{\alpha}u(s)]^q\,ds, & 0<n\leqslant\frac{1}{\alpha-\beta},\ \mu<1-\frac{1}{q}. \end{cases} \tag{10}$$

where,

$$\begin{aligned} \tilde{R}_{\alpha-\beta}(t,u,n) &= 2^{n-1}\ u^{n}_{1-\alpha}(a)\ R^{n}_{\alpha-\beta}(t-a) = B_{\alpha,\beta,u,n}\ t^{-n(1-\alpha+\beta)} \\ B_{\alpha,\beta,u,n} &= \frac{2^{n-1}\ u^{n}_{1-\alpha}(a)}{\Gamma^{n}(\alpha-\beta)} \\ \tilde{S}_{\alpha-\beta}(t,u,n,q,\mu) &= \tilde{R}_{\alpha-\beta}(t,u,n)+\left(1-\frac{n}{q}\right)2^{n-1}S^{\frac{nq}{q-n}}_{\alpha-\beta}(t,q,\mu) \\ &= B_{\alpha,\beta,u,n}\ t^{-n(1-\alpha+\beta)}+C_{\alpha-\beta,q,n,\mu}\ (t-a)^{\frac{n[q(\alpha-\beta-\mu)-1]}{q-n}} \\ C_{\alpha-\beta,q,n,\mu} &= \left(1-\frac{n}{q}\right)2^{n-1}\ A^{\frac{nq}{q-n}}_{\alpha-\beta,q,\mu} \end{aligned}$$

Proof. The result follows directly from Corollary 6, Lemma 13, and Lemma 19. □

REMARK 2. Note that under our assumptions, the functions S_{α}, $\tilde{R}_{\alpha-\beta}$, and $\tilde{S}_{\alpha-\beta}$ are non-negative.

3. The Results

In this section we state and prove our results. Namely, we will provide some bounds for solutions of the differential inequalities in (a) – (c). Let us start with the linear case (a).

THEOREM 21. *Let $a(t)$ and $b_i(t)$, $i=1,\dots,k$ be nonnegative continuous functions on $[0,T]$, $0<T\leqslant\infty$. Suppose that $u(t)$ is a nonnegative function having a summable nonnegative fractional derivative $D^{\alpha}u$, $0<\alpha<1$, and satisfying*

$$D^{\alpha}u(t)\leqslant a(t)+\sum_{i=1}^{k}b_i(t)\,D^{\beta_i}u(t), \qquad 0\leqslant\beta_i<\alpha<1, \quad i=1,\dots,k. \tag{11}$$

Then,

$$D^{\alpha}u(t)\leqslant t^{-\mu}\ \left(\tilde{a}(t)e^{\int_0^t\tilde{b}(s)ds}\right)^{\frac{1}{q}},$$

where

$$\tilde{a}(t) = 2^{q-1}\ t^{q\mu}\ \left(a(t) + u_{1-\alpha}(0)\sum_{i=1}^{k} b_i(t)\ R_{\alpha-\beta_i}(t)\right)^q,$$

and

$$\tilde{b}(t) = 2^{q-1}\ t^{q\mu}\ \left(\sum_{i=1}^{k} b_i(t)\ S_{\alpha-\beta_i}(t,q,\mu)\right)^q,$$

for any $q > \frac{1}{\alpha-\beta_i}$, $i = 1,\dots,k$ *and* $\mu < 1 - \frac{1}{q}$.

Proof. By Lemma 20 (first inequality) we have

$$\begin{aligned} D^{\alpha}u(t) &\leqslant a(t) + \sum_{i=1}^{k} b_i(t)\left[\tilde{R}_{\alpha-\beta_i}(t,u,1) + S_{\alpha-\beta_i}(t,q,\mu)\ \left(\int_0^t [s^{\mu}D^{\alpha}u(s)]^q\,ds\right)^{\frac{1}{q}}\right] \\ &\leqslant \hat{a}(t) + \hat{b}(t)\ \left(\int_0^t [s^{\mu}D^{\alpha}u(s)]^q\,ds\right)^{\frac{1}{q}}, \end{aligned}$$

with

$$\hat{a}(t) = a(t) + \sum_{i=1}^{k} b_i(t)\ \tilde{R}_{\alpha-\beta_i}(t,u,1), \qquad \hat{b}(t) = \sum_{i=1}^{k} b_i(t) S_{\alpha-\beta_i}(t,q,\mu).$$

Multiplying both sides by t^{μ}, raising them to the power q, and using (3), we obtain

$$[t^{\mu}D^{\alpha}u(t)]^q \leqslant \tilde{a}(t) + \tilde{b}(t)\int_0^t [s^{\mu}D^{\alpha}u(s)]^q ds. \tag{12}$$

The result follows from Gronwall inequality. □

The next theorem is concerned with the inequality in (b).

THEOREM 22. *Let* $a(t)$ *and* $b_i(t)$, $i = 1,\dots,k$ *be nonnegative continuous functions on* $[0,T]$, $0 < T \leqslant \infty$. *Suppose that* $u(t)$ *is a nonnegative function having a summable nonnegative fractional derivative* $D^{\alpha}u$, $0 < \alpha < 1$, *and satisfying*

$$D^{\alpha}u(t) \leqslant a(t) + \sum_{i=1}^{k} b_i(t)\left[D^{\beta_i}u(t)\right]^n, \qquad 0 \leqslant \beta_i < \alpha < 1, \quad i = 1,\dots,k, \tag{13}$$

with a positive integer $n > 1$. *Then we have*

$$D^{\alpha}u(t) \leqslant t^{-\mu}\ \left[A^{1-q}(t) - (q-1)\,tB(t)\right]^{\frac{1}{1-q}}, \qquad t \in [0,T], \tag{14}$$

where $T > 0$ *is such that*

$$A^{1-q}(t) - (q-1)\,t\,B(t) > 0,$$

for all $t \in [0,T)$, *and*

$$A(t) = \max_{0\leqslant s\leqslant t} \tilde{a}(s), \qquad B(t) = \max_{0\leqslant s\leqslant t} \tilde{b}(s).$$

The functions $\tilde{a}$, $\tilde{b}$, *and the powers* q, μ *are as follows:*

(a) If $n > \frac{1}{\alpha-\beta_i}$, $1 \leqslant i \leqslant k$, *then* $q = n$, $\mu < 1 - \frac{1}{n}$, *and*

$$\tilde{a}(t) = t^{\mu} \left[a(t) + \sum_{i=1}^{k} b_i(t)\, \tilde{R}_{\alpha-\beta_i}(t,u,n) \right],$$

$$\tilde{b}(t) = 2^{n-1}\, t^{\mu} \sum_{i=1}^{k} b_i(t)\, S^n_{\alpha-\beta_i}(t,n,\mu).$$

(b) If $1 < n \leqslant \frac{1}{\alpha-\beta_i}$, $1 \leqslant i \leqslant k$, *then* q *is any value* $> \frac{1}{\alpha-\beta}$, *with* $\beta = m_i n_{1 \leqslant i \leqslant k} \beta_i$, $\mu < 1 - \frac{1}{q}$, *and*

$$\tilde{a}(t) = t^{\mu} \left[a(t) + \sum_{i=1}^{k} b_i(t)\, \tilde{S}_{\alpha-\beta_i}(t,u,n,q,\mu) \right],$$

$$\tilde{b}(t) = \frac{n 2^{n-1}}{q}\, t^{\mu} \sum_{i=1}^{k} b_i(t).$$

Proof. (a) From Lemma 20 (second inequality) we have

$$\begin{aligned} D^{\alpha}u(t) &\leqslant a(t) + \sum_{i=1}^{k} b_i(t) \left[\tilde{R}_{\alpha-\beta_i}(t,u,n) + 2^{n-1} S^n_{\alpha-\beta_i}(t,n,\mu) \int_0^t [s^{\mu} D^{\alpha} u(s)]^n\, ds \right] \\ &\leqslant a(t) + \sum_{i=1}^{k} b_i(t) \tilde{R}_{\alpha-\beta_i}(t,u,n) \\ &\quad + \left(2^{n-1} \sum_{i=1}^{k} b_i(t) S^n_{\alpha-\beta_i}(t,n,\mu) \right) \int_0^t [s^{\mu} D^{\alpha} u(s)]^n\, ds. \end{aligned}$$

Multiplying both sides by t^{μ}, we obtain

$$t^{\mu} D^{\alpha} u(t) \leqslant \tilde{a}(t) + \tilde{b}(t) \int_0^t [s^{\mu} D^{\alpha} u(s)]^n\, ds.$$

The result follows from Lemma 8.

(b) From Lemma 20 (third inequality)

$$\begin{aligned} D^{\alpha}u(t) &\leqslant a(t) + \sum_{i=1}^{k} b_i(t) \left[\tilde{S}_{\alpha-\beta_i}(t,u,n,q,\mu) + \frac{n 2^{n-1}}{q} \int_0^t [s^{\mu} D^{\alpha} u(s)]^q\, ds \right] \\ &= a(t) + \sum_{i=1}^{k} b_i(t) \tilde{S}_{\alpha-\beta_i}(t,u,n,q,\mu) + \frac{n 2^{n-1}}{q} \sum_{i=1}^{k} b_i(t) \int_0^t [s^{\mu} D^{\alpha} u(s)]^q\, ds \end{aligned}$$

and thus

$$t^{\mu}\, D^{\alpha} u(t) \leqslant \tilde{a}(t) + \tilde{b}(t) \int_0^t [s^{\mu} D^{\alpha} u(s)]^q\, ds.$$

We conclude using Lemma 8. □

REMARK 3. In case n satisfy (a) for some i and satisfy (b) for the others, we combine the proofs in (a) and (b) and use the corresponding i-th term in the series of $\tilde{b}(t)$.

THEOREM 23. *Let $a(t)$ and $b_i(t)$, $i = 1, \ldots, k$ be nonnegative continuous functions on $[0,T]$, $0 < T \leqslant \infty$. Suppose that $u(t)$ is a nonnegative function having a summable nonnegative fractional derivative $D^\alpha u$, $0 \leqslant \alpha \leqslant 1$, and satisfying*

$$D^\alpha u(t) \leqslant a(t) + \sum_{i=1}^{k} b_i(t) \left[D^{\beta_i} u(t)\right]^{n_i}, \qquad 0 \leqslant \beta_i < \alpha < 1, \quad i = 1, \ldots, k, \quad (15)$$

with positive integers $n_i > 1$, $1 \leqslant i \leqslant k$. Then we have

$$D^\alpha u(t) \leqslant t^{-\mu} c_k(t), \qquad 0 \leqslant t < T,$$

where $c_0(t) := \max_{0 \leqslant s \leqslant t} \tilde{a}(s)$,

$$c_i(t) = \left[c_{i-1}^{1-q_i}(t) - (q_i - 1)\, t\, \tilde{b}_i(t)\right]^{\frac{1}{1-q_i}}, \qquad i = 1, \ldots, k,$$

and T is chosen so that the functions $c_i(t)$, $i = 1, \ldots, k$, are defined for $0 \leqslant t < T$.

The functions $\tilde{a}$, $\tilde{b}_i$, and the powers q_i, μ are as follows:

(a) If $n_i > \frac{1}{\alpha - \beta_i}$, $i = 1, \ldots, k$, then $q_i = n_i$, $\mu < 1 - \frac{1}{n_i}$, $i = 1, \ldots, k$, and

$$\tilde{a}(t) = \left[a(t) + \sum_{i=1}^{k} b_i(t)\, \tilde{R}_{\alpha-\beta_i}(t, u, n_i)\right] t^\mu,$$

$$\tilde{b}_i(t) = 2^{n_i - 1}\, b_i(t)\, S_{\alpha-\beta_i}^{n_i}(t, n_i, \mu)\; t^\mu.$$

(b) If $1 < n_i \leqslant \frac{1}{\alpha - \beta_i}$, $i = 1, \ldots, k$, then q_i is any value $> \frac{1}{\alpha-\beta_i}$, $\mu < 1 - \frac{1}{q_i}$, $i = 1, \ldots, k$ and

$$\tilde{a}(t) = \left[a(t) + \sum_{i=1}^{k} b_i(t)\; \tilde{S}_{\alpha-\beta_i}(t, u, n_i, q_i, \mu)\right] t^\mu,$$

$$\tilde{b}_i(t) = \frac{n_i 2^{n_i - 1}}{q_i}\; b_i(t)\; t^\mu.$$

Proof. (a) From Lemma 20 (second inequality) we have

$$\begin{aligned} D^\alpha u(t) \;&\leqslant\; a(t) + \sum_{i=1}^{k} b_i(t) \left[\tilde{R}_{\alpha-\beta_i}(t, u, n_i) + 2^{n_i-1}\, S_{\alpha-\beta_i}^{n_i}(t, n_i, \mu) \;\int_0^t [s^\mu D^\alpha u(s)]^{n_i}\, ds\right] \\ &\leqslant\; a(t) + \sum_{i=1}^{k} b_i(t)\, \tilde{R}_{\alpha-\beta_i}(t, u, n_i) \\ &\qquad + \sum_{i=1}^{k} 2^{n_i-1}\, b_i(t)\, S_{\alpha-\beta_i}^{n_i}(t, n_i, \mu) \;\int_0^t [s^\mu D^\alpha u(s)]^{n_i}\, ds \end{aligned}$$

Multiplying both sides by t^μ, we obtain

$$t^\mu D^\alpha u(t) \leqslant \tilde{a}(t) + \sum_{i=1}^{k} \tilde{b}_i(t) \int_0^t [s^\mu D^\alpha u(s)]^{n_i}\, ds,$$

where $\tilde{a}(t)$ and $\tilde{b}(t)$ are as in the statement of part (a) of the theorem. Next, we appeal to Lemma 10.

(b) From Lemma 20 (third inequality) we have

$$\begin{aligned} D^\alpha u(t) &\leqslant a(t) + \sum_{i=1}^{k} b_i(t) \left[\tilde{S}_{\alpha-\beta_i}(t,u,n_i,q_i,\mu) + \frac{n_i 2^{n_i-1}}{q_i} \int_0^t [s^\mu D^\alpha u(s)]^{q_i}\, ds,\right] \\ &\leqslant a(t) + \sum_{i=1}^{k} b_i(t)\ \tilde{S}_{\alpha-\beta_i}(t,u,n_i,q_i,\mu) + \sum_{i=1}^{k} \frac{n_i 2^{n_i-1}}{q_i}\ b_i(t) \int_0^t [s^\mu D^\alpha u(s)]^{q_i}\, ds. \end{aligned}$$

Multiplying both sides by t^μ, we obtain

$$t^\mu D^\alpha u(t) \leqslant \tilde{a}(t) + \sum_{i=1}^{k} \tilde{b}_i(t) \int_0^t [s^\mu D^\alpha u(s)]^{q_i}\, ds.$$

We assumes the exponents q_i to be in nondecreasing order. If not, we can always reorder them. Lemma 10 allows us to derive the result. □

REMARK 4. In case some of the n_i satisfy (a) and the remaining satisfy (b), we combine the proofs in (a) and (b) and reorder n_i, q_i in the nondecreasing order.

4. Application

In this section we present an application of the results proved in the previous section. We use the bounds to show that when the coefficients are power functions, then so is the fractional derivative of highest order α.

Consider the problem

$$\begin{cases} D^\alpha u(t) = f\left(t, \{D^{\beta_i} u(t)\}_{i=1}^k\right), & 0 \leqslant \beta_i < \alpha < 1, \quad i = 1,\ldots,k \\ u(0) = u_0 \end{cases} \tag{16}$$

with

$$\left|f\left(t, \{D^{\beta_i} u(t)\}_{i=1}^k\right)\right| \leqslant a(t) + \sum_{i=1}^{k} b_i(t) \left|D^{\beta_i} u(t)\right|^{n_i}, \qquad t > 0,\ n_i > 1.$$

Suppose there exists $T_0 > 0$ such that for $t > T_0$,

$$a(t) \leqslant A_a t^{-r}, \qquad b_i(t) \leqslant A_b t^{-s_i}$$

for some $r, s_i, A_a, A_b > 0$. Then we have the following results.

THEOREM 24. *For each* $i = 1, \ldots, k$, *suppose that* $n_i > \frac{1}{\alpha-\beta_i}$, *and for any* $\epsilon > 0$,

$$s_i > n_i(\alpha - \beta_i) - (n_i - 1)\mu_1, \tag{17}$$

with

$$\mu_1 = \min\left\{\frac{r}{1+\epsilon}, \frac{1}{2+\epsilon}\right\}. \tag{18}$$

Let

$$\mu_2 = \min_{1 \leqslant i \leqslant k}\left\{r, 1 - \frac{1}{n_i}\right\}. \tag{19}$$

Then

$$D^\alpha u(t) \leqslant ct^{-\mu}, \qquad t > T_0,$$

for

$$\mu_1 < \mu < \mu_2, \tag{20}$$

and some $c > 0$.

Proof. Note that $\mu_1 < \mu_2$ since $n_i > 1$, and thus μ is well defined. It follows from (19) that $\mu < 1 - 1/n_i$, $i = 1, \ldots k$. From Theorem 23(a) we have

$$D^\alpha u(t) \leqslant t^{-\mu} c_k(t) \tag{21}$$

$$c_0(t) = \max_{0 \leqslant s \leqslant t} \tilde{a}(s) \tag{22}$$

$$c_i(t) = \left[c_{i-1}^{1-n_i}(t) - (n_i - 1)\, t\, \tilde{b}_i(t)\right]^{\frac{1}{1-n_i}} = \left[\frac{c_{i-1}^{n_i-1}}{1 - (n_i-1)\, t\, \tilde{b}_i(t)\, c_{i-1}^{n_i-1}}\right]^{\frac{1}{n_i-1}}, \tag{23}$$

with

$$\begin{aligned} \tilde{a}(t) &= \left[a(t) + \sum_{i=1}^{k} b_i(t)\, \tilde{R}_{\alpha-\beta_i}(t, u, n_i)\right] t^\mu & (24)\\ &= \left[a(t) + \sum_{i=1}^{k} b_i(t)\, B_{\alpha,\beta_i,u,n_i}\, t^{-n_i(1-\alpha+\beta_i)}\right] t^\mu & (25)\\ &\leqslant A_a\, t^{\mu-r} + B_a \sum_{i=1}^{k} t^{-\eta_i}, & (26) \end{aligned}$$

and

$$\begin{aligned} t\,\tilde{b}_i(t) &= 2^{n_i-1}\, b_i(t)\, S_{\alpha-\beta_i}^{n_i}(t, n_i, \mu)\; t^{\mu+1} & (27)\\ &= 2^{n_i-1}\, b_i(t)\, \left(A_{\alpha-\beta_i,n_i,\mu}\; t^{\alpha-\beta_i-\mu-\frac{1}{n_i}}\right)^{n_i}\; t^{\mu+1} & (28)\\ &\leqslant \tilde{B}_i\, t^{-\rho_i}, & (29) \end{aligned}$$

where

$$B_{\alpha,\beta_i,u,n_i} = \frac{2^{n_i-1} u_{1-\alpha}^{n_i}(0)}{\Gamma^{n_i}(\alpha-\beta_i)}, \quad B_a = A_b \max_{1\leqslant i\leqslant k} B_{\alpha,\beta_i,u,n_i}, \quad \tilde{B}_i = 2^{n_i-1}\, A_b\, A_{\alpha-\beta_i,n_i,\mu}^{n_i},$$

$$\eta_i = n_i(1-\alpha+\beta_i)+s_i-\mu,$$

and

$$\rho_i = n_i(\mu-\alpha+\beta_i)+s_i-\mu.$$

It follows from (17), (18), and (20) that $\mu < r$, $\rho_i > 0$ and $\eta_i > 1$, $1 \leqslant i \leqslant k$, since

$$\begin{aligned}\rho_i &= (n_i-1)\mu - n_i(\alpha-\beta_i)+s_i\\ &> (n_i-1)\mu - n_i(\alpha-\beta_i)+n_i(\alpha-\beta_i)-(n_i-1)\mu_1\\ &= (n_i-1)\mu-(n_i-1)\mu_1 = (n_i-1)(\mu-\mu_1) > 0,\end{aligned}$$

and

$$\eta_i-\rho_i = n_i(1-\mu) > n_i\left(1-1+\frac{1}{n_i}\right) = 1.$$

Therefore, $\tilde{a}(t)$ is uniformly bounded and $t\,\tilde{b}_i(t)$ has a power-type decay. The result follows from Corollary 12. □

THEOREM 25. *Suppose for* $1 \leqslant i \leqslant k$,

$$n_i < \frac{1}{\alpha-\beta_i}, \tag{30}$$

and

$$s_i > 1. \tag{31}$$

Let

$$\mu < \min_{1\leqslant i\leqslant k}\left\{r, s_i-1, 1-\frac{1}{n_i}\right\}. \tag{32}$$

Then

$$D^\alpha u(t) \leqslant ct^{-\mu}, \qquad t > T_0,$$

for some $c > 0$.

Proof. Let $q_i > \frac{1}{\alpha-\beta_i}$, $1 \leqslant i \leqslant k$, then from Theorem 23,

$$D^\alpha u(t) \leqslant t^{-\mu} c_k(t) \tag{33}$$

$$c_0(t) = \max_{0\leqslant s\leqslant t} \tilde{a}(s) \tag{34}$$

$$c_i(t) = \left[c_{i-1}^{1-n_i}(t) - (n_i-1)\,t\,\tilde{b}_i(t)\right]^{\frac{1}{1-n_i}} = \left[\frac{c_{i-1}^{n_i-1}}{1-(n_i-1)\,t\,\tilde{b}_i(t)\,c_{i-1}^{n_i-1}}\right]^{\frac{1}{n_i-1}}, \tag{35}$$

with

$$\begin{aligned}\tilde{a}(t)&\left[a(t)+\sum_{i=1}^{k} b_i(t)\,\tilde{S}_{\alpha-\beta_i}(t,u,n_i,q_i,\mu)\right]t^\mu\\ &= \left[a(t)+\sum_{i=1}^{k} b_i(t)\left\{B_{\alpha,\beta_i,u,n_i}\,t^{-n_i(1-\alpha+\beta_i)} + C_{\alpha-\beta_i,q_i,n_i,\mu}\,t^{\frac{n_i[q_i(\alpha-\beta_i-\mu)-1]}{q_i-n_i}}\right\}\right]t^\mu\end{aligned}$$

$$
\begin{aligned}
&< \left[a(t) + A_b \sum_{i=1}^{k} \left\{ B_{\alpha,\beta_i,u,n_i}\, t^{-n_i(1-\alpha+\beta_i)-s_i} + C_{\alpha-\beta_i,q_i,n_i,\mu}\, t^{\frac{n_i[q_i(\alpha-\beta_i-\mu)-1]}{q_i-n_i}-s_i} \right\} \right] t^{\mu} \\
&\leqslant A_a t^{\mu-r} + A_b \sum_{i=1}^{k} \left\{ B_{\alpha,\beta_i,u,n_i} t^{-n_i(1-\alpha+\beta_i)-s_i+\mu} + C_{\alpha-\beta_i,q_i,n_i,\mu} t^{\frac{n_i[q_i(\alpha-\beta_i-\mu)-1]}{q_i-n_i}-s_i+\mu} \right\} \\
&\leqslant A_a\; t^{\mu-r} + B_a \sum_{i=1}^{k} \left[t^{-\eta_i} + t^{-\delta_i} \right]
\end{aligned}
$$

and

$$
\begin{aligned}
t\,\tilde{b}_i(t) &= \frac{n_i 2^{n_i-1}}{q_i}\, b_i(t)\; t^{\mu+1} \\
&\leqslant \frac{n_i 2^{n_i-1}}{q_i}\; A_b\, t^{\mu-s_i+1} \\
&\leqslant \tilde{B}_i\, t^{\mu-s_i+1},
\end{aligned}
$$

where

$$
B_a = A_b \max_{1\leqslant i\leqslant k} \left\{ B_{\alpha,\beta_i,u,n_i}, C_{\alpha-\beta_i,q_i,n_i,\mu} \right\}, \qquad \tilde{B}_i = \frac{n_i 2^{n_i-1}}{q_i}\; A_b,
$$

$$
\eta_i = n_i(1-\alpha+\beta_i) + s_i - \mu,
$$

and

$$
\delta_i = \frac{-n_i\left[q_i(\alpha-\beta_i-\mu)-1\right]}{q_i-n_i} + s_i - \mu.
$$

It follows from (30), (31) and (32) that

$$
\mu - r < 0, \qquad \mu - s_i + 1 < 0,
$$

$$
\begin{aligned}
\eta_i &= n_i(1-\alpha-\beta_i) + s_i - \mu \\
&> n_i - n_i(\alpha-\beta_i) + 1 \\
&> n_i - 1 + 1 = n_i \geqslant 2,
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_i &= \frac{n_i}{q_i-n_i}\left[1 - q_i(\alpha-\beta_i-\mu)\right] + s_i - \mu \\
&> \frac{n_i}{q_i-n_i}\left[1 + q_i\mu - q_i(\alpha-\beta_i)\right] + 1 \\
&= \frac{n_i}{q_i-n_i}\left[1 + q_i\mu - q_i(\alpha-\beta_i) + \frac{q_i}{n_i} - 1\right] \\
&= \frac{n_i}{q_i-n_i}\left[q_i\mu - q_i(\alpha-\beta_i) + \frac{q_i}{n_i}\right] \\
&= \frac{q_i}{q_i-n_i}\left[n_i\mu - n_i(\alpha-\beta_i) + 1\right]
\end{aligned}
$$

$$> \frac{q_i}{q_i - n_i}\left[n_i\mu - 1 + 1\right] = \frac{q_i n_i \mu}{q_i - n_i} > 0.$$

Therefore, $\tilde{a}(t)$ is uniformly bounded and $t\,\tilde{b}_i(t)$ has a power-type decay. The result follows from Corollary 12. □

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