Mathematical Inequalities & Applications

Volume 12, Number 2 (2009), 377–390

EQUALITY CONDITIONS FOR NORM INEQUALITIES IN REPRODUCING KERNEL HILBERT SPACES

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Dedicated to the memory of Professor Nobuyuki Suita

(*Communicated by S. Saitoh*)

Abstract. We consider norm inequalities arising from nonlinear maps between reproducing kernel Hilbert spaces. In many cases we know that equality for such inequalities occurs only for the reproducing kernels. To investigate this phenomenon we introduce a new class of RKHSs and give fairly general equality conditions for norm inequalities.

1. Introduction

In 1965 Lebedev and Milin [8] found the following inequality: If $f(z)$ is an analytic function in the unit disk $\Delta = \{|z| < 1\}$ with the expansions $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $e^{f(z)} = \sum_{n=0}^{\infty} b_n z^n$, then

$$\sum_{n=0}^{\infty} |b_n|^2 \leqslant \exp \sum_{n=1}^{\infty} n|a_n|^2, \tag{1}$$

where equality occurs if and only if there exists $\rho \in \Delta$ with $a_n = \rho^n/n$ for all n. This is a prototype of the inequalities treated in this paper, that is, we can rewrite the above inequality as

$$\|e^f\|_{H^2}^2 \leqslant e^{\|f\|_D^2},$$

where $\|f\|_D = (1/\pi \iint_\Delta |f'(z)|^2 dxdy)^{1/2}$ is the Dirichlet norm, and $\|e^f\|_{H^2} = (\sum_{n=0}^{\infty} |b_n|^2)^{1/2}$ is the Hardy H^2 norm of the function e^f. Also we remark that in this case we have the identity

$$k_{H^2}(z,w) = e^{k_D(z,w)} = 1/(1 - z\overline{w}), \ z, w \in \Delta,$$

where k_{H^2} and k_D are the kernel functions for the Hardy H^2 space and the Dirichlet space D on Δ normalized by $f(0) = 0$, respectively. Moreover, the above equality condition is equivalent to $f(z) = k_D(z,\rho)$ for some $\rho \in \Delta$. Thus it is obvious that there exists a deep connection between the Lebedev-Milin inequality and reproducing kernels. For reproducing kernel Hilbert spaces (RKHSs), the reader is referred to [1, 16].

Mathematics subject classification (2000): 30D50, 30C85.

Keywords and phrases: reproducing kernel, Hilbert space, norm inequality, tensor product.

Paper MIA-12-29

Let $\varphi(z) = \sum_{n=1}^{\infty} c_n z^n$ and $\psi(z) = \sum_{n=1}^{\infty} p_n z^n$ be entire functions such that 1) $p_n \geqslant 0$ for all n and 2) $c_n = 0$ whenever $p_n = 0$. If H_K is a complex RKHS on a set E with the reproducing kernel K and the norm $\|\cdot\|_K$, then from the general theory of RKHSs there exists a unique RKHS $H_{\psi(K)}$ on E with the reproducing kernel $\psi(K)$. Putting $\varphi_\psi(z) = \sum_{p_n>0} \frac{|c_n|^2}{p_n} z^n$, we have the following norm inequality which generalizes the Lebedev-Milin inequality (1) [2, 17]: For all $f \in H_K$,

$$\|\varphi(f)\|_{\psi(K)}^2 \leqslant \varphi_\psi(\|f\|_K^2). \tag{2}$$

In the special case where $\varphi(z) = \psi(z)$, it is easy to see [2] that if $f = K_q (= K(\cdot, q))$ for some $q \in E$, then equality holds in (2). Although the converse of this does not hold in general [13], by studying various special but important RKHSs, there are many papers [2, 3, 4, 5, 9, 10, 11, 12, 14, 15, 20] asserting that equality occurs in (2) if and only if $f = K_q$ for some $q \in E$. Unfortunately, in order to investigate the condition for equality, all these papers relied on case-by-case arguments. The aim of this paper is to obtain more general and satisfactory theory of equality conditions for such norm inequalities. To this end, we shall introduce a class of RKHSs called "algebra-dense" and study relations between equality conditions and $\mathbb{C}$-algebra homomorphisms. We hope that our results will yield a unified approach for solving the equality problems as treated in the papers cited above.

The paper is organized as follows. In Section 2 we define a tensor product of RKHSs on E to be *regular* or *weakly regular* in connection with equality conditions. We introduce the class of RKHSs called *algebra-dense* and study algebra-dense RKHSs which is "maximal". In Section 3 we study equality conditions for norm inequalities of nonlinear maps by using results obtained in Section 2. In Sections 4 and 5 we consider more concrete cases with special algebras such as polynomial rings. In Section 6 as an application we prove that some typical RKHSs on compact bordered Riemann surfaces are algebra-dense and maximal.

The author would like to thank Professor Saburou Saitoh for his encouragement and valuable comments on this paper.

2. Main results

Let m be a fixed integer greater than or equal to two. Let H_j $(j = 1, 2, \ldots, m)$ be a complex RKHS on the set E with the reproducing kernel $K_x^{(j)}$ at $x \in E$. Then the Hilbert tensor product $H = \otimes_{j=1}^m H_j$ is a RKHS on $E^m = \prod_{j=1}^m E$. (We note that in [1, p. 358] the term "direct product" is used instead of "tensor product".) Denote by $(H)_0$ the subspace of H defined by $\{f \in H : f|_{E_d^m} = 0\}$, where E_d^m is the diagonal $\{(x, x, \ldots, x) : x \in E\}$ of the set E^m. For $f, g \in H$ define an equivalence relation "$\sim$" by $f \sim g$ if and only if $f|_{E_d^m} = g|_{E_d^m}$. An element $\phi \in H$ is said to be *extremal* if $\phi \in (H)_0^\perp$. Thus, ϕ is extremal if and only if $f \sim g$ implies $\langle f, \phi \rangle = \langle g, \phi \rangle$.

REMARK 2.1. The definition of the extremality above is closely related to equality conditions of norm inequalities for the tensor product. As is well known, if H' denotes the unique RKHS with the kernel functions $\prod_{j=1}^m K_x^{(j)}$, $x \in E$, then H' consists of functions on E induced from the restrictions of functions in H to the diagonal E_d^m. For $\phi_j \in H_j$ $(j = 1, 2 \ldots, m)$, we have

$$\|\phi_1 \cdots \phi_m\|_{H'} \leqslant \|\phi_1 \otimes \cdots \otimes \phi_m\|_H = \|\phi_1\|_{H_1} \cdots \|\phi_m\|_{H_m},$$

where equality occurs if and only if $\otimes_{j=1}^m \phi_j$ is extremal (cf. Section 3).

DEFINITION 2.2. Let H_j $(j = 1, \ldots, m)$ be RKHSs on E. Then the tensor product $H = \otimes_j H_j$ is called *regular* (cf. [13]) if for every nonzero extremal $\otimes_{j=1}^m \phi_j \in H$, there exist a point $q \in E$ and constants $c_j \in \mathbb{C}$ such that $\phi_j = c_j K_q^{(j)}$ $(j = 1, 2, \ldots, m)$. Also, H is called *weakly regular* if for every nonzero extremal $\otimes_{j=1}^m \phi_j \in H$, there exists a point $q \in E$ such that for each j $(j = 1, 2, \ldots, m)$, either of the following holds:

(i) q is a common zero of the functions in H_j, or

(ii) $\phi_j = c_j K_q^{(j)}$ for some constant $c_j \in \mathbb{C}$.

For later reference, we shall call Case $\mathscr{I}$(i) above the *exceptional case*.

In what follows, we always assume that R denotes a $\mathbb{C}$-subalgebra of $\mathbb{C}^E$, the algebra of complex-valued functions on E. The existence of the identity of R is not assumed. For a complex subspace H of $\mathbb{C}^E$, let $R^{-1}H$ denote the subspace of H defined by $\{f \in H \colon rf \in H \text{ for all } r \in R\}$.

LEMMA 2.3. *Assume that* $\phi = \otimes_{j=1}^m \phi_j \in \otimes_{j=1}^m H_j$ *is nonzero extremal. If each* $R^{-1}H_j$ *is dense in* H_j $(j = 1, 2, \ldots, m)$, *then there exists a unique* $\mathbb{C}$*-algebra homomorphism* $\Lambda_\phi \colon R \to \mathbb{C}$ *satisfying, for every* $f \in R$ *and for every* $u \in R^{-1}H_j$ $(j = 1, 2, \ldots, m)$,

$$\langle f u, \phi_j \rangle = \Lambda_\phi(f) \langle u, \phi_j \rangle . \tag{3}$$

Proof. Recall that the inner product of the tensor product satisfies

$$\langle \otimes_{j=1}^m \varphi_j, \otimes_{j=1}^m \psi_j \rangle = \prod_{j=1}^m \langle \varphi_j, \psi_j \rangle$$

for any $\otimes_{j=1}^m \varphi_j, \otimes_{j=1}^m \psi_j \in H$. Hence, for $f_j,\ g_j \in H_j$ $(j = 1, 2, \ldots, m)$ we have

$$\prod_{j=1}^m f_j = \prod_{j=1}^m g_j \text{ on } E \implies \prod_{j=1}^m \langle f_j, \phi_j \rangle = \prod_{j=1}^m \langle g_j, \phi_j \rangle , \tag{4}$$

since ϕ is extremal and $\otimes_{j=1}^m f_j \sim \otimes_{j=1}^m g_j$.

Since $R^{-1}H_j$ is dense in H_j and $\phi_j \neq 0$, there exists an element $u_j \in R^{-1}H_j$ with $\langle u_j, \phi_j \rangle \neq 0$. Fixing such an element u_j for each j, we define Λ_ϕ for $f \in R$ by

$$\Lambda_\phi(f) = \frac{\langle f u_j, \phi_j \rangle}{\langle u_j, \phi_j \rangle}. \tag{5}$$

Now we show that this definition is well-defined, that is, Λ_ϕ is determined independently of the choice of j and u_j. For $f \in R$ set $f_k,\ g_k \in H_k$ $(k = 1, 2, \ldots, m)$ by

$$f_k = \begin{cases} f u_i & (k = i) \\ u_k & (k \neq i) \end{cases}, \quad g_k = \begin{cases} f u_j & (k = j) \\ u_k & (k \neq j) \end{cases}.$$

From (4) we have

$$\langle f u_i, \phi_i \rangle \langle u_j, \phi_j \rangle = \langle u_i, \phi_i \rangle \langle f u_j, \phi_j \rangle . \tag{6}$$

Thus, for all $f \in R$ and i, j,

$$\frac{\langle f u_i, \phi_i \rangle}{\langle u_i, \phi_i \rangle} = \frac{\langle f u_j, \phi_j \rangle}{\langle u_j, \phi_j \rangle}.$$

Similarly, for $f, g \in R$, setting

$$f_k = \begin{cases} f u_i & (k = i) \\ g u_j & (k = j) \\ u_k & (k \neq i, j) \end{cases} , \quad g_k = \begin{cases} f g u_i & (k = i) \\ u_k & (k \neq i) \end{cases} ,$$

we have

$$\langle f u_i, \phi_i \rangle \langle g u_j, \phi_j \rangle = \langle f g u_i, \phi_i \rangle \langle u_j, \phi_j \rangle .$$

Hence,

$$\frac{\langle f g u_i, \phi_i \rangle}{\langle u_i, \phi_i \rangle} = \frac{\langle f u_i, \phi_i \rangle \langle g u_j, \phi_j \rangle}{\langle u_i, \phi_i \rangle \langle u_j, \phi_j \rangle} = \frac{\langle f u_i, \phi_i \rangle \langle g u_i, \phi_i \rangle}{\langle u_i, \phi_i \rangle \langle u_i, \phi_i \rangle}.$$

Therefore, we have proved that the linear functional Λ_ϕ on R is multiplicative and that its definition is well-defined. Since the right hand side of (5) is unchanged if we multiply ϕ_j by any nonzero constant, Λ_ϕ is dependent only on the tensor product ϕ. Hence $\Lambda_\phi : R \to \mathbb{C}$ is a well-defined $\mathbb{C}$-algebra homomorphism. The uniqueness of Λ_ϕ is clear from definition.

Finally, to show that the identity (3) holds for every $u \in R^{-1} H_j$, it suffices only to show that if $\langle u, \phi_j \rangle = 0$, then $\langle f u, \phi_j \rangle = 0$. Indeed, from (6) we have, for $k \neq j$,

$$\langle f u, \phi_j \rangle \langle u_k, \phi_k \rangle = \langle u, \phi_j \rangle \langle f u_k, \phi_k \rangle = 0.$$

Thus $\langle f u, \phi_j \rangle = 0$, since $\langle u_k, \phi_k \rangle \neq 0$. □

REMARK 2.4. If R has the identity, then from (3) we have $\Lambda_\phi(1) = 1$.

Given complex subspaces R_1 and R_2 of R, let $R_1 \cdot R_2$ denote the complex subspace of R given by

$$R_1 \cdot R_2 = \{\sum_{i=1}^{n} a_i b_i : a_i \in R_1 \text{ and } b_i \in R_2\}.$$

DEFINITION 2.5. Let H be a RKHS on E. H is called *R-dense* if $R \cdot (R \cap H)$ is a dense subspace of H. If H is R-dense for some $\mathbb{C}$-algebra R on E, H is called *algebra-dense*.

REMARK 2.6. If H is R-dense, then $R \cap H$ is both 1) a dense subspace of H and 2) an ideal of R. Moreover, if $1 \in R$, then H is R-dense if and only if 1) and 2) hold.

DEFINITION 2.7. Let H be an R-dense RKHS on E. Let $\chi : R \to \mathbb{C}$ be a $\mathbb{C}$-algebra homomorphism. If there exists a constant $C > 0$ such that

$$|\chi(f)| \leqslant C \|f\| \text{ for all } f \in R \cap H,$$

χ is called an *H-bounded homomorphism* of R. The set of nonzero H-bounded homomorphisms of R is called an *H-hull* of E and is denoted by $\hat{E}_H$.

If H_j $(j = 1,2,\dots,m)$ is R-dense, then $R \cap H_j$ is a dense R-invariant subspace of H_j and we can apply Lemma 2.3 to the tensor product $\otimes_j H_j$. The following is our main

THEOREM 2.8. *For each $j = 1,2,\dots,m$, assume that H_j is an R-dense RKHS on E. If $\phi = \otimes_{j=1}^m \phi_j \in \otimes_{j=1}^m H_j$ is nonzero extremal, then there exists a unique $\mathbb{C}$-algebra homomorphism $\Lambda_\phi \in \bigcap_{j=1}^m \hat{E}_{H_j}$ satisfying* (3). *Furthermore, for each $j = 1,2,\dots,m$, either of the following holds:*

(i) $\Lambda_\phi|_{R\cap H_j} = 0$.

(ii) *There exists a constant $C_j \neq 0$ such that for each $f \in R \cap H_j$, we have $\langle f, \phi_j \rangle = C_j \Lambda_\phi(f)$.*

Proof. First, we prove that either $\mathscr{I}$(i) or $\mathscr{I}$(ii) holds. Assume that Case $\mathscr{I}$(i) does not hold. Then there exists an element $f_0 \in R \cap H_j$ such that $\Lambda_\phi(f_0) \neq 0$. Since $R \cap H_j$ is dense in H_j, there exists an element $g \in R \cap H_j$ with $\langle g, \phi_j \rangle \neq 0$. For every $f \in R \cap H_j$, Lemma 2.3 implies the identity

$$\langle fg, \phi_j \rangle = \Lambda_\phi(f)\langle g, \phi_j \rangle = \Lambda_\phi(g)\langle f, \phi_j \rangle .$$

Putting $f = f_0$ above, we obtain $\Lambda_\phi(g) \neq 0$. Hence, Case $\mathscr{I}$(ii) holds if we set $C_j = \langle g, \phi_j \rangle / \Lambda_\phi(g) \neq 0$.

Now it is clear that Λ_ϕ is an H_j-bounded homomorphism of R. For, if Case $\mathscr{I}$(i) holds then this is trivial, otherwise this follows from Schwarz's inequality. We must show that $\Lambda_\phi \neq 0$. Since $R \cdot (R \cap H_j)$ is dense in H_j, there exist $f_i \in R$ and $g_i \in R \cap H_j$ such that $\langle \sum_i f_i g_i, \phi_j \rangle = \sum_i \langle f_i g_i, \phi_j \rangle \neq 0$. Thus, there exists an index i such that $\langle f_i g_i, \phi_j \rangle = \Lambda_\phi(f_i)\langle g_i, \phi_j \rangle \neq 0$. Hence, $\Lambda_\phi(f_i) \neq 0$ and so $\Lambda_\phi \neq 0$. Thus, $\Lambda_\phi \in \hat{E}_{H_j}$ for every j $(j = 1,2,\dots,m)$. □

DEFINITION 2.9. Let H be an R-dense RKHS on E. Then H is called *maximal* if every nonzero H-bounded homomorphism of R is a point evaluation of R at some point in E. If we need to distinguish the algebra R when H is maximal, then H is called *R-maximal*.

EXAMPLE 2.10. Let $\ell^2(E)$ be the complex Hilbert space $\{f \in \mathbb{C}^E \mid \sum_{x\in E} |f(x)|^2 < \infty\}$ equipped with the inner product $\langle f, g \rangle = \sum_{x\in E} f(x)\overline{g(x)}$, $f, g \in \ell^2(E)$. The identity $\langle f, \delta_p \rangle = f(p)$ implies that the function δ_p is the reproducing kernel of $\ell^2(E)$ at $p \in E$ where δ_p is the function defined by $\delta_p(x) = \delta_{xp}$ (Kronecker's delta). Hence, $\ell^2(E)$ is a RKHS on E. It is easy to see that $\ell^2(E)$ is a $\mathbb{C}$-subalgebra of $\mathbb{C}^E$ (without the identity). Putting $R = \ell^2(E)$ we shall show that $\ell^2(E)$ is R-dense and maximal. From $\delta_x^2 = \delta_x$ we see that $R \cdot R$ is dense in R, which implies that $\ell^2(E)$ is R-dense. Let $\chi\colon R \to \mathbb{C}$ be a nonzero $\mathbb{C}$-algebra homomorphism. From $\delta_x^2 = \delta_x$, $\chi(\delta_x)$ is equal to 0 or 1. Also, $\delta_x\delta_y = 0$ $(x \neq y)$ implies that $\chi(\delta_x)\chi(\delta_y) = 0$ $(x \neq y)$. From $\chi \neq 0$, we conclude that there exists a point $q \in E$ such that $\chi(\delta_x) = \delta_x(q)$ for $x \in E$. Since the span of δ_x's is dense in R, $\chi(f) = f(q)$ for all $f \in R$. Thus $\ell^2(E)$ is maximal.

As a corollary to Theorem 2.8, we have

COROLLARY 2.11. *Let H_j $(j = 1,2,\dots,m)$ be R-dense RKHSs on E. If H_j is maximal for some j, then their tensor product $\otimes_{j=1}^m H_j$ is weakly regular.*

Proof. Let $\phi = \otimes_j \phi_j$ be nonzero extremal in $\otimes_{j=1}^m H_j$. Then by Theorem 2.8 the algebra homomorphism Λ_ϕ is an H_j-bounded homomorphism of R $(j = 1,2,\dots,m)$. Since H_j is maximal for some j, Λ_ϕ is a point evaluation of R at some point $q \in E$. For fixed $j = 1,2,\dots,m$ assume that q is not a common zero of H_j. Since $R \cap H_j$ is dense in H_j, Case $\mathscr{I}$(i) of Theorem 2.8 does not hold. Thus there exists a constant $C_j \neq 0$ such that $\langle f, \phi_j \rangle = C_j f(q)$ for all $f \in R \cap H_j$. Since H_j is R-dense, for each $f \in H_j$ there exists a sequence $f_n \in R \cap H_j$ $(n = 1,2,\dots)$ such that $f_n \to f$ $(n \to \infty)$ (strong convergence). From $\langle f_n, \phi_j \rangle = C_j f_n(q)$, letting $n \to \infty$ we have $\langle f, \phi_j \rangle = C_j f(q)$ for $f \in H_j$. Since $C_j \neq 0$, ϕ_j induces a constant multiple of the point evaluation of H_j at $q \in E$. Thus, ϕ_j is the reproducing kernel of H_j at q up to a nonzero multiplicative constant. Hence $\otimes_{j=1}^m H_j$ is weakly regular. □

3. Equality conditions for the norm inequality

Before studying equality conditions we first recall some facts from the theory of kernel functions. Let $A\colon H_1 \to H_2$ be a linear map from a Hilbert space H_1 into a linear space H_2 with closed kernel $\ker A$ $(= A^{-1}(0))$. The *range norm* of $A(H_1)$ is the norm which makes A a partial isometry from H_1 onto $A(H_1)$. In fact, the range $A(H_1)$ equipped with this range norm is a Hilbert space isomorphic to $H_1 \ominus \ker A$ and is called the *operator range* of the map A. With this terminologies, the RKHS $H_{K_1+K_2}$ on E is the operator range of the map $H_{K_1} \oplus H_{K_2} \ni f \oplus g \mapsto f + g \in H_{K_1+K_2}$. Hence we have the Pythagorean inequality

$$\|f + g\|^2_{K_1+K_2} \leqslant \|f\|^2_{K_1} + \|g\|^2_{K_2}, \tag{7}$$

for all $f_1 \in H_{K_1}$ and $f_2 \in H_{K_2}$, where equality holds if and only if

$$\langle f_1, h \rangle_{K_1} = \langle f_2, h \rangle_{K_2}, \quad \forall h \in H_{K_1} \cap H_{K_2}. \tag{8}$$

Also, the RKHS $H_{K_1K_2}$ on E is the operator range of the map $H_{K_1} \otimes H_{K_2} \ni \sum_i f_i \otimes g_i \mapsto \sum_i f_i g_i \in H_{K_1K_2}$ which is induced by the restriction map from $E \times E$ to its diagonal. Hence, for all $f_1 \in H_{K_1}$ and $f_2 \in H_{K_2}$

$$\|fg\|_{K_1K_2} \leqslant \|f \otimes g\|_{K_1 \otimes K_2} = \|f\|_{K_1} \|g\|_{K_2}, \tag{9}$$

where equality holds if and only if $f \otimes g \in (H_{K_1} \otimes H_{K_2})_0^\perp$. As a special case we note that

$$\|f\|_{cK} = \|f\|_K / \sqrt{c} \tag{10}$$

for any positive constant c. For proofs of these statements, see, for instance, [6, p. 32] and [16].

Applying (7), (9) and (10), for $f \in H_K$ we have

$$\|\varphi(f)\|^2_{\psi(K)} \leqslant \sum_{p_n > 0} \|c_n f^n\|^2_{p_n K^n} = \sum_{p_n > 0} \frac{|c_n|^2}{p_n} \|f^n\|^2_{K^n} \leqslant \sum_{p_n > 0} \frac{|c_n|^2}{p_n} \|f\|^{2n}_K. \tag{11}$$

Thus we obtain the inequality (2) stated in the introduction.

The next Theorem asserts that if the RKHS H_K is algebra-dense and maximal, then "usually" its kernel functions are, up to constants, the only functions which attain equality in (2), and hence the exceptional case does not occur.

THEOREM 3.1. *Let H_K be a RKHS on E which is R-dense and maximal. Assume that $\varphi(z)=\sum_{n=1}^{\infty}c_nz^n$ and $\psi(z)=\sum_{n=1}^{\infty}p_nz^n$ are entire functions with the properties:*
(i) $p_n \geqslant 0$ *for all* n.
(ii) $c_n=0$ *whenever* $p_n=0$.
(iii) $c_ic_j \neq 0$ *for some* i,j *with* $1 \leqslant i<j$.
Then, equality holds in the inequality (2) *if and only if there exist a point* $q \in E$ *and constants* C, C' *such that* $\varphi(Cz)=C'\psi(z)$ *for all* $z \in \mathbb{C}$ *and* $f=CK_q$.

Proof. If $\varphi(Cz)=C'\psi(z)$, then $c_nC^n=C'p_n$ for all $n \geqslant 1$. Thus $\varphi_\psi(|C|^2z)=|C'|^2\psi(z)$. Hence we have

$$\|\varphi(CK_q)\|^2_{\psi(K)}=\|C'\psi(K_q)\|^2_{\psi(K)}=|C'|^2\sum_n p_nK^n(q,q)$$
$$=|C'|^2\psi(\|K_q\|^2_K)=\varphi_\psi(\|CK_q\|^2_K),$$

proving the sufficiency part of Theorem.

To prove the necessity assume that equality holds in (2) for $f \in H_K$. Noting that the case $f=0$ corresponds to the choice with $C=C'=0$, we may assume that $f \neq 0$. By the hypotheses $\mathscr{I}$(ii) and $\mathscr{I}$(iii) there exist indices i and j with $p_ip_j \neq 0,\ 1 \leqslant i<j$. From the chain of inequalities (11) and since $j \geqslant 2$, $f^{\otimes j}$ must be nonzero extremal in $H_K^{\otimes j}$. By Corollary 2.11 $H_K^{\otimes j}$ is weakly regular. Thus, there exists a point $q \in E$ such that $\Lambda_{f^{\otimes j}}(g)=g(q)$ for all $g \in R$, and that q is either a common zero of H_K or $f=CK_q$ for some nonzero constant C.

Next we shall show that q is not a common zero of H_K. To prove this we note that if $\otimes_n\phi_n$ is extremal in $\otimes_nH_n$, then by the basic properties of the range norm and the tensor product, for any $\otimes_ng_n \in \otimes_nH_n$ we have

$$\langle \Pi_ng_n,\Pi_n\phi_n\rangle_{\Pi_nK_n}=\langle \otimes_ng_n,\otimes_n\phi_n\rangle_{\otimes_nK_n}=\Pi_n\langle g_n,\phi_n\rangle_{K_n}.$$

Since H_K is R-dense and $f \neq 0$, there exists $u \in R \cap H_K$ with $\langle f,u\rangle_K \neq 0$. Since $f^{\otimes j}$ is extremal in $H_K^{\otimes j}$, we have

$$\langle c_jf^j,u^j\rangle_{p_jK^j}=\frac{c_j}{p_j}\langle f^j,u^j\rangle_{K^j}=\frac{c_j}{p_j}\langle f,u\rangle^j_K. \tag{12}$$

On the other hand, if $i \geqslant 2$, $f^{\otimes i}$ is extremal in $H_K^{\otimes i}$, and so by decomposing u^j as a product $u^{j-i+1}\cdot u^{i-1}$, we have

$$\langle c_if^i,u^j\rangle_{p_iK^i}=\frac{c_i}{p_i}\langle f,u^{j-i+1}\rangle_K\langle f,u\rangle^{i-1}_K. \tag{13}$$

Obviously, this identity also holds for $i=1$. Since $f^{\otimes j}$ is nonzero extremal, it follows from Lemma 2.3

$$\langle u^{j-i+1},f\rangle_K=u^{j-i}(q)\langle u,f\rangle_K. \tag{14}$$

Decomposing u^j as above we see that $u^j \in H_{p_iK^i}\cap H_{p_jK^j}$. The equality condition (8) implies, for any k, l with $p_kp_l \neq 0$,

$$\langle c_kf^k,u^l\rangle_{p_kK^k}=\langle c_lf^l,u^l\rangle_{p_lK^l}. \tag{15}$$

Combining (12)–(15), we have

$$\overline{u^{j-i}(q)} = \frac{c_j p_i}{c_i p_j} \langle f, u \rangle_K^{j-i} \neq 0.$$

Since $j > i$, the point q is not a common zero of H_K, as desired.

Thus, $f = CK_q$ for some constant $C \neq 0$. By (15) the reproducing property of f yields, for any k, l with $p_k p_l \neq 0$,

$$\frac{c_k C^k}{p_k} = \frac{c_l C^l}{p_l}.$$

Putting $C' = c_k C^k / p_k$, we immediately obtain the identity $\varphi(Cz) = C'\psi(z)$. □

4. The case of polynomial ring

As an important example, we first consider the case where E is a subset of the complex n-dimensional space $\mathbb{C}^n$, and R is a restriction to E of the polynomial ring $\mathbb{C}[z_1, \dots, z_n]$. For simplicity, we use the standard multi-index notation: If $z = (z_1, z_2, \dots, z_n)$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, then $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $z^\alpha = \prod_{i=1}^n z_i^{\alpha_i}$. A power series with center at the origin is denoted by $\sum_\alpha a_\alpha z^\alpha$.

DEFINITION 4.1. Let H be a RKHS on a subset E of $\mathbb{C}^n$. If H is $\mathbb{C}[z_1, \dots, z_n]|_E$-dense, then H is called *polynomially dense*.

Let $\chi\colon \mathbb{C}[z_1, \dots, z_n] \to \mathbb{C}$ be a $\mathbb{C}$-algebra homomorphism with $\chi(1) = 1$. For any polynomial $f(z) = \sum_\alpha a_\alpha z^\alpha$, we have

$$\chi(f) = \sum_\alpha a_\alpha \chi(z)^\alpha = f(\chi(z)),$$

where $\chi(z) = (\chi(z_1), \chi(z_2), \dots, \chi(z_n)) \in \mathbb{C}^n$. Hence we conclude that any nonzero $\mathbb{C}$-algebra homomorphism of $\mathbb{C}[z_1, \dots, z_n]|_E$ is a point evaluation at some point of $\mathbb{C}^n$. Thus we have immediately

PROPOSITION 4.2. *Let H be a polynomially dense RKHS on E. Then H is maximal if and only if the following holds: Given a point $q \in \mathbb{C}^n$, if there exists a constant $C > 0$ with $|f(q)| \leqslant C\|f\|$ for all $f \in \mathbb{C}[z_1, \dots, z_n] \cap H$, then $q \in E$.*

We next give an example of polynomially dense RKHSs and provide a sufficient condition for these RKHSs to be maximal.

EXAMPLE 4.3. ([5]) For $z, \zeta \in \mathbb{C}^n$ we put $z\zeta = (z_1\zeta_1, \dots, z_n\zeta_n) \in \mathbb{C}^n$. Fix a power series with positive coefficients $\eta(z) = \sum_\alpha c_\alpha z^\alpha$, $(c_\alpha > 0,\ \alpha \in \mathbb{Z}_+^n)$, and assume that the domain of convergence D of the function $\eta(zz)$ is nonempty. A function f holomorphic in the domain D has a power series expansion $f(z) = \sum_\alpha a_\alpha z^\alpha$ on D. Define the norm of f by

$$\|f\|^2 = \sum_\alpha \frac{|a_\alpha|^2}{c_\alpha},$$

and let $\mathscr{H}_\eta$ denote the space of holomorphic functions in D with $\|f\| < \infty$. Define an inner product of f and $g \in \mathscr{H}_\eta$ by

$$\langle f, g\rangle = \sum_\alpha \frac{a_\alpha \bar{b}_\alpha}{c_\alpha}, \quad g(z) = \sum_\alpha b_\alpha z^\alpha,$$

then $\mathscr{H}_\eta$ is a Hilbert space. For $\zeta \in D$ let $k_\zeta(z)$ denote the function $\eta(z\bar{\zeta})$. Then we easily see that $k_\zeta \in \mathscr{H}_\eta$ and that $f(\zeta) = \langle f, k_\zeta\rangle$ for all $f \in \mathscr{H}_\eta$. Thus, k_ζ is the reproducing kernel at ζ for the space $\mathscr{H}_\eta$, and hence $\mathscr{H}_\eta$ is a RKHS on D. By definition of the norm, $\mathscr{H}_\eta$ is clearly polynomially dense.

PROPOSITION 4.4. *If $\eta(z\bar{z}) = \infty$ for every $z \in \partial D$, then $\mathscr{H}_\eta$ is polynomially dense and maximal.*

Proof. Since $\eta(z\bar{z})$ is a series with nonnegative terms, if $\eta(z\bar{z}) < \infty$, then $\eta(tz\overline{tz}) < \infty$ for all t with $0 < t < 1$. Thus, from the hypothesis it is easy to see that $\eta(z\bar{z}) = \infty$ for every $z \notin D$. For $\zeta \notin D$ and $n \in \mathbb{N}$, let $k_\zeta^{(n)}(z) = \sum_{|\alpha| \leqslant n} c_\alpha \bar{\zeta}^\alpha z^\alpha \in \mathscr{H}_\eta$ be the n-th partial sum of $k_\zeta(z)$. Then,

$$\frac{|k_\zeta^{(n)}(\zeta)|}{\|k_\zeta^{(n)}\|} = \sqrt{k_\zeta^{(n)}(\zeta)} \to \sqrt{\eta(\zeta\bar{\zeta})} = \infty \ (n \to \infty).$$

Thus, the point evaluation at $\zeta \notin D$ is not $\mathscr{H}_\eta$-bounded. In view of Proposition 4.2 this implies that $\mathscr{H}_\eta$ is maximal. □

REMARK 4.5. Theorems in Sections 5 and 6 of [5] are immediate consequences of our Theorem 3.1 and Proposition 4.4.

5. Algebra of meromorphic functions

Throughout this section, let E be a regular subregion of a compact Riemann surface S. Here, a proper subregion E of S is called *regular* if E and its exterior have the same boundary consisting of a finite number of analytic Jordan curves. Let $\mathscr{R}_E$ denote the complex algebra of meromorphic functions on S which are holomorphic on $\overline{E}$.

DEFINITION 5.1. A RKHS H on E is called *meromorphically dense* if H is $\mathscr{R}_E$-dense.

We prepare the following Proposition which is useful for testing the maximality of meromorphically dense RKHSs.

PROPOSITION 5.2. *Let $\chi\colon \mathscr{R}_E \to \mathbb{C}$ be a $\mathbb{C}$-algebra homomorphism with $\chi(1) = 1$. Then there exists a unique point $q \in \overline{E}$ such that $\chi(f) = f(q)$ for all $f \in \mathscr{R}_E$.*

Proof. First we show that $\chi(f) \in f(\overline{E})$ for any $f \in \mathscr{R}_E$. If $f \neq 0$ on $\overline{E}$, then $1/f \in \mathscr{R}_E$, and from the identity $\chi(f)\chi(1/f) = \chi(1) = 1$ we have $\chi(f) \neq 0$.

Therefore, $\chi(f-\chi(f))=0$ implies that $f-\chi(f)$ vanishes for some point in $\overline{E}$. Thus $\chi(f)\in f(\overline{E})$.

Choose any $f\in\mathscr{R}_E\setminus\mathbb{C}$ and set $\zeta=\chi(f)$. Let $f^{-1}(\zeta)$ consists of distinct points $q_1,\dots,q_r$ $(1\leqslant r\leqslant n)$ where n is the degree of the meromorphic function f on the compact Riemann surface S. From the above remark, $f^{-1}(\zeta)\cap\overline{E}\neq\emptyset$. Let us choose a point $\zeta_0\in\mathbb{C}$ such that $\zeta_0\neq\zeta$ and $f^{-1}(\zeta_0)$ consists of n distinct points. Applying the Riemann-Roch theorem, one verifies easily that there exists $g_0\in\mathscr{R}_E$ with the following properties:

i) g_0 takes different values at different points of $f^{-1}(\zeta_0)$.

ii) $g_0(p)\neq 0$ for any $p\in\overline{E}$.

iii) $g_0(q_j)=0$ for all $q_j\notin\overline{E}$.

Then, for sufficiently small $\epsilon\neq 0$, the function $g=1/(g_0+\epsilon)$ is an element of $\mathscr{R}_E$ holomorphic at all points q_j $(j=1,\dots,r)$ and satisfies the inequality

$$\sup_{z\in\overline{E}}|g(z)|<|g(q_j)|\quad\text{for any } q_j\notin\overline{E}. \tag{16}$$

By the general theory of compact Riemann surfaces [7], there exist rational functions a_k $(k=0,\dots,n)$ such that

$$\sum_{k=0}^{n}a_k(f)g^k=0,\quad (a_n(z)=1). \tag{17}$$

Since g is holomorphic at all points $q_1,\dots,q_r$, it is clear from well known constructions of (17) that the rational functions $a_k(z)$ $(k=0,1,\dots,n)$ are holomorphic at $z=\zeta$. Thus, we may assume that a_k is of the form b_k/c_k such that $b_k,c_k\in\mathbb{C}[z]$ and $c_k(\zeta)\neq 0$. Put $s=\prod_{k=0}^{n}c_k$ and multiply the identity (17) by $s(f)$. Since $g\in\mathscr{R}_E$ and each sa_k $(k=0,\dots,n)$ is a polynomial, we can apply the homomorphism χ to the resulting identity. Thus

$$\sum_{k=0}^{n}sa_k(\zeta)\chi(g)^k=0.$$

Since $sa_k(\zeta)=s(\zeta)a_k(\zeta)$ with $s(\zeta)\neq 0$, we have $\sum_{k=0}^{n}a_k(\zeta)\chi(g)^k=0$. Hence by construction of (17), $\chi(g)=g(q_j)$ for some j $(1\leqslant j\leqslant r)$. Therefore, we have proved that there exists a point q such that $\chi(f)=f(q)$ and $\chi(g)=g(q)$. From (16) we conclude that $q\in\overline{E}$, since $\chi(g)\in g(\overline{E})$.

We now show that the homomorphism χ is the point evaluation at q. By the property i) above, the meromorphic functions f and g form a primitive pair (cf. [7, p. 233]). Thus, it is well known that for any meromorphic function h on S, there exist rational functions A_k $(k=0,\dots,n-1)$ such that

$$h=\sum_{k=0}^{n-1}A_k(f)g^k. \tag{18}$$

First, consider the case where $h\in\mathscr{R}_E$ is holomorphic at each point of the set $f^{-1}(\zeta)$. Then, from well known constructions of (18), every coefficient A_k is holomorphic at

ζ. Again, by multiplying suitable polynomial as above, we obtain

$$\chi(h) = \sum_{k=0}^{n-1} A_k(\zeta) g(q)^k = h(q).$$

Second, for general $h \in \mathscr{R}_E$, choose a function $h_2 \in \mathscr{R}_E$ such that $h_2(q) \neq 0$ and that the product $h_2 h$ is holomorphic at each point of $f^{-1}(\zeta)$. This is possible if h_2 has sufficiently large order of zeros at every $q_j \notin \overline{E}$. From

$$h_2(q)h(q) = \chi(h_2 h) = \chi(h_2)\chi(h) = h_2(q)\chi(h),$$

we conclude that $\chi(h) = h(q)$ for all $h \in \mathscr{R}_E$, as desired. The uniqueness of the point q is obvious, since the algebra of functions $\mathscr{R}_E$ separates points of S. □

From Proposition 5.2, we immediately have

PROPOSITION 5.3. *Let H be a meromorphically dense RKHS on E. Then H is maximal if and only if the following holds: Given a point $q \in \overline{E}$, if there exists a constant $C > 0$ with $|f(q)| \leqslant C\|f\|$ for all $f \in \mathscr{R}_E \cap H$, then $q \in E$.*

6. Application

As an application of the results obtained in the previous sections we study the regularity of tensor products of RKHSs consisting of analytic functions or analytic differentials on a compact bordered Riemann surface. Let E be the interior of a compact bordered Riemann surface $\overline{E} = E \cup \partial E$ with nonempty boundary ∂E. Let $\hat{E}$ be the Schottky double [19] of $\overline{E}$. Then E can be viewed as a regular subregion of the compact Riemann surface $\hat{E}$. Define the $\mathbb{C}$-algebra $\mathscr{R}_E$ in this context. Consider the following RKHSs on E:

(i) $\mathscr{H}_1(E,\rho)$: (Weighted Szegö space) The Hardy H^2 space of analytic functions f on E with norm $\|f\|^2 = \int_{\partial E} |f|^2 \rho |dz|$ where $\rho|dz|$ is a positive continuous metric on ∂E. In the integrand f denotes the nontangential boundary value of f on ∂E.

(ii) $\mathscr{H}_2(E,\rho)$: (Weighted Dirichlet space) The space of analytic functions f on E with finite Dirichlet norm $\|f\|^2 = \frac{i}{2\pi} \iint_E \rho\, df \wedge \overline{df}$ satisfying $f(a) = 0$ for a fixed point $a \in E$, where ρ is a positive continuous function on $\overline{E}$.

(iii) $\mathscr{H}_3(E,\rho)$: (Weighted Bergman space) The Bergman space of analytic differentials f on E with norm $\|f\|^2 = \frac{i}{2\pi} \iint_E \rho f \wedge \overline{f}$, where ρ is a positive continuous function on $\overline{E}$.

For the proof of Theorem below we need a weaker form of the result in [18, Theorem 8] on uniform approximation.

PROPOSITION 6.1. (S. Scheinberg) *Let E be a regular subregion of a compact Riemann surface S. For every holomorphic function f on $\overline{E}$ and for every positive constant ϵ, there exists a function $g \in \mathscr{R}_E$ such that $\|f - g\|_\infty < \epsilon$ on $\overline{E}$, where $\|\cdot\|_\infty$ denotes the sup-norm on $\overline{E}$.*

THEOREM 6.2. *The following hold:*

(i) *RKHS* $\mathscr{H}_j(E,\rho)$ $(j=1,2,3)$ *is meromorphically dense and maximal.*

(ii) *For any integer* $n \geqslant 2$, $\mathscr{H}_j(E,\rho)^{\otimes n}$ $(j \neq 2)$ *is regular and* $\mathscr{H}_2(E,\rho)^{\otimes n}$ *is weakly regular.*

(iii) *Let* Δ *be the unit disk* $\{|z|<1\}$ *and set* $a=0$. *Then* $\phi^{\otimes 2}$ *is extremal in* $\mathscr{H}_2(\Delta,1)^{\otimes 2}$ *if and only if* $\phi(z)=cz$ *or* $\phi = ck_q$ $(q \in \Delta \setminus \{0\})$ *for some* $c \in \mathbb{C}$, *where* $k_q(z) = -\log(1-\bar{q}z)$ *is the reproducing kernel of* $\mathscr{H}_2(\Delta,1)$ *at* q. *Thus,* $\mathscr{H}_2(\Delta,1)^{\otimes 2}$ *is not regular.*

Proof. $\mathscr{I}$(i) First we remark that the norm of the RKHS $\mathscr{H}_j(E,\rho)$ is equivalent for each weight ρ and, as a set, the space $\mathscr{H}_j(E,\rho)$ is independent of ρ, since ρ is positive and continuous on compact set $\overline{E}$. Thus we may assume $\rho=1$ without loss of generality. Let $\mathscr{H}_j(E)=\mathscr{H}_j(E,1)$ for simplicity.

Since $1 \in \mathscr{R}_E$, to prove that $\mathscr{H}_j(E)$ is meromorphically dense, it suffices to show that $\mathscr{R}_E \cap \mathscr{H}_j(E)$ is dense in $\mathscr{H}_j(E)$ and is an ideal of $\mathscr{R}_E$. To establish that the space $\mathscr{R}_E \cap \mathscr{H}_j(E)$ is dense in $\mathscr{H}_j(E)$ $(j=1,2,3)$, we recall that the Szegö kernels, the exact Bergman kernels and the Bergman kernels are analytically continued to a neighborhood of $\overline{E}$ [19, 16]. Since the linear span of the kernel functions is dense, the set of functions holomorphic on $\overline{E}$ are dense in $\mathscr{H}_j(E)$, and by Proposition 6.1 we see easily that $\mathscr{R}_E \cap \mathscr{H}_j(E)$ is dense in $\mathscr{H}_j(E)$ $(j=1,2,3)$. On the other hand, it is clear that $\mathscr{R}_E \cap \mathscr{H}_j(E)$ is an ideal of $\mathscr{R}_E$. Thus, $\mathscr{H}_j(E)$ $(j=1,2,3)$ is meromorphically dense.

Next, we show that all the $\mathscr{H}_j(E)$ $(j=1,2,3)$ are maximal. In view of Proposition 5.3 it suffices to show that, for fixed $b \in \partial E$, there exists a family of functions $\{f_p\}$, $f_p \in \mathscr{R}_E \cap \mathscr{H}_j(E)$ such that $|f_p(b)|/\|f_p\|$ tends to ∞ as $p \to b$. Consider the case of the Dirichlet space $\mathscr{H}_2(E)$. The proof of the other cases is similar but more easy. Now we claim that we need only to show that there exists a family of functions $\{f_p\}$ holomorphic on $\overline{E}$ with the above property. This is seen as follows. Given a holomorphic function f on $\overline{E}$, there exists a regular subregion E_1 with $\overline{E} \subset E_1$ such that f is holomorphic on $\overline{E_1}$. Applying Cauchy's integral formula we see that there exists a constant $C>0$ with $\|f\| \leqslant C\|f\|_{\infty,E_1}$ where $\|f\|_{\infty,E_1}$ denotes $\sup_{x\in E_1}|f(x)|$. From Proposition 6.1 there exists $g \in \mathscr{R}_{E_1}$ such that $\|f-g\|_{\infty,E_1} \leqslant \epsilon\|f\|$. Then

$$|g(b)| \geqslant |f(b)| - \|f-g\|_{\infty,E_1} \geqslant |f(b)| - \epsilon\|f\|,$$
$$\|g\| \leqslant \|f\| + \|f-g\| \leqslant \|f\| + C\|f-g\|_{\infty,E_1} \leqslant (1+\epsilon C)\|f\|.$$

By choosing ϵ so small that $0<\epsilon<\min\{1,1/C\}$ is satisfied, we have

$$\frac{|g(b)|}{\|g\|} \geqslant \frac{1}{1+\epsilon C}\frac{|f(b)|}{\|f\|} - \epsilon > \frac{|f(b)|}{2\|f\|} - 1,$$

which implies our claim, as desired.

By definition we have the identity

$$k_B(x,y) = \frac{\partial^2}{\partial x \partial \overline{y}} k_D(x,y),$$

where $k_D(x,y)$ is the kernel function for the Dirichlet space $\mathscr{H}_2(E)$ and $k_B(x,y)dx\overline{dy}$ is the exact Bergman kernel for E. Let ϕ be the canonical anti-conformal involution for the double $\hat{E}$ fixing ∂E. It is well known [19, p. 118] that the exact Bergman kernel $k_B(x,y)$ is extended to a meromorphic bilinear differential on $\hat{E}$ with double pole only at $x=\phi(y)$, and that $k_B(x,y)$ has the expansion

$$k_B(x,y) = -\frac{1}{\pi(x-\phi(y))^2} + \text{regular terms} \tag{19}$$

for x,y in a coordinate neighborhood U centered at $b \in \partial E$. Integrating (19) we see that for $p \in E$ the Dirichlet kernel $k_D(x,p)$ is extended holomorphically to a neighborhood of $\overline{E}$ with the expansion

$$k_D(x,y) = \frac{1}{\pi}\log\frac{1}{x-\phi(y)} + \text{regular terms} \tag{20}$$

for $x,y \in U \cap E$. Setting $f_p(x) = k_D(x,p)$ for $p \in E$ near b, we have

$$|f_p(b)|/\|f_p\| = |k_D(b,p)|/\sqrt{k_D(p,p)}.$$

By (20) this tends to ∞ as $p \to b \in \partial E$ nontangentially. Thus $\mathscr{H}_2(E)$ is maximal.

$\mathscr{I}$(ii) By Corollary 2.11 and $\mathscr{I}$(i) all the tensor products of these spaces are weakly regular. Moreover, since both $\mathscr{H}_1(E,\rho)$ and $\mathscr{H}_3(E,\rho)$ have no common zeros, the products $\mathscr{H}_j(E,\rho)^{\otimes n}$ $(j=1,3;\ n \geqslant 2)$ are regular.

$\mathscr{I}$(iii) By definition, if $\phi = ck_q$, it is clear that $\phi^{\otimes 2}$ is extremal. We shall show that $z^{\otimes 2} \in (\mathscr{H}_2(\Delta,1)^{\otimes 2})_0^{\perp}$. Since the set of functions $\{z^i/\sqrt{i}\}_{i=1}^{\infty}$ is a complete orthonormal system (CONS) for $\mathscr{H}_2(\Delta,1)$, the set $\{z^i \otimes z^j/\sqrt{ij}\}_{i,j=1}^{\infty}$ is a CONS for the tensor product $\mathscr{H}_2(\Delta,1)^{\otimes 2}$. Hence, any $f \in \mathscr{H}_2(\Delta,1)^{\otimes 2}$ is given by

$$f = \sum_{i,j=1}^{\infty} \frac{c_{ij}}{\sqrt{ij}} z^i \otimes z^j, \quad \text{with} \sum_{i,j=1}^{\infty} |c_{ij}|^2 < \infty.$$

Then we see that $f \in (\mathscr{H}_2(\Delta,1)^{\otimes 2})_0$ if and only if $\sum_{i+j=n} c_{ij}/\sqrt{ij} = 0$ for all $n \geqslant 2$. In particular, $f \in (\mathscr{H}_2(\Delta,1)^{\otimes 2})_0$ implies $c_{11}=0$. Since $\langle f, z^{\otimes 2}\rangle = c_{11}$, $z^{\otimes 2} \in (\mathscr{H}_2(\Delta,1)^{\otimes 2})_0^{\perp}$, that is, $z^{\otimes 2}$ is extremal

Contrarily, suppose that $\phi^{\otimes 2} \in (\mathscr{H}_2(\Delta,1)^{\otimes 2})_0^{\perp}$. We may assume without loss of generality $\phi \neq 0$. From Lemma 2.3 there exists a point $q \in \Delta$ such that

$$\langle zf, \phi\rangle = f(q)\langle z, \phi\rangle. \tag{21}$$

If $q \neq 0$, then q is not a common zero of $\mathscr{H}_2(\Delta,1)$. Since $\mathscr{H}_2(\Delta,1)^{\otimes 2}$ is weakly regular, $\phi = ck_q$ for some constant $c \in \mathbb{C}$. On the other hand, if $q=0$, then by (21) $\phi \perp z^n$ for all $n \geqslant 2$. Therefore, ϕ is a constant multiple of the function z. Thus the first assertion of $\mathscr{I}$(iii) is proved. $\mathscr{H}_2(\Delta,1)^{\otimes 2}$ is not regular, since the function z cannot be a constant multiple of the kernel function k_q for any $q \in \Delta$. $\square$

REMARK 6.3. The paper [9, p. 73] overlooked that $z^{\otimes 2}$ is also extremal. Therefore, the proof of equality condition in [9] needs a slight modification.

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(Received May 23, 2007)

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