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INTEGRAL REPRESENTATIONS OF GENERALIZED WHITELEY MEANS AND RELATED INEQUALITIES

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Abstract. The main purpose of this paper is to give two integral representations of generalized Whiteley means which are natural generalization of complete symmetric means. Various applications are given, mainly towards obtaining inequalities characteristic for complete symmetric means such as Schur's inequality. The Schur convexity of this means and related complete symmetric functions is also discussed.

1. Introduction

The elementary and complete symmetric polynomials are used to define means that generalize the geometric and arithmetic means in a very natural way. Inequalities arising in studying relations between these means generalize many classical inequalities as for example geometric mean-arithmetic mean inequality. A history of means defined by elementary and complete symmetric polynomials goes back to I. Newton. Interested reader in this subject should consult Chapter V of an excellent book by P. S. Bullen [4].

Our primary concern in this paper are complete symmetric polynomials (called also complete symmetric functions). The complete symmetric polynomial of the rth degree is defined by

$$C_n^{[r]}(\mathbf{x}) = \sum_{i_1+\cdots i_n=r} \prod_{j=1}^{n} x_j^{i_j},$$

where $r \in \mathbb{N}$, $i_1, \ldots, i_n$ are nonnegative integers and $\mathbf{x} \in \mathbb{R}_+^n = \{(x_1, \ldots, x_n) : x_i \geqslant 0$, $i = 1, \ldots, n\}$. It is customary to define $C_n^{[0]}(\mathbf{x}) = 1$ for every $\mathbf{x} \in \mathbb{R}_+^n$.

The complete symmetric mean of the rth degree of an $n-$ tuple $\mathbf{x} \in \mathbb{R}_+^n$ is defined by

$$c_n^{[r]}(\mathbf{x}) = \frac{1}{\binom{n+r-1}{r}} C_n^{[r]}(\mathbf{x}).$$

The basic inequality for the complete symmetric means is

$$\left(c_n^{[r]}(\mathbf{x})\right)^2 \leqslant c_n^{[r-1]}(\mathbf{x}) \cdot c_n^{[r+1]}(\mathbf{x}), \tag{1.1}$$

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which holds for every $r \in \mathbb{N}$ and every $\mathbf{x} \in \mathbb{R}_+^n$. This inequality was proved by T. Popoviciu in [16] and I. Schur (see [7, p. 164]). The proof of I. Schur was based on an integral representation

$$c_n^{[r]}(\mathbf{x}) = (n-1)! \int_{\Delta_{n-1}} \left(\sum_{i=1}^{n} x_i t_i \right)^r dt_1 \dots dt_{n-1}, \tag{1.2}$$

and Cauchy inequality, where $\Delta_{n-1} = \{(t_1, \dots, t_{n-1}) : t_i \geqslant 0, i = 1, \dots, n-1, \sum_{i=1}^{n-1} t_i \leqslant 1\}$ and $t_n = 1 - \sum_{i=1}^{n-1} t_i$.

Schur's inequality (1.1) in a simple manner implies the following inequalities:

$$\left(c_n^{[r]}(\mathbf{x})\right)^{\frac{1}{r}} \leqslant \left(c_n^{[r+1]}(\mathbf{x})\right)^{\frac{1}{r+1}}, \quad r \in \mathbb{N}, \tag{1.3}$$

$$c_n^{[r-s]}(\mathbf{x}) c_n^{[r+s]}(\mathbf{x}) \leqslant c_n^{[r-s-1]}(\mathbf{x}) c_n^{[r+s+1]}(\mathbf{x}), \quad 0 \leqslant s < r. \tag{1.4}$$

The inequality (1.3) is classical (see [4] and references therein). The inequality (1.4), although trivial consequence of (1.1), drew some attention after curious error in [12] where the reverse inequality was proved for $s = 1$ and $n = 2$. In the same paper K. V. Menon proved (1.1) for $r = 1, 2, 3$ and every $n \in \mathbb{N}$. In [6] D. W. Detemple and J. M. Robertson have proved (1.1) for $n = 2$ and every $r \in \mathbb{N}$. Finally, K. Guan in [8] has proved (1.1), in [9] (1.4) for $s = 1$ and in [10] (1.4) generally.

The main purpose of this paper is to give integral representations of generalized Whiteley means which are natural generalization of complete symmetric means (see Section 2.). The first representation is in the sense of [1, p. 36] and the second one is a generalization of (1.2), and using this we prove for the generalized Whiteley means inequalities analogous to (1.1), (1.3), (1.4) and some related inequalities. It is also interesting to point out that Schur's idea of using integral representation was somehow neglected in above mentioned papers.

2. Integral representations of generalized Whiteley means

Let $r \in \mathbb{N} \cup \{0\}$, $\mathbf{x} \in \mathbb{R}_+^n$ and let $\mathbf{s} = (s_1, \dots, s_n)$ be such that $s_i > 0$ for every $i = 1, \dots, n$. The generalized complete symmetric polynomials $W_n^{[r,\mathbf{s}]}(\mathbf{x})$ of degree r are defined by

$$\sum_{r=0}^{\infty} W_n^{[r,\mathbf{s}]}(\mathbf{x}) t^r = \prod_{i=1}^{n} \frac{1}{(1 - x_i t)^{s_i}}$$

for $|t|$ small enough.

REMARK 2.1. Notice that for $\mathbf{s} = (1, \dots, 1) \doteq \mathbf{1}$, $W_n^{[r,\mathbf{1}]} = C_n^{[r]}$.

REMARK 2.2. It is easy to see that generalized complete symmetric polynomials can be written alternatively as

$$W_n^{[r,\mathbf{s}]}(\mathbf{x}) = \sum_{i_1 + \dots + i_n = r} \prod_{j=1}^{n} \binom{s_j + i_j - 1}{i_j} x_j^{i_j}$$

(compare [14]).

The functions $W_n^{[r,\mathbf{s}]}(\mathbf{x})$ and their associate means were introduced by C. Gini (see [4]).

The Whiteley means $w_n^{[r,s]}$ were defined using generalized complete symmetric polynomials $W_n^{[r,s]}$, with the convention $\mathbf{s} = s$ meaning $s_i = s,\ i = 1,\dots,n,$ by

$$w_n^{[r,s]}(\mathbf{x}) = \frac{1}{\binom{ns+r-1}{r}} W_n^{[r,s]}, \quad r \in \mathbb{N} \cup \{0\}, \quad s > 0. \tag{2.1}$$

In [1] one can find an integral representation of $W_n^{[r,s]}$ in the form

$$W_n^{[r,s]}(\mathbf{x}) = \frac{1}{r!\,[\Gamma(s)]^n} \int_{\mathbb{R}_+^n} \left(\sum_{i=1}^n x_i u_i\right)^r \exp\left(-\sum_{i=1}^n u_i\right) \prod_{i=1}^n u_i^{s-1}\, du_1 \dots du_n. \tag{2.2}$$

Similarly, using

$$\frac{1}{(1-x_i t)^{s_i}} = \frac{1}{\Gamma(s_i)} \int_0^\infty e^{-u_i(1-x_i t)} u_i^{s_i-1}\, du_i, \quad i = 1,\dots,n,$$

for $|t|$ small enough, it follows

$$\sum_{r=0}^\infty W_n^{[r,\mathbf{s}]}(\mathbf{x}) t^r = \prod_{i=1}^n \frac{1}{(1-x_i t)^{s_i}} = \frac{1}{\prod_{i=1}^n \Gamma(s_i)} \int_{\mathbb{R}_+^n} e^{-\sum_{i=1}^n u_i(1-x_i t)} \prod_{i=1}^n u_i^{s_i-1}\, du_1 \dots du_n. \tag{2.3}$$

Derivating (2.3) $r-$ times with respect to variable t and setting $t = 0$ we obtain

$$W_n^{[r,\mathbf{s}]}(\mathbf{x}) = \frac{1}{r! \prod_{i=1}^n \Gamma(s_i)} \int_{\mathbb{R}_+^n} \left(\sum_{i=1}^n x_i u_i\right)^r e^{-\sum_{i=1}^n u_i} \prod_{i=1}^n u_i^{s_i-1}\, du_1 \dots du_n.$$

We define generalized Whiteley means by setting

$$w_n^{[r,\mathbf{s}]}(\mathbf{x}) = \frac{1}{C} W_n^{[r,\mathbf{s}]}(\mathbf{x}),$$

where $C = C(r,n,\mathbf{s})$ is determined by normalization in the sense that if $x_i = x,\ i = 1,\dots,n,$ then $w_n^{[r,\mathbf{x}]}(\mathbf{x}) = x^r$. It follows that

$$C = C(r,n,\mathbf{s}) = \frac{1}{r! \prod_{i=1}^n \Gamma(s_i)} \int_{\mathbb{R}_+^n} \left(\sum_{i=1}^n u_i\right)^r e^{-\sum_{i=1}^n u_i} \prod_{i=1}^n u_i^{s_i-1}\, du_1 \dots du_n.$$

REMARK 2.3. Notice that for $\mathbf{s} = s$, $C(r,n,s) = \binom{ns+r-1}{r}$ (compare (2.1)).

We give two forms of expressing $C(r,n,\mathbf{s})$ in an analogous way.

LEMMA 2.1. *Let $r \in \mathbb{N} \cup \{0\}$ and $\mathbf{s}$ be such that $s_i > 0,\ i = 1,\dots n$. Then* (1)

$$C(r,n,\mathbf{s}) = \sum_{r_1+\dots+r_n=r} \prod_{i=1}^n \binom{s_i + r_i - 1}{r_i},$$

where $r_1,\dots,r_n$ are nonnegative integers.

(2)
$$C(r,n,\mathbf{s}) = \binom{\sum_{i=1}^n s_i + r - 1}{r}.$$

Proof. Although the first expression follows from Remark 2.2, we give a short proof. Using Multinomial theorem and properties of the gamma function we have:

$$\begin{aligned}
r!\prod_{i=1}^n \Gamma(s_i) C(r,n,\mathbf{s}) &= \int_{\mathbb{R}_+^n} \left(\sum_{i=1}^n u_i\right)^r e^{-\sum_{i=1}^n u_i} \prod_{i=1}^n u_i^{s_i-1}\, du_1 \dots du_n \\
&= \sum_{r_1+\dots+r_n=r} \binom{r}{r_1,\dots,r_n} \prod_{i=1}^n \Gamma(s_i + r_i) \\
&= \sum_{r_1+\dots+r_n=r} \binom{r}{r_1,\dots,r_n} \prod_{i=1}^n \Gamma(s_i) \binom{s_i + r_i - 1}{r_i} r_i! \\
&= r!\prod_{i=1}^n \Gamma(s_i) \sum_{r_1+\dots+r_n=r} \prod_{i=1}^n \binom{s_i + r_i - 1}{r_i}.
\end{aligned}$$

The proof of the second expression is by induction. The equality trivially holds for $r = 0$. Suppose that the equality holds for some $r \in \mathbb{N}$ and every choice of $\mathbf{s}$. To shorten notation we denote by $d\mathbf{u}$ the product measure on $\mathbb{R}_+^n$. We have

$$\begin{aligned}
C(r+1,n,\mathbf{s}) &= \frac{1}{(r+1)!\prod_{i=1}^n \Gamma(s_i)} \int_{\mathbb{R}_+^n} \left(\sum_{i=1}^n u_i\right)^{r+1} e^{-\sum_{i=1}^n u_i} \prod_{i=1}^n u_i^{s_i-1}\, d\mathbf{u} \\
&= \frac{1}{(r+1)!\prod_{i=1}^n \Gamma(s_i)} \sum_{i=1}^n \int_{\mathbb{R}_+^n} u_i \left(\sum_{i=1}^n u_i\right)^{r} e^{-\sum_{i=1}^n u_i} \prod_{j=1}^n u_j^{s_j-1}\, d\mathbf{u} \\
&= \frac{1}{(r+1)!\prod_{i=1}^n \Gamma(s_i)} \sum_{i=1}^n \int_{\mathbb{R}_+^n} \left(\sum_{i=1}^n u_i\right)^{r} e^{-\sum_{i=1}^n u_i} u_i^{s_i} \prod_{j\neq i} u_j^{s_j-1}\, d\mathbf{u}.
\end{aligned}$$

Using inductive assumption we have

$$\begin{aligned}
C(r+1,n,\mathbf{s}) &= \frac{1}{(r+1)!\prod_{i=1}^n \Gamma(s_i)} \sum_{i=1}^n r!\Gamma(s_i+1) \binom{\sum_{j\neq i} s_j + s_i + r}{r} \prod_{j\neq i} \Gamma(s_j) \\
&= \frac{1}{(r+1)\prod_{i=1}^n \Gamma(s_i)} \sum_{i=1}^n s_i \binom{\sum_{j=1}^n s_j + r}{r} \prod_{j=1}^n \Gamma(s_j) \\
&= \frac{1}{r+1} \binom{\sum_{i=1}^n s_i + r}{r} \sum_{i=1}^n s_i = \binom{\sum_{i=1}^n s_i + r}{r+1}.
\end{aligned}$$

□

REMARK 2.4. From Lemma 2.1 we get an interesting combinatorial identity:

$$\sum_{r_1+\cdots+r_n=r}\prod_{i=1}^{n}\binom{s_i+r_i-1}{r_i}=\binom{\sum_{i=1}^{n}s_i+r-1}{r}$$

where $r_1,\dots r_n$ are nonnegative integers.

From Lemma 2.1 follows

$$\int_{\mathbb{R}_+^n}\left(\sum_{i=1}^{n}u_i\right)^r e^{-\sum_{i=1}^{n}u_i}\prod_{i=1}^{n}u_i^{s_i-1}d\mathbf{u}=r!\prod_{i=1}^{n}\Gamma(s_i)\binom{\sum_{i=1}^{n}s_i+r-1}{r}. \tag{2.4}$$

The proof of this equality depends on the assumption $r\in\mathbb{N}\cup\{0\}$. In the following lemma we extend this equality on the domain of convergency of the involved integral.

The following standard notation for the extended beta function will be useful:

$$\mathrm{B}(s_1,s_2,\dots,s_n)=\frac{\prod_{i=1}^{n}\Gamma(s_i)}{\Gamma\left(\sum_{i=1}^{n}s_i\right)},\quad s_i>0,\quad i=1,\dots,n.$$

LEMMA 2.2. *Let* $\mathbf{s}\in\mathbb{R}_+^n$ *and* $r\in\mathbb{R}$ *be such that* $s_i>0,\ i=1,\dots,n$ *and* $r>-\sum_{i=1}^{n}s_i$. *Then*

$$\int_{\mathbb{R}_+^n}\left(\sum_{i=1}^{n}u_i\right)^r e^{-\sum_{i=1}^{n}u_i}\prod_{i=1}^{n}u_i^{s_i-1}d\mathbf{u}=\Gamma\left(\sum_{i=1}^{n}s_i+r\right)\mathrm{B}(s_1,s_2,\dots,s_n). \tag{2.5}$$

Proof. Denote the integral in (2.5) by $I_n(r;s_1,\dots,s_n)$. Using change of variables: $t_i=u_i,\ i\neq n-1,\ t_{n-1}=u_{n-1}+u_n$ we have:

$$\begin{aligned}
I_n(r;s_1,\dots,s_n)&=\int_{\mathbb{R}_+^n}\left(\sum_{i=1}^{n}u_i\right)^r e^{-\sum_{i=1}^{n}u_i}\prod_{i=1}^{n}u_i^{s_i-1}d\mathbf{u}\\
&=\int_{t_n\leqslant t_{n-1}}\left(\sum_{i=1}^{n-1}t_i\right)^r e^{-\sum_{i=1}^{n-1}t_i}\prod_{i\neq n-1}t_i^{s_i-1}\left(t_{n-1}-t_n\right)^{s_{n-1}-1}\ dt_n d\mathbf{t}\\
&=\int_{\mathbb{R}_+^{n-1}}\left(\sum_{i=1}^{n-1}t_i\right)^r e^{-\sum_{i=1}^{n-1}t_i}\prod_{i\neq n}t_i^{s_i-1}\\
&\qquad\qquad\int_0^{t_{n-1}}t_n^{s_n-1}\left(1-\frac{t_n}{t_{n-1}}\right)^{s_{n-1}-1}dt_n d\mathbf{t}\\
&=\int_{\mathbb{R}_+^{n-1}}\left(\sum_{i=1}^{n-1}t_i\right)^r e^{-\sum_{i=1}^{n-1}t_i}\prod_{i=1}^{n-2}t_i^{s_i-1}t_{n-1}^{s_{n-1}+s_n-1}\\
&\qquad\qquad\int_0^1 u^{s_n-1}\left(1-u\right)^{s_{n-1}-1}du d\mathbf{t}\\
&=\mathrm{B}(s_{n-1},s_n)I_{n-1}(r;s_1,\dots,s_{n-1}+s_n).
\end{aligned}$$

Using this recursion formula and the definition of the extended beta function it follows

$$\begin{aligned} I_n(r; s_1, \dots, s_n) &= \mathrm{B}(s_1, \dots, s_{n-1}, s_n) I_1(r; s_1 + \dots + s_{n-1} + s_n) \\ &= \mathrm{B}(s_1, \dots, s_{n-1}, s_n)\Gamma(r + \sum_{i=1}^{n} s_i). \end{aligned}$$

□

Defining the measure μ_r on $\mathbb{R}_+^n$ by

$$d\mu_r(\mathbf{u}) = \frac{1}{\Gamma(r + \sum_{i=1}^n s_i)\mathrm{B}(s_1, \dots, s_n)} \left(\sum_{i=1}^n u_i\right)^r e^{-\sum_{i=1}^n u_i} \prod_{i=1}^n u_i^{s_i - 1} d\mathbf{u},$$

Lemma 2.2 gives that μ_r is a probability measure on $\mathbb{R}_+^n$. Using the measure μ_r the generalized Whiteley mean $w_n^{[r,\mathbf{s}]}$ can be written as follows:

$$w_n^{[r,\mathbf{s}]}(\mathbf{x}) = \int_{\mathbb{R}_+^n} (\mathrm{A}(\mathbf{x}; \mathbf{u}))^r \, d\mu_r(\mathbf{u}),$$

where $\mathrm{A}(\mathbf{x}; \mathbf{u}) = \frac{\sum_{i=1}^n u_i x_i}{\sum_{i=1}^n u_i}$ is the arithmetic mean of an $\mathbf{x} \in \mathbb{R}_+^n$ with weight $\mathbf{u}$.

The measure μ_r is not suitable for applying some standard inequalities (for example power mean inequality) because it depends on r. We give an integral representation of generalized Whiteley mean analogous to Schur's representation (1.2) which is more suitable for such purposes.

LEMMA 2.3. *Let $\mathbf{s} \in \mathbb{R}_+^n$ and $r \in \mathbb{R}$ be such that $s_i > 0$, $i = 1, \dots, n$ and $r > -\sum_{i=1}^n s_i$. Then*

$$w_n^{[r,\mathbf{s}]}(\mathbf{x}) = \frac{1}{\mathrm{B}(s_1, \dots, s_n)} \int_{\Delta_{n-1}} \left(\sum_{i=1}^n x_i t_i\right)^r \prod_{i=1}^n t_i^{s_i - 1} \, dt_1 \dots dt_{n-1}, \tag{2.6}$$

where $\Delta_{n-1} = \{(t_1, \dots, t_{n-1}); t_i \geqslant 0, \ i = 1, \dots, n-1, \ \sum_{i=1}^{n-1} t_i \leqslant 1\}$ and $t_n = 1 - \sum_{i=1}^{n-1} t_i$.

Proof. Using change of variables $u_i = v_i^2$, $i = 1, \dots, n$ we have:

$$\int_{\mathbb{R}_+^n} \left(\sum_{i=1}^n x_i u_i\right)^r e^{-\sum_{i=1}^n u_i} \prod_{i=1}^n u_i^{s_i - 1} d\mathbf{u} = 2^n \int_{\mathbb{R}_+^n} \left(\sum_{i=1}^n x_i v_i^2\right)^r e^{-\sum_{i=1}^n v_i^2} \prod_{i=1}^n v_i^{2s_i - 1} d\mathbf{v}. \tag{2.7}$$

The next step is to transform (2.7) to polar coordinates. Set:

$$v_1 = t \prod_{i=1}^{n-1} \sin \vartheta_i, \quad v_k = t \cos \vartheta_{k-1} \prod_{i=k}^{n-1} \sin \vartheta_i, \quad k = 2, \dots, n-1, \quad v_n = t \cos \vartheta_{n-1},$$

so

$$d\mathbf{v} = t^{n-1} \prod_{k=2}^{n-1} \sin^{k-1} \vartheta_k dt d\vartheta_1 \dots d\vartheta_{n-1} = t^{n-1} dt d\mathbf{S}_{n-1} = t^{n-1} \sin^{n-2} \vartheta_{n-1} dt d\vartheta_{n-1} d\mathbf{S}_{n-2},$$

where $t \geqslant 0$, $\vartheta_i \in [0,\pi/2]$, $i=1,\dots,n-1$ and $d\mathbf{S}_{n-1}$, $d\mathbf{S}_{n-2}$ are induced Lebesgue measures on unit spheres $\mathbf{S}^{n-1} \subset \mathbb{R}^n$ and $\mathbf{S}^{n-2} \subset \mathbb{R}^{n-1}$, respectively. Also set:

$$\sigma_1 = \prod_{i=1}^{n-1} \sin\vartheta_i, \quad \sigma_k = \cos\vartheta_{k-1} \prod_{i=k}^{n-1} \sin\vartheta_i, \quad k=2,\dots,n-1, \quad \sigma_n = \cos\vartheta_{n-1}.$$

Notice that $\sigma = (\sigma_1,\dots,\sigma_n) \in \mathbf{S}^{n-1} \subset \mathbb{R}^n$. Using this and simplifying, (2.7) is equal to

$$\begin{aligned}
&2^n \int_0^\infty t^{2\left(r+\sum_{i=1}^n s_i-1\right)} e^{-t^2} t dt \\
&\quad \times \int_{\mathbf{S}_+^{n-1}} \left(\sum_{i=1}^n x_i\sigma_i^2\right)^r \prod_{i=1}^n \sigma_i^{2s_i-1} \prod_{k=2}^{n-1} \sin^{k-1}\vartheta_k d\vartheta_1 \dots d\vartheta_{n-1} \\
&\quad = 2^{n-1}\Gamma(r+\sum_{i=1}^n s_i) \cdot \int_{\mathbf{S}_+^{n-1}} \left(\sum_{i=1}^n x_i\sigma_i^2\right)^r \prod_{i=1}^n \sigma_i^{2s_i-1} \prod_{k=2}^{n-1} \sin^{k-1}\vartheta_k d\vartheta_1 \dots d\vartheta_{n-1} \\
&\quad = 2^{n-1}\Gamma(r+\sum_{i=1}^n s_i) \cdot \int_{\mathbf{S}_+^{n-1}} \left(\sum_{i=1}^n x_i\sigma_i^2\right)^r \prod_{i=1}^n \sigma_i^{2(s_i-1)} \prod_{i=1}^n \sigma_i \sin^{n-2}\vartheta_{n-1} d\vartheta_{n-1} d\mathbf{S}_{n-2}.
\end{aligned} \tag{2.8}$$

Define new variables by $t_i = \sigma_i^2,\ i=1,\dots,n-1$. Notice that

$$\frac{\partial t_i}{\partial \vartheta_j} = 2\sigma_i \frac{\partial \sigma_i}{\partial \vartheta_j},\ i=1,\dots,n-1,\ j=1,\dots,n-1.$$

Also set $\sigma_i = \sin\vartheta_{n-1}\overline{\sigma}_i,\ i=1,\dots,n-1$. Notice that $(\overline{\sigma}_1,\dots,\overline{\sigma}_{n-1}) \in \mathbf{S}^{n-2}$. The elementary properties of determinants give us the following sequence of equalities for the Jacobian:

$$\begin{aligned}
\frac{\partial(t_1,\dots,t_{n-1})}{\partial(\vartheta_1,\dots,\vartheta_{n-1})} &= 2^{n-1} \prod_{i=1}^{n-1} \sigma_i \frac{\partial(\sigma_1,\dots,\sigma_{n-1})}{\partial(\vartheta_1,\dots,\vartheta_{n-1})} \\
&= 2^{n-1} \prod_{i=1}^{n-1} \sigma_i \sin^{n-2}\vartheta_{n-1} \cos\vartheta_{n-1} \left.\frac{\partial(t\overline{\sigma}_1,\dots,t\overline{\sigma}_{n-1})}{\partial(\vartheta_1,\dots,\vartheta_{n-2},t)}\right|_{t=1} \\
&= 2^{n-1} \prod_{i=1}^{n} \sigma_i \sin^{n-2}\vartheta_{n-1} \left.\frac{\partial(t\overline{\sigma}_1,\dots,t\overline{\sigma}_{n-1})}{\partial(\vartheta_1,\dots,\vartheta_{n-2},t)}\right|_{t=1},
\end{aligned}$$

where the second equality follows by extracting $\sin\vartheta_{n-1}$ from the first $(n-2)-$ columns and $\cos\vartheta_{n-1}$ from the last column of the determinant $\frac{\partial(\sigma_1,\dots,\sigma_{n-1})}{\partial(\vartheta_1,\dots,\vartheta_{n-1})}$. It follows:

$$dt_1\dots dt_{n-1} = \frac{\partial(t_1,\dots,t_{n-1})}{\partial(\vartheta_1,\dots,\vartheta_{n-1})} d\vartheta_1\dots d\vartheta_{n-1} = 2^{n-1}\prod_{i=1}^n \sigma_i \sin^{n-2}\vartheta_{n-1} d\vartheta_{n-1} d\mathbf{S}_{n-2}.$$

Finally, denoting $t_n = \sigma_n^2$, (2.8) is equal to

$$\Gamma\left(r+\sum_{i=1}^{n} s_i\right)\int_{\Delta_{n-1}}\left(\sum_{i=1}^{n} x_i t_i\right)^r \prod_{i=1}^{n} t_i^{s_i-1} dt_1 \ldots dt_{n-1}$$

which obviously gives the equality (2.6). □

Defining the measure ν on Δ_{n-1} by

$$d\nu(\mathbf{u}) = \frac{1}{\mathrm{B}(s_1,\ldots,s_n)}\prod_{i=1}^{n} u_i^{s_i-1} d\mathbf{u},$$

Lemma 2.3 gives that ν is a probability measure on Δ_{n-1}. Using the measure ν the generalized Whiteley mean $w_n^{[r,\mathbf{s}]}$ can be written as follows:

$$w_n^{[r,\mathbf{s}]}(\mathbf{x}) = \int_{\Delta_{n-1}}\left(\sum_{i=1}^{n} x_i u_i\right)^r d\nu(\mathbf{u}). \tag{2.9}$$

Notice that the integral in this equality is defined for every $r \in \mathbb{R}$.

The measure ν is in [5] called Dirichlet measure. For the connection of Dirichlet measure and Stolarsky-Tobey mean see [17], where the Stolarsky-Tobey mean $E_{r,s}(\mathbf{x};\mathbf{s})$, for $r(s-r) \neq 0$, is (in our notations) defined by:

$$E_{r,s}(\mathbf{x},\mathbf{s}) = \left[w_n^{[(s-r)/r,\mathbf{s}]}(\mathbf{x}^r)\right]^{1/(s-r)},$$

where $\mathbf{x}^r = (x_1^r,\ldots,x_n^r)$.

3. Inequalities

THEOREM 3.1. *If $r_1 < r_2$, $r_1, r_2 \in \mathbb{R}$, then*

$$\left(w_n^{[r_1,\mathbf{s}]}(\mathbf{x})\right)^{\frac{1}{r_1}} \leqslant \left(w_n^{[r_2,\mathbf{s}]}(\mathbf{x})\right)^{\frac{1}{r_2}} \tag{3.1}$$

where the cases $r_1 = 0$ or $r_2 = 0$ are treated in a standard way using limiting process.

If $r \in \mathbb{R}$, then

$$\left(w_n^{[r,\mathbf{s}]}(\mathbf{x})\right)^2 \leqslant w_n^{[r-1,\mathbf{s}]}(\mathbf{x}) w_n^{[r+1,\mathbf{s}]}(\mathbf{x}). \tag{3.2}$$

Proof. By (2.9) we can write (3.1) as

$$\left(\int_{\Delta_{n-1}}\left(\sum_{i=1}^{n} x_i u_i\right)^{r_1} d\nu(\mathbf{u})\right)^{\frac{1}{r_1}} \leqslant \left(\int_{\Delta_{n-1}}\left(\sum_{i=1}^{n} x_i u_i\right)^{r_2} d\nu(\mathbf{u})\right)^{\frac{1}{r_2}}$$

what is true by the integral power mean inequality.

Similarly, by (2.9) we can write (3.2) as

$$\left(\int_{\Delta_{n-1}}\left(\sum_{i=1}^{n}x_iu_i\right)\left(\sum_{i=1}^{n}x_iu_i\right)^{r-1}d\nu(\mathbf{u})\right)^2$$
$$\leqslant\left(\int_{\Delta_{n-1}}\left(\sum_{i=1}^{n}x_iu_i\right)^{r-1}d\nu(\mathbf{u})\right)\left(\int_{\Delta_{n-1}}\left(\sum_{i=1}^{n}x_iu_i\right)^{2}\left(\sum_{i=1}^{n}x_iu_i\right)^{r-1}d\nu(\mathbf{u})\right)$$

what is true by integral Cauchy's inequality as it can be easily seen if we denote $d\overline{\nu}(\mathbf{u})=\left(\sum_{i=1}^{n}x_iu_i\right)^{r-1}d\nu(\mathbf{u})$. □

REMARK 3.1. For $s_1=\ldots=s_n=s=1$, inequality (3.2) becomes inequality (1.1).

THEOREM 3.2. *Let* p_1,p_2,q_1,q_2 *be real numbers such that* $p_1\leqslant q_1$, $p_2\leqslant q_2$, $p_1<p_2$, $q_1<q_2$. *Then the following inequality is valid*

$$\left(\frac{w_n^{[p_2,\mathbf{s}]}(\mathbf{x})}{w_n^{[p_1,\mathbf{s}]}(\mathbf{x})}\right)^{\frac{1}{p_2-p_1}}\leqslant\left(\frac{w_n^{[q_2,\mathbf{s}]}(\mathbf{x})}{w_n^{[q_1,\mathbf{s}]}(\mathbf{x})}\right)^{\frac{1}{q_2-q_1}}. \tag{3.3}$$

Proof. Inequality (3.3) can be written by (2.9) as

$$\left(\frac{\int_{\Delta_{n-1}}\left(\sum_{i=1}^{n}x_iu_i\right)^{p_2}d\nu(\mathbf{u})}{\int_{\Delta_{n-1}}\left(\sum_{i=1}^{n}x_iu_i\right)^{p_1}d\nu(\mathbf{u})}\right)^{\frac{1}{p_2-p_1}}\leqslant\left(\frac{\int_{\Delta_{n-1}}\left(\sum_{i=1}^{n}x_iu_i\right)^{q_2}d\nu(\mathbf{u})}{\int_{\Delta_{n-1}}\left(\sum_{i=1}^{n}x_iu_i\right)^{q_1}d\nu(\mathbf{u})}\right)^{\frac{1}{q_2-q_1}}$$

what is true by the well-known inequality between Gini's integral means. □

REMARK 3.2. Set $p_1=r$, $p_2=r+1$, $q_1=s$, $q_2=s+1$. We have for $r\leqslant s$

$$\frac{w_n^{[r+1,\mathbf{s}]}(\mathbf{x})}{w_n^{[r,\mathbf{s}]}(\mathbf{x})}\leqslant\frac{w_n^{[s+1,\mathbf{s}]}(\mathbf{x})}{w_n^{[s,\mathbf{s}]}(\mathbf{x})}$$

i.e.

$$w_n^{[r,\mathbf{s}]}(\mathbf{x})w_n^{[s+1,\mathbf{s}]}(\mathbf{x})\geqslant w_n^{[r+1,\mathbf{s}]}(\mathbf{x})w_n^{[s,\mathbf{s}]}(\mathbf{x}).$$

Further let $r\to r-s-1$, $s\to r+s$. We get

$$w_n^{[r-s-1,\mathbf{s}]}(\mathbf{x})w_n^{[r+s+1,\mathbf{s}]}(\mathbf{x})\geqslant w_n^{[r-s,\mathbf{s}]}(\mathbf{x})w_n^{[r+s,\mathbf{s}]}(\mathbf{x})\qquad s\geqslant -1/2.$$

This is a generalization of (1.4).

Of course, using integral representation (2.9) we can give some further extensions of Cauchy's inequality.

THEOREM 3.3. *The following inequality is valid for positive n-tuple* $\mathbf{x}$*:*

$$\left|w_n^{[a_0+a_i+a_j,\mathbf{s}]}(\mathbf{x})\right|_{i,j=1}^{m}\geqslant 0 \tag{3.4}$$

where $|A_{ij}|_{i,j=1}^{m}\geqslant 0$ *denotes the determinant of order* m *and where* $a_0,\ a_i,\ a_j\in\mathbb{R}$, $i,j=1,\ldots m$.

Proof. By (2.9), inequality (3.4) is equivalent to

$$\left| \int_{\Delta_{n-1}} \left(\sum_{i=1}^{n} x_i u_i\right)^{a_0} \left(\sum_{i=1}^{n} x_i u_i\right)^{a_i} \left(\sum_{i=1}^{n} x_i u_i\right)^{a_j} d\nu(\mathbf{u}) \right|_{i,j=1}^{m} \geqslant 0,$$

what is true by the well-known Gram's inequality

$$|\langle f_i, f_j \rangle|_{i,j=1}^{m} \geqslant 0$$

where the scalar product is defined as

$$\langle f_i, f_j \rangle = \int_{\Delta_{n-1}} f_i(\mathbf{u}) f_j(\mathbf{u}) d\overline{\nu}(\mathbf{u}).$$

It can be easily seen if we denote

$$d\overline{\nu}(\mathbf{u}) = \left(\sum_{i=1}^{n} x_i u_i\right)^{a_0} d\nu(\mathbf{u})$$
$$f_i(\mathbf{u}) = \left(\sum_{i=1}^{n} x_i u_i\right)^{a_i} \qquad i = 1, ..., m.$$

□

Inequality (3.4) for $m = 2$, $a_0 = 0$, $a_1 = (r - r_1)/2$, $a_2 = (r + r_1)/2$, $r, r_1 \in \mathbb{R}$, implies

$$\left(w_n^{[r,\mathbf{s}]}\right)^2 \leqslant w_n^{[r-r_1,\mathbf{s}]} w_n^{[r+r_1,\mathbf{s}]}. \tag{3.5}$$

This is again a generalization of (1.1).

THEOREM 3.4. *Let* $\mathbf{x}$ *be a positive n-tuple and let* p_i, $i = 1, ..., m$, *be positive numbers such that* $\sum_{i=1}^{m} \frac{1}{p_i} = 1$. *Further, let* a_i $(i = 0, 1, ..., m)$ *be real numbers and* $a = \sum_{i=0}^{m} a_i$. *Then the following inequality is valid*

$$w_n^{[a,\mathbf{s}]}(\mathbf{x}) \leqslant \prod_{i=1}^{m} \left[w_n^{[a_0 + a_i p_i, \mathbf{s}]}(\mathbf{x}) \right]^{\frac{1}{p_i}}. \tag{3.6}$$

Proof. It is a simple consequence of Hölder's inequality since (3.6) is equivalent to

$$\int_{\Delta_{n-1}} \prod_{i=1}^{m} \left(\sum_{i=1}^{n} x_i u_i\right)^{a_i} \left(\sum_{i=1}^{n} x_i u_i\right)^{a_0} d\nu(\mathbf{u})$$
$$\leqslant \prod_{i=1}^{m} \left[\int_{\Delta_{n-1}} \left(\sum_{i=1}^{n} x_i u_i\right)^{a_i p_i} \cdot \left(\sum_{i=1}^{n} x_i u_i\right)^{a_0} d\nu(\mathbf{u}) \right]^{\frac{1}{p_i}}.$$

It can be easily seen if we denote

$$d\overline{\nu}(\mathbf{u}) = \left(\sum_{i=1}^{n} x_i u_i\right)^{a_0} d\nu(\mathbf{u}).$$

□

Of course, generalization (3.5) of (1.1) follows from (3.6) for $m=2$, $p_1=p_2=2$, $a_0 = 0$, $a_1 = (r-r_1)/2$, $a_2 = (r+r_1)/2$, $r, r_1 \in \mathbb{R}$.

THEOREM 3.5. *Let* $\mathbf{x}$ *be a positive n-tuple. Let* α, β, a, b *be real numbers such that* $ab > 0$. *Then the following inequality is valid*

$$\begin{aligned} &w_n^{[\alpha+a,\mathbf{s}]}(\mathbf{x})w_n^{[\beta+b,\mathbf{s}]}(\mathbf{x}) + w_n^{[\alpha+b,\mathbf{s}]}(\mathbf{x})w_n^{[\beta+a,\mathbf{s}]}(\mathbf{x}) \\ &\quad \leqslant w_n^{[\alpha,\mathbf{s}]}(\mathbf{x})w_n^{[\beta+a+b,\mathbf{s}]}(\mathbf{x}) + w_n^{[\beta,\mathbf{s}]}(\mathbf{x})w_n^{[\alpha+a+b,\mathbf{s}]}(\mathbf{x}). \end{aligned} \tag{3.7}$$

The sign of inequality in (3.7) is reversed for $ab < 0$.

Proof. The idea is to follow the proof of the Chebyshev inequality. For $ab > 0$ and fixed $\mathbf{x} \in \mathbb{R}_+^n$ and every $\mathbf{u}, \mathbf{v} \in \mathbb{R}_+^n$ we obviously have

$$\left(\left(\sum_{i=1}^{n} x_i u_i\right)^a - \left(\sum_{i=1}^{n} x_i v_i\right)^a\right)\left(\left(\sum_{i=1}^{n} x_i u_i\right)^b - \left(\sum_{i=1}^{n} x_i v_i\right)^b\right) \geqslant 0,$$

from which by multiplying with $\left(\sum_{i=1}^{n} x_i u_i\right)^{\alpha}\left(\sum_{i=1}^{n} x_i v_i\right)^{\beta}$ and integrating over $\Delta_{n-1} \times \Delta_{n-1}$ with respect to $d\nu(\mathbf{u})d\nu(\mathbf{v})$ (3.7) follows. □

From (3.7) (and its reversed form) for $a = -1$, $b = s + 1/2$, $\alpha = r - s$, $\beta = r + 1/2$ it follows

$$w_n^{[r-s-1,\mathbf{s}]}(\mathbf{x})w_n^{[r+s+1,\mathbf{s}]}(\mathbf{x}) \leqslant w_n^{[r-s,\mathbf{s}]}(\mathbf{x})w_n^{[r+s,\mathbf{s}]}(\mathbf{x})$$

for $s < -1/2$, and the reversed inequality for $s > -1/2$. This is a generalization of inequality (1.4).

Also, from (3.7) for $a = r - r_1$, $b = r + r_1$, $\alpha = \beta = 0$ we have

$$w_n^{[r-r_1,\mathbf{s}]}(\mathbf{x})w_n^{[r+r_1,\mathbf{s}]}(\mathbf{x}) \leqslant w_n^{[2r,\mathbf{s}]}(\mathbf{x})$$

assuming $|r_1| < |r|$, and the reversed inequality assuming $|r_1| > |r|$. Using also (3.5) we obtain

$$\left(w_n^{[r,\mathbf{s}]}(\mathbf{x})\right)^2 \leqslant w_n^{[r-r_1,\mathbf{s}]}(\mathbf{x})w_n^{[r+r_1,\mathbf{s}]}(\mathbf{x}) \leqslant w_n^{[2r,\mathbf{s}]}(\mathbf{x}), \quad |r_1| < |r|,$$

and

$$\left(w_n^{[r,\mathbf{s}]}(\mathbf{x})\right)^2 \leqslant w_n^{[2r,\mathbf{s}]}(\mathbf{x}) \leqslant w_n^{[r-r_1,\mathbf{s}]}(\mathbf{x})w_n^{[r+r_1,\mathbf{s}]}(\mathbf{x}), \quad |r_1| > |r|,$$

where the first inequality in the last sequence of inequalities follows again from (3.7) for $a = b = r$, $\alpha = \beta = 0$ and holds for every $r \in \mathbb{R}$.

THEOREM 3.6. *Let* $\mathbf{x}$ *be a positive n-tuple and let* r,s,t *be real numbers such that* $r>s>t>0$. *Then the following inequality is valid*

$$\left(w_n^{[s,\mathbf{s}]}(\mathbf{x})\right)^{r-t} \leqslant \left(w_n^{[r,\mathbf{s}]}(\mathbf{x})\right)^{s-t} \cdot \left(w_n^{[t,\mathbf{s}]}(\mathbf{x})\right)^{r-s}. \tag{3.8}$$

Proof. By (2.9) inequality (3.8) is true by the integral analogue of the Liapunoff inequality. □

REMARK 3.3. Inequality (3.8) is equivalent to

$$w_n^{[s,\mathbf{s}]}(\mathbf{x}) \leqslant \left(w_n^{[r,\mathbf{s}]}(\mathbf{x})\right)^{\frac{s-t}{r-t}} \cdot \left(w_n^{[t,\mathbf{s}]}(\mathbf{x})\right)^{\frac{r-s}{r-t}}, \qquad r>s>t>0. \tag{3.9}$$

From (3.9) by AG-inequality we get

$$w_n^{[s,\mathbf{s}]}(\mathbf{x}) \leqslant \frac{s-t}{r-t} \cdot w_n^{[r,\mathbf{s}]}(\mathbf{x}) + \frac{r-s}{r-t} \cdot w_n^{[t,\mathbf{s}]}(\mathbf{x}), \qquad r>s>t>0, \tag{3.10}$$

or

$$w_n^{[s,\mathbf{s}]}(\mathbf{x}) - w_n^{[t,\mathbf{s}]}(\mathbf{x}) \leqslant \frac{s-t}{r-s}\left(w_n^{[r,\mathbf{s}]}(\mathbf{x}) - w_n^{[s,\mathbf{s}]}(\mathbf{x})\right), \qquad r>s>t>0.$$

THEOREM 3.7. *Let* $\mathbf{x}$ *be a positive n-tuple. Let* a,b,p,q *be real numbers such that* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$. *Then the following inequality is valid*

$$(M-m)\, w_n^{[ap,\mathbf{s}]}(\mathbf{x}) + (mM^p - Mm^p)\, w_n^{[bq,\mathbf{s}]}(\mathbf{x}) \leqslant (M^p - m^p)\, w_n^{[a+b,\mathbf{s}]}(\mathbf{x}) \tag{3.11}$$

where m *and* M *denote minimum and maximum values of* $x_i^{\frac{ap-bq}{p}}$ $(i=1,\dots,n)$.

Proof. The reversed Hölder's inequality for positive functional

$$A(f) = \int_{\Delta_{n-1}} f(\mathbf{u})d\nu(\mathbf{u})$$

(see [2] or [15, p. 136]) says that if $0<m\leqslant f(\mathbf{u})g(\mathbf{u})^{-\frac{q}{p}} \leqslant M$, then the following inequality is valid

$$(M-m)\,A(f^p) + (mM^p - Mm^p)\,A(g^q) \leqslant (M^p - m^p)\,A(fg). \tag{3.12}$$

Inequality (3.11) follows if we set the functions $f(\mathbf{u}) = \left(\sum_{i=1}^n x_i u_i\right)^a$, $g(\mathbf{u}) = \left(\sum_{i=1}^n x_i u_i\right)^b$ in (3.12). □

REMARK 3.4. If $m<M$, then the inequality (3.11) is equivalent to

$$w_n^{[a+b,\mathbf{s}]}(\mathbf{x}) \geqslant \frac{M-m}{M^p - m^p} \cdot w_n^{[ap,\mathbf{s}]}(\mathbf{x}) + \frac{mM^p - Mm^p}{M^p - m^p} \cdot w_n^{[bq,\mathbf{s}]}(\mathbf{x}), \;\; \frac{1}{p}+\frac{1}{q}=1.$$

THEOREM 3.8. *Let* $\mathbf{x}$ *be a positive n-tuple. Let* a,b,p,q *be real numbers such that* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$. *Then the following inequality is valid*

$$w_n^{[a+b,\mathbf{s}]}(\mathbf{x}) \geqslant \lambda \cdot \left(w_n^{[ap,\mathbf{s}]}(\mathbf{x})\right)^{\frac{1}{p}} \cdot \left(w_n^{[bq,\mathbf{s}]}(\mathbf{x})\right)^{\frac{1}{q}} \tag{3.13}$$

where

$$\lambda = p^{\frac{1}{p}} q^{\frac{1}{q}} (M-m)^{\frac{1}{p}} \left(mM^p - Mm^p\right)^{\frac{1}{q}} \frac{1}{M^p - m^p}$$

and m *and* M *denote minimum and maximum values of* $x_i^{\frac{ap-bq}{p}}$, $i=1,...,n$.

Proof. Using the appropriate form of the reversed Hölders's inequality (see [2] or [15, p. 136]) the proof is analogous to the proof of the previous theorem. □

REMARK 3.5. As it holds by Hölder's inequality that

$$w_n^{[a+b,\mathbf{s}]}(\mathbf{x}) \leqslant \left(w_n^{[ap,\mathbf{s}]}(\mathbf{x})\right)^{\frac{1}{p}} \cdot \left(w_n^{[bq,\mathbf{s}]}(\mathbf{x})\right)^{\frac{1}{q}}$$

we now have

$$\lambda \cdot \left(w_n^{[ap,\mathbf{s}]}(\mathbf{x})\right)^{\frac{1}{p}} \cdot \left(w_n^{[bq,\mathbf{s}]}(\mathbf{x})\right)^{\frac{1}{q}} \leqslant w_n^{[a+b,\mathbf{s}]}(\mathbf{x}) \leqslant \left(w_n^{[ap,\mathbf{s}]}(\mathbf{x})\right)^{\frac{1}{p}} \cdot \left(w_n^{[bq,\mathbf{s}]}(\mathbf{x})\right)^{\frac{1}{q}}.$$

REMARK 3.6. In the special case $p=q=2$ we get

$$\frac{4mM}{(m+M)^2} \cdot w_n^{[2a,\mathbf{s}]}(\mathbf{x}) \cdot w_n^{[2b,\mathbf{s}]}(\mathbf{x}) \leqslant \left(w_n^{[a+b,\mathbf{s}]}(\mathbf{x})\right)^2 \leqslant w_n^{[2a,\mathbf{s}]}(\mathbf{x}) \cdot w_n^{[2b,\mathbf{s}]}(\mathbf{x})$$

where m and M are minimum and maximum values of x_i^{a-b} $(i=1,...,n)$.

Setting $a=\frac{r+1}{2}$, $b=\frac{r-1}{2}$ we get the generalization of (3.2)

$$\frac{4mM}{(m+M)^2} \cdot w_n^{[r+1,\mathbf{s}]}(\mathbf{x}) \cdot w_n^{[r-1,\mathbf{s}]}(\mathbf{x}) \leqslant \left(w_n^{[r,\mathbf{s}]}(\mathbf{x})\right)^2 \leqslant w_n^{[r+1,\mathbf{s}]}(\mathbf{x}) \cdot w_n^{[r-1,\mathbf{s}]}(\mathbf{x})$$

where m and M are minimum and maximum of x_i $(i=1,...,n)$.

4. Remarks on the Schur convexity of the generalized complete symmetric functions

For the basic definitions and properties of the Schur convex functions the interested reader can consult Marshall-Olkin's book [11].

It is obvious from the Minkowski inequality and integral representations of $W_n^{[r,\mathbf{s}]}$ that for $r \geqslant 1$

$$\left[W_n^{[r,\mathbf{s}]}(\mathbf{x}+\mathbf{y})\right]^{\frac{1}{r}} \leqslant \left[W_n^{[r,\mathbf{s}]}(\mathbf{x})\right]^{\frac{1}{r}} + \left[W_n^{[r,\mathbf{s}]}(\mathbf{y})\right]^{\frac{1}{r}}. \tag{4.1}$$

Inequality (4.1) is reversed for $r<1$. Because $\left[W_n^{[r,\mathbf{s}]}(\mathbf{x})\right]^{\frac{1}{r}}$ is homogeneous of degree one in $\mathbf{x}$, inequality (4.1) says that this function is convex for $r \geqslant 1$ (concave for

$r<1$). It follows that $W_n^{[r,\mathbf{s}]}(\mathbf{x})$ is convex function for $r \geqslant 1$ or $r<0$, and concave for $0<r<1$. For $s_1=\ldots=s_n=s$, $W_n^{[r,s]}(\mathbf{x})$ is symmetric, so it is Schur convex for $r \geqslant 1$ or $r<0$ and Schur concave for $0<r<1$. This is a result from [18] for $r \in \mathbb{N}$. Guan in [9] and [10] proved this for $W_n^{[r,1]}$ and $r \in \mathbb{N}$ by different method.

Guan in the same papers has proved that $W_n^{[r,1]}(\mathbf{x})/W_n^{[r-1,1]}(\mathbf{x})$ is Schur convex for $r \in \mathbb{N}$. We notice that from Dresher's inequality (see [1]) we have for $1 \leqslant r \leqslant 2$

$$\frac{W_n^{[r,\mathbf{s}]}(\mathbf{x}+\mathbf{y})}{W_n^{[r-1,\mathbf{s}]}(\mathbf{x}+\mathbf{y})} \leqslant \frac{W_n^{[r,\mathbf{s}]}(\mathbf{x})}{W_n^{[r-1,\mathbf{s}]}(\mathbf{x})} + \frac{W_n^{[r,\mathbf{s}]}(\mathbf{y})}{W_n^{[r-1,\mathbf{s}]}(\mathbf{y})}. \tag{4.2}$$

Arguing as above we conclude that $W_n^{[r,s]}(\mathbf{x})/W_n^{[r-1,s]}(\mathbf{x})$ is Schur convex for $1 \leqslant r \leqslant 2$.

Finally, following Guan (see [9] and [10]), we give a sketch of the proof that $\Phi(\mathbf{x}) = W_n^{[r,s]}(\mathbf{x})/W_n^{[r-1,s]}(\mathbf{x})$ is Schur convex for $r \in \mathbb{N}$. It is enough to prove that

$$(x_i - x_j)\left(\frac{\partial \Phi}{\partial x_i} - \frac{\partial \Phi}{\partial x_j}\right) \geqslant 0 \tag{4.3}$$

for $i \neq j$ and $\mathbf{x} \in \mathbb{R}_+^n$ (see [11]). Easy calculation gives:

$$\frac{\partial \Phi}{\partial x_i} - \frac{\partial \Phi}{\partial x_j} = \frac{1}{\left(W_n^{[r-1,s]}\right)^2} \times \left[W_n^{[r-1,s]}\left(\frac{\partial W_n^{[r,s]}}{\partial x_i} - \frac{\partial W_n^{[r,s]}}{\partial x_j}\right) - W_n^{[r,s]}\left(\frac{\partial W_n^{[r-1,s]}}{\partial x_i} - \frac{\partial W_n^{[r-1,s]}}{\partial x_j}\right)\right]. \tag{4.4}$$

Repeated use of the identity

$$\frac{\partial W_n^{[r,s]}}{\partial x_j} = sW_n^{[r-1,s]} + x_j \frac{\partial W_n^{[r-1,s]}}{\partial x_j}, \quad j=1,\ldots,n,$$

which is deduced in [14] for $r \in \mathbb{N}$ (this can be also obtained for real $r \geqslant 1$ from integral representation of $W_n^{[r,s]}$ and integration by parts), gives

$$\frac{\partial W_n^{[r,s]}}{\partial x_j} = s\sum_{k=0}^{r-1} x_j^k W_n^{[r-k-1,s]}, \quad j=1,\ldots,n.$$

Using this in (4.4) and rearranging we obtain

$$\frac{\partial \Phi}{\partial x_i} - \frac{\partial \Phi}{\partial x_j} = \frac{1}{\left(W_n^{[r-1,s]}\right)^2} \times \left[\sum_{k=0}^{r-2}\left(W_n^{[r-1,s]}W_n^{[r-1-k,s]} - W_n^{[r,s]}W_n^{[r-2-k,s]}\right)\left(x_i^k - x_j^k\right) + W_n^{[r-1,s]}\left(x_i^{r-1} - x_j^{r-1}\right)\right]. \tag{4.5}$$

Having in mind the inequality

$$\left(W_n^{[r,s]}\right)^2 \geqslant W_n^{[r-1,s]}W_n^{[r+1,s]}$$

proven by Menon in [14], (4.5) implies (4.3).

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