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ON THE BOUNDS FOR THE NORMALIZED JENSEN FUNCTIONAL AND JENSEN–STEFFENSEN INEQUALITY

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Abstract. We consider the inequalities of type

$$M\mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{q}) \geqslant \mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{p}) \geqslant m\mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{q}),$$

where f is a convex function and $\mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{p}) = \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right)$, recently introduced by S.S. Dragomir. We give an alternative proof of such inequalities and prove another similar result for the case when f is a convex function on an interval in the real line, while $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfy the conditions for Jensen-Steffensen inequality. We show that our result improves the result of Dragomir in this special case. We also prove the integral versions of all our results, including those related to Boas' generalization of Jensen-Steffensen integral inequality.

1. Introduction

Jensen's inequality for convex functions is probably one of the most important inequalities which is extensively used in almost all areas of mathematics. For a comprehensive inspection of the classical and recent results related to this inequality the reader is referred to [4]. There are many forms of this inequality (discrete and integral). Here we recall classical discrete form:

Suppose X is a real linear space, $C \subseteq X$ is a convex set in X and $f : C \to \mathbb{R}$ is a convex function defined on C. If $z_1, z_2, \ldots, z_n \in C$, $n \geqslant 2$ are any vectors and $r_i \geqslant 0$, $i = 1, 2, \ldots, n$ are nonnegative real numbers such that $R_n > 0$, $R_n := \sum_{i=1}^n r_i$, then the weighted Jensen's inequality

$$f\left(\frac{1}{R_n}\sum_{i=1}^n r_i z_i\right) \leqslant \frac{1}{R_n}\sum_{i=1}^n r_i f(z_i) \tag{J}$$

is valid. If f is a strictly convex function on C and $r_i > 0$, $i = 1, 2, \ldots, n$, then the equality case holds in (J) if and only if $z_1 = z_2 = \ldots = z_n$.

It is interesting that in the case when $X = \mathbb{R}$ that is when $f : I \to \mathbb{R}$ is a convex function defined on an interval $I \subseteq \mathbb{R}$, if $\boldsymbol{z} = (z_1, \ldots, z_n) \in I^n$ is a monotonic (either nondecreasing or nonincreasing) n-tuple, then (J) remains valid even in the case when

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the condition "$\boldsymbol{r} = (r_1,\ldots,r_n)$ is nonnegative n-tuple" is somewhat relaxed. More precisely the following is true [4, p. 57]:

THEOREM A. (Jensen-Steffensen) *Let $I \subseteq R$ be an interval and let $\boldsymbol{z} = (z_1,\ldots,z_n) \in I^n$ be a monotonic n-tuple. Let $\boldsymbol{r} = (r_1,\ldots,r_n)$ be a real n-tuple and denote $R_k := \sum_{i=1}^{k} r_i,\ k = 1,2,\ldots,n$. If*

$$0 \leqslant R_k \leqslant R_n, \quad k = 1,2,\ldots,n, \quad R_n > 0,$$

then (J) holds for any convex function $f : I \to \mathbb{R}$.

When (J) is considered under assumptions of Theorem A we refer it as (JS) and call it Jensen-Steffensen inequality. A detailed discussion of the equality case in (JS) for a strictly convex function $f : I \to \mathbb{R}$ can be found in [1].

Another inequality closely related to (J) and which we shall use in this paper is the reversal of Jensen's inequality [4, p. 83]:

THEOREM B. (Jensen reversed) *Let $\boldsymbol{r} = (r_1,\ldots,r_n)$ be a real n-tuple such that*

$$r_1 > 0, \quad r_i \leqslant 0, \quad i = 2,\ldots,n, \quad R_n = \sum_{i=1}^{n} r_i > 0.$$

Let $C \subseteq X$ be a convex set in a real linear space X and let $z_i \in C,\ i = 1,\ldots,n$ be a vectors such that $\frac{1}{R_n}\sum_{i=1}^{n} r_i z_i \in C$. If $f : C \to \mathbb{R}$ is a convex function, then the reversed Jensen's inequality

$$f\left(\frac{1}{R_n}\sum_{i=1}^{n} r_i z_i\right) \geqslant \frac{1}{R_n}\sum_{i=1}^{n} r_i f(z_i) \tag{JR}$$

holds.

Recently Dragomir [3] proved an interesting result for the difference between the right side and the left side of (J).

To be more specific, let $\mathscr{P}_n$ denotes the set of all nonnegative real n-tuples $(p_1,\ldots,p_n)$ with the property that $\sum_{i=1}^{n} p_i = 1$. For any convex function $f : C \to \mathbb{R}$ defined on a convex set C in a real linear space X and for any choice of n-tuples $\boldsymbol{x} = (x_1,\ldots,x_n) \in C^n$ and $\boldsymbol{p} = (p_1,\ldots,p_n) \in \mathscr{P}_n$ we define

$$\mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{p}) := \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right)$$

and we call it the normalized Jensen functional. For a fixed function f and n-tuple $\boldsymbol{x}$, $\mathscr{J}_n(f,\boldsymbol{x},\cdot)$ can be observed as a function on $\mathscr{P}_n$. Note that $\mathscr{P}_n$ is obviously a convex subset in $\mathbb{R}^n$ and that because of (J) we have $\mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{p}) \geqslant 0$, for all $\boldsymbol{p} \in \mathscr{P}_n$. Now we state Dragomir's result [3, Th. 1]:

THEOREM C. (Dragomir) *If $p,q \in \mathscr{P}_n$ and $q_i > 0,\ i = 1,\ldots,n$, then*

$$\max_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} \mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{q}) \geqslant \mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{p}) \geqslant \min_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} \mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{q}). \tag{JB}$$

Dragomir's proof of (JB) is based on a direct application of (J) for appropriate choices of the elements in (J).

In this paper we give an alternative proof of (JB) based on a direct applications of (J) and (JR). This proof is more suitable for further generalizations. To be more specific our proof of (JB) allows us to prove another result analogous to (JB) in the case when $f : I \to \mathbb{R}$ is a convex function defined on an interval $I \subseteq \mathbb{R}$ and n-tuple $\boldsymbol{p} = (p_1, \ldots, p_n)$ satisfies the conditions for Jensen-Steffensen inequality, i.e. p_i need not be nonnegative. Moreover we show that our result in that case is an improvement of (JB). We also prove the integral versions of all our results, including those related to Boas' generalization of Jensen-Steffensen integral inequality.

2. Alternative proof of (JB)

Proof. (of THEOREM C) We assume the notations from introduction. For given $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ from $\mathscr{P}_n$ such that $q_i > 0$, $i = 1, \ldots, n$, let us denote

$$m = m(\boldsymbol{p}, \boldsymbol{q}) := \min_{1 \leqslant i \leqslant n} \left\{ \tfrac{p_i}{q_i} \right\}, \qquad M = M(\boldsymbol{p}, \boldsymbol{q}) := \max_{1 \leqslant i \leqslant n} \left\{ \tfrac{p_i}{q_i} \right\}.$$

Obviously we have

$$\tfrac{p_i}{q_i} - m \geqslant 0, \quad M - \tfrac{p_i}{q_i} \geqslant 0, \quad i = 1, \ldots, n,$$

which implies

$$p_i - mq_i \geqslant 0, \quad Mq_i - p_i \geqslant 0, \quad i = 1, \ldots, n. \tag{2.1}$$

If $m \geqslant 1$, then $p_i - q_i \geqslant p_i - mq_i \geqslant 0$, $i = 1, \ldots, n$. On the other hand $\sum_{i=1}^n (p_i - q_i) = 0$ since both $\boldsymbol{p}$ and $\boldsymbol{q}$ are from $\mathscr{P}_n$. This implies $p_i - q_i = 0$, $i = 1, \ldots, n$ that is $\boldsymbol{q} = \boldsymbol{p}$ and $\mathscr{J}_n(f, \boldsymbol{x}, \boldsymbol{q}) = \mathscr{J}_n(f, \boldsymbol{x}, \boldsymbol{p})$. Since $\boldsymbol{q} = \boldsymbol{p}$, the condition (2.1) is possible only with $m = 1$ and $M \geqslant 1$ so that (JB) obviously holds. Similarly, if $M \leqslant 1$, then we have $q_i - p_i \geqslant Mq_i - p_i \geqslant 0$, $i = 1, \ldots, n$ and by the same argument we conclude that $\boldsymbol{q} = \boldsymbol{p}$ and $\mathscr{J}_n(f, \boldsymbol{x}, \boldsymbol{q}) = \mathscr{J}_n(f, \boldsymbol{x}, \boldsymbol{p})$. In this case (2.1) is possible only with $M = 1$ and $m \leqslant 1$ so that (JB) obviously holds.

It remains to consider the case when $m < 1$ and $M > 1$. To prove the right inequality in (JB) note that $p_i - mq_i \geqslant 0$, $i = 1, \ldots, n$ and $\sum_{i=1}^n (p_i - mq_i) = 1 - m > 0$. Now, applying (J) twice we get

$$\begin{aligned}
m \left[f\left(\sum_{i=1}^n q_i x_i \right) - \sum_{i=1}^n q_i f(x_i) \right] + \sum_{i=1}^n p_i f(x_i) & \\
= mf\left(\sum_{i=1}^n q_i x_i \right) + \sum_{i=1}^n (p_i - mq_i) f(x_i) & \\
\geqslant mf\left(\sum_{i=1}^n q_i x_i \right) + \sum_{j=1}^n (p_j - mq_j) \cdot f\left(\frac{\sum_{i=1}^n (p_i - mq_i) x_i}{\sum_{j=1}^n (p_j - mq_j)} \right) &
\end{aligned}$$

$$=mf\left(\sum_{i=1}^{n}q_ix_i\right)+(1-m)f\left(\frac{\sum_{i=1}^{n}p_ix_i-m\sum_{i=1}^{n}q_ix_i}{1-m}\right)$$
$$\geqslant f\left(m\sum_{i=1}^{n}q_ix_i+\sum_{i=1}^{n}p_ix_i-m\sum_{i=1}^{n}q_ix_i\right)$$
$$=f\left(\sum_{i=1}^{n}p_ix_i\right),$$

that is

$$m\left[f\left(\sum_{i=1}^{n}q_ix_i\right)-\sum_{i=1}^{n}q_if\left(x_i\right)\right]+\sum_{i=1}^{n}p_if\left(x_i\right)\geqslant f\left(\sum_{i=1}^{n}p_ix_i\right)$$

and this is equivalent to the right inequality in (JB).

To prove the left inequality in (JB) note that $Mq_i-p_i\geqslant 0,\ i=1,\dots,n$ and $\sum_{i=1}^{n}(Mq_i-p_i)=M-1>0$. Now, applying (J) once and (JR) after that, we get

$$M\left[f\left(\sum_{i=1}^{n}q_ix_i\right)-\sum_{i=1}^{n}q_if\left(x_i\right)\right]+\sum_{i=1}^{n}p_if\left(x_i\right)$$
$$=Mf\left(\sum_{i=1}^{n}q_ix_i\right)-\sum_{i=1}^{n}(Mq_i-p_i)f\left(x_i\right)$$
$$\leqslant Mf\left(\sum_{i=1}^{n}q_ix_i\right)-\sum_{j=1}^{n}(Mq_j-p_j)\cdot f\left(\frac{\sum_{i=1}^{n}(Mq_i-p_i)x_i}{\sum_{j=1}^{n}\left(Mq_j-p_j\right)}\right)$$
$$=Mf\left(\sum_{i=1}^{n}q_ix_i\right)+(1-M)f\left(\frac{M\sum_{i=1}^{n}q_ix_i-\sum_{i=1}^{n}p_ix_i}{M-1}\right)$$
$$\leqslant f\left(M\sum_{i=1}^{n}q_ix_i-M\sum_{i=1}^{n}q_ix_i+\sum_{i=1}^{n}p_ix_i\right)$$
$$=f\left(\sum_{i=1}^{n}p_ix_i\right),$$

that is

$$M\left[f\left(\sum_{i=1}^{n}q_ix_i\right)-\sum_{i=1}^{n}q_if\left(x_i\right)\right]+\sum_{i=1}^{n}p_if\left(x_i\right)\leqslant f\left(\sum_{i=1}^{n}p_ix_i\right)$$

and this is equivalent to the left inequality in (JB). □

By careful inspection of the above proof we see that in fact we proved somewhat more general result in which Dragomir's condition $q_i>0,\ i=1,\dots,n$ is eliminated:

THEOREM 1. *Assume that* $\boldsymbol{p},\boldsymbol{q}\in\mathscr{P}_n$. *If* m *and* M *are any real constants such that*

$$p_i-mq_i\geqslant 0,\quad Mq_i-p_i\geqslant 0,\quad i=1,\dots,n,$$

then

$$M\mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{q})\geqslant\mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{p})\geqslant m\mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{q}).$$

3. Bounds for the normalized Jensen functional under Jensen-Steffensen conditions

For any real n-tuple $\boldsymbol{p} = (p_1, \ldots, p_n)$ and for $k \in \{1, 2, \ldots, n\}$ let us denote $P_k := \sum_{i=1}^{k} p_i$. Let $\tilde{\mathscr{P}}_n$ denotes the set of all real n-tuples $(p_1, \ldots, p_n)$ satisfying the following Jensen-Steffensen conditions

$$0 \leqslant P_k \leqslant 1, \quad k = 1, \ldots, n-1, \quad P_n = 1. \tag{3.1}$$

Any n-tuple $\boldsymbol{p}$ from $\mathscr{P}_n$ obviously satisfies (3.1) so that $\mathscr{P}_n \subseteq \tilde{\mathscr{P}}_n$. Also it is easy to see that $\tilde{\mathscr{P}}_n$ is a convex subset of $\mathbb{R}^n$,

Let $f : I \to \mathbb{R}$ be a convex function defined on an interval $I \subseteq \mathbb{R}$. If $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ is any monotonic n-tuple and $\boldsymbol{p} = (p_1, \ldots, p_n) \in \tilde{\mathscr{P}}_n$, then $\sum_{i=1}^{n} p_i x_i \in I$ (see the proof of (JS) [4, p. 57]) and

$$\mathscr{J}_n(f, \boldsymbol{x}, \boldsymbol{p}) := \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right)$$

is well defined. Also, because of (JS) we have $\mathscr{J}_n(f, \boldsymbol{x}, \boldsymbol{p}) \geqslant 0$, for all $\boldsymbol{p} \in \tilde{\mathscr{P}}_n$.

THEOREM 2. *Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ be two n-tuples from $\tilde{\mathscr{P}}_n$. For $k \in \{1, \ldots, n\}$ denote $P_k := \sum_{i=1}^{k} p_i$, $Q_k := \sum_{i=1}^{k} q_i$. Let m and M be any real constants such that*

$$P_k - mQ_k \geqslant 0, \quad (1 - P_k) - m(1 - Q_k) \geqslant 0, \quad k = 1, \ldots, n-1 \tag{3.2}$$

and

$$MQ_k - P_k \geqslant 0, \quad M(1 - Q_k) - (1 - P_k) \geqslant 0, \quad k = 1, \ldots, n-1. \tag{3.3}$$

If $f : I \to \mathbb{R}$ is a convex function defined on an interval $I \subseteq \mathbb{R}$ and if $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ is any monotonic n-tuple, then

$$M \mathscr{J}_n(f, \boldsymbol{x}, \boldsymbol{q}) \geqslant \mathscr{J}_n(f, \boldsymbol{x}, \boldsymbol{p}) \geqslant m \mathscr{J}_n(f, \boldsymbol{x}, \boldsymbol{q}). \tag{3.4}$$

Proof. Let us first consider the case $m \geqslant 1$. Since $0 \leqslant Q_k \leqslant 1$, from (3.2) we get for all $k \in \{1, \ldots, n-1\}$

$$P_k - Q_k \geqslant P_k - mQ_k \geqslant 0,$$
$$Q_k - P_k = (1 - P_k) - (1 - Q_k) \geqslant (1 - P_k) - m(1 - Q_k) \geqslant 0,$$

which implies that $P_k = Q_k$ for all $k \in \{1, \ldots, n-1\}$. This, together with $P_n = Q_n = 1$ implies that $\boldsymbol{p} = \boldsymbol{q}$ and $\mathscr{J}_n(f, \boldsymbol{x}, \boldsymbol{q}) = \mathscr{J}_n(f, \boldsymbol{x}, \boldsymbol{p})$. Since $\boldsymbol{q} = \boldsymbol{p}$, the condition (3.2) is possible only with $m = 1$, while the condition (3.3) is possible only with $M \geqslant 1$ so that (3.4) obviously holds. Similarly, if we consider the case $M \leqslant 1$, then from (3.3) we get for all $k \in \{1, \ldots, n-1\}$

$$Q_k - P_k \geqslant MQ_k - P_k \geqslant 0,$$
$$P_k - Q_k = (1 - Q_k) - (1 - P_k) \geqslant M(1 - Q_k) - (1 - P_k) \geqslant 0,$$

which implies that $P_k = Q_k$ for all $k \in \{1,\dots,n-1\}$ and by the same argument we again conclude that $\boldsymbol{p} = \boldsymbol{q}$ and $\mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{q}) = \mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{p})$. Also the condition (3.3) is possible only with $M = 1$, while the condition (3.2) is possible only with $m \leqslant 1$ so that (3.4) obviously holds.

It remains to consider the case when $m < 1$ and $M > 1$. To prove the right inequality in (3.4) we consider the n-tuple $\boldsymbol{r} = (r_1,\dots,r_n)$ defined by $r_i := p_i - mq_i$, $i = 1,\dots,n$. For $k \in \{1,\dots,n\}$ we have

$$R_k = \sum_{i=1}^{k} r_i = P_k - mQ_k, \;\; k = 1,\dots,n-1, \;\; R_n = 1 - m > 0.$$

Now, using (3.2) we get

$$\begin{aligned} R_k &= P_k - mQ_k \geqslant 0, \\ R_n - R_k &= 1 - m - P_k + mQ_k = (1-P_k) - m(1-Q_k) \geqslant 0, \end{aligned}$$

for all $k \in \{1,\dots,n-1\}$. Now, we follow our proof of Theorem C, but instead of using (J) twice we first use (JS) and then we use (J). So we get

$$m\left[f\left(\sum_{i=1}^{n} q_i x_i\right) - \sum_{i=1}^{n} q_i f(x_i)\right] + \sum_{i=1}^{n} p_i f(x_i) \geqslant f\left(\sum_{i=1}^{n} p_i x_i\right),$$

which is equivalent to the right inequality in (3.4).

To prove the left inequality in (3.4) we consider the n-tuple $\boldsymbol{s} = (s_1,\dots,s_n)$ defined by $s_i := Mq_i - p_i$, $i = 1,\dots,n$. For $k \in \{1,\dots,n\}$ we have

$$S_k = \sum_{i=1}^{k} s_i = MQ_k - P_k, \;\; k = 1,\dots,n-1, \;\; S_n = M - 1 > 0.$$

Using (3.3) we get

$$\begin{aligned} S_k &= MQ_k - P_k \geqslant 0, \\ S_n - S_k &= M - 1 - MQ_k + P_k = M(1-Q_k) - (1-P_k) \geqslant 0, \end{aligned}$$

for all $k \in \{1,\dots,n-1\}$. We follow again our proof of Theorem C, first using (JS) instead of (J) and then using (JR). So we get

$$M\left[f\left(\sum_{i=1}^{n} q_i x_i\right) - \sum_{i=1}^{n} q_i f(x_i)\right] + \sum_{i=1}^{n} p_i f(x_i) \leqslant f\left(\sum_{i=1}^{n} p_i x_i\right)$$

and this is equivalent to the left inequality in (3.4). □

COROLLARY 1. *Let* $\boldsymbol{p} = (p_1,\dots,p_n)$ *and* $\boldsymbol{q} = (q_1,\dots,q_n)$ *be two* n*-tuples from* $\tilde{\mathscr{P}}_n$. *For* $k \in \{1,\dots,n\}$ *denote* $P_k := \sum_{i=1}^{k} p_i$, $Q_k := \sum_{i=1}^{k} q_i$. *Assume that* $0 < Q_k < 1$ *for all* $k \in \{1,\dots,n-1\}$ *and define*

$$\tilde{m} = \tilde{m}(\boldsymbol{p},\boldsymbol{q}) := \min\left\{\frac{P_k}{Q_k}, \frac{1-P_k}{1-Q_k} : k = 1,\dots,n-1\right\}, \tag{3.5}$$

$$\tilde{M} = \tilde{M}(\boldsymbol{p},\boldsymbol{q}) := \max\left\{\frac{P_k}{Q_k}, \frac{1-P_k}{1-Q_k} : k = 1,\ldots,n-1\right\}. \tag{3.6}$$

If $f : I \to \mathbb{R}$ *is a convex function defined on an interval* $I \subseteq \mathbb{R}$ *and if* $\boldsymbol{x} = (x_1,\ldots,x_n) \in I^n$ *is any monotonic* n*-tuple, then*

$$\tilde{M}\mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{q}) \geqslant \mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{p}) \geqslant \tilde{m}\mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{q}). \tag{3.7}$$

Proof. Since $0 < Q_k < 1$ for all $k \in \{1,\ldots,n-1\}$, $\tilde{m}$ and $\tilde{M}$ are well defined and obviously (3.2) and (3.3) are satisfied for $m = \tilde{m}$ and $M = \tilde{M}$. Therefore we can apply Theorem 2 to obtain (3.7). □

We can consider the uniform distribution $\boldsymbol{u} = \left(\frac{1}{n},\ldots,\frac{1}{n}\right)$ and corresponding nonweighted Jensen functional

$$\mathscr{J}_n(f,\boldsymbol{x}) := \mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{u}) = \frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

Then we can state the following special case of the above Corollary:

COROLLARY 2. *Let* $\boldsymbol{p} = (p_1,\ldots,p_n)$ *be* n*-tuple from* $\tilde{\mathscr{P}}_n$ *. For* $k \in \{1,\ldots,n\}$ *denote* $P_k := \sum_{i=1}^{k} p_i$ *and define*

$$\tilde{m}_0 := n\cdot\min\left\{\frac{P_k}{k}, \frac{1-P_k}{n-k} : k = 1,\ldots,n-1\right\},$$
$$\tilde{M}_0 := n\cdot\max\left\{\frac{P_k}{k}, \frac{1-P_k}{n-k} : k = 1,\ldots,n-1\right\}.$$

If $f : I \to \mathbb{R}$ *is a convex function defined on an interval* $I \subseteq \mathbb{R}$ *and if* $\boldsymbol{x} = (x_1,\ldots,x_n) \in I^n$ *is any monotonic* n*-tuple, then*

$$\tilde{M}_0\mathscr{J}_n(f,\boldsymbol{x}) \geqslant \mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{p}) \geqslant \tilde{m}_0\mathscr{J}_n(f,\boldsymbol{x}).$$

Proof. We apply Corollary 1 with arbitrary $\boldsymbol{p} \in \mathscr{P}_n$ and $\boldsymbol{q} = \boldsymbol{u}$. In this case we have $Q_k = \frac{k}{n}$, $k = 1,\ldots,n$ so that we get $\tilde{m} = \tilde{m}_0$ and $\tilde{M} = \tilde{M}_0$. □

Next we show that Theorem 2 in some way provides improvement of Theorem C in case when $X = \mathbb{R}$.

Denote by Π_n the set off all permutations of $(1,2,\ldots,n)$. Suppose $\pi = (\pi(1),\pi(2),\ldots,\pi(n)) \in \Pi_n$. If $\boldsymbol{a} = (a_1,a_2,\ldots,a_n)$ is any n-tuple (anywhere), then we denote

$$\boldsymbol{a}_\pi := (a_{\pi(1)},a_{\pi(2)},\ldots,a_{\pi(n)}).$$

First we prove one simple auxiliary result.

LEMMA 1. *Let* $\boldsymbol{p} = (p_1,\ldots,p_n)$ *and* $\boldsymbol{q} = (q_1,\ldots,q_n)$ *be two nonnegative* n*-tuples from* $\mathscr{P}_n$*. If* $q_i > 0$ *for all* $i \in \{1,\ldots,n\}$*, then* $\tilde{m}(\boldsymbol{p},\boldsymbol{q})$ *and* $\tilde{M}(\boldsymbol{p},\boldsymbol{q})$ *are well defined by (3.5) and (3.6) and*

$$\max_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\} \geqslant \tilde{M}(\boldsymbol{p},\boldsymbol{q}), \quad \tilde{m}(\boldsymbol{p},\boldsymbol{q}) \geqslant \min_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\}.$$

Proof. Since $q_i > 0$ for all i it is obvious that $0 < Q_k < 1$ for all $k \in \{1,\dots,n-1\}$, so that $\tilde{m}(\boldsymbol{p},\boldsymbol{q})$ and $\tilde{M}(\boldsymbol{p},\boldsymbol{q})$ are well defined by (3.5) and (3.6). Also for any $k \in \{1,\dots,n-1\}$ we can write

$$P_k := \sum_{i=1}^{k} p_i = \sum_{i=1}^{k} \frac{p_i}{q_i} q_i, \quad 1 - P_k = \sum_{i=k+1}^{n} p_i = \sum_{i=k+1}^{n} \frac{p_i}{q_i} q_i.$$

Now

$$\max_{1 \leqslant i \leqslant n} \left\{\frac{p_i}{q_i}\right\} Q_k = \max_{1 \leqslant i \leqslant n} \left\{\frac{p_i}{q_i}\right\} \sum_{i=1}^{k} q_i \geqslant P_k \geqslant \min_{1 \leqslant i \leqslant n} \left\{\frac{p_i}{q_i}\right\} \sum_{i=1}^{k} q_i = \min_{1 \leqslant i \leqslant n} \left\{\frac{p_i}{q_i}\right\} Q_k,$$

i.e.

$$\max_{1 \leqslant i \leqslant n} \left\{\frac{p_i}{q_i}\right\} \geqslant \frac{P_k}{Q_k} \geqslant \min_{1 \leqslant i \leqslant n} \left\{\frac{p_i}{q_i}\right\}.$$

Similarly we get for all $k \in \{1,\dots,n-1\}$

$$\max_{1 \leqslant i \leqslant n} \left\{\frac{p_i}{q_i}\right\} \geqslant \frac{1-P_k}{1-Q_k} \geqslant \min_{1 \leqslant i \leqslant n} \left\{\frac{p_i}{q_i}\right\}$$

and desired conclusion follows. □

REMARK 1. It is clear that inequalities stated in Lemma 1 can be strict. For example if $n = 4$, $\boldsymbol{p} = (\frac{1}{5}, \frac{2}{5}, \frac{1}{10}, \frac{3}{10})$ and $\boldsymbol{q} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, then we get

$$\max_{1 \leqslant i \leqslant n} \left\{\frac{p_i}{q_i}\right\} = \frac{8}{5} > \tilde{M}(\boldsymbol{p},\boldsymbol{q}) = \frac{6}{5}, \quad \tilde{m}(\boldsymbol{p},\boldsymbol{q}) = \frac{4}{5} > \min_{1 \leqslant i \leqslant n} \left\{\frac{p_i}{q_i}\right\} = \frac{2}{5}.$$

It is not hard to see that generally

$$\max_{1 \leqslant i \leqslant n} \left\{\frac{p_i}{q_i}\right\} = \max_{\pi \in \Pi_n} \tilde{M}(\boldsymbol{p}_\pi, \boldsymbol{q}_\pi), \quad \min_{1 \leqslant i \leqslant n} \left\{\frac{p_i}{q_i}\right\} = \min_{\pi \in \Pi_n} \tilde{m}(\boldsymbol{p}_\pi, \boldsymbol{q}_\pi).$$

THEOREM 3. *Let $f : I \to \mathbb{R}$ be a convex function defined on an interval $I \subseteq \mathbb{R}$ and let $\boldsymbol{x} = (x_1,\dots,x_n) \in I^n$ be any n-tuple. Let $\pi = (\pi(1),\pi(2),\dots,\pi(n))$ be a permutation of $(1,2,\dots,n)$ such that $\boldsymbol{x}_\pi$ is monotonic (nondecreasing or nonincreasing). If $\boldsymbol{p} = (p_1,\dots,p_n)$ and $\boldsymbol{q} = (q_1,\dots,q_n)$ are two n-tuples from $\mathscr{P}_n$ such that $q_i > 0$ for all $i \in \{1,\dots,n\}$, then*

$$\begin{aligned}\max_{1 \leqslant i \leqslant n} \left\{\frac{p_i}{q_i}\right\} \mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{q}) &\geqslant \tilde{M}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)\mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{q}) \geqslant \mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{p}) \\ &\geqslant \tilde{m}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)\mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{q}) \geqslant \min_{1 \leqslant i \leqslant n} \left\{\frac{p_i}{q_i}\right\} \mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{q}),\end{aligned}$$

where $\tilde{m}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)$ and $\tilde{M}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)$ are defined as in (3.5) and (3.6). The first and the last inequality can be strict.

Proof. Since π is chosen so that $\boldsymbol{x}_\pi$ is monotonic we can apply Corollary 1 and Lemma 1 to the n-tuples $\boldsymbol{p}_\pi$ and $\boldsymbol{q}_\pi$ to get

$$\max_{1\leqslant i\leqslant n}\left\{\frac{p_{\pi(i)}}{q_{\pi(i)}}\right\}\mathscr{J}_n(f,\boldsymbol{x}_\pi,\boldsymbol{q}_\pi)\geqslant \tilde{M}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)\mathscr{J}_n(f,\boldsymbol{x}_\pi,\boldsymbol{q}_\pi)\geqslant \mathscr{J}_n(f,\boldsymbol{x}_\pi,\boldsymbol{p}_\pi)$$
$$\geqslant \tilde{m}(\boldsymbol{p}_\pi,\boldsymbol{q}_\pi)\mathscr{J}_n(f,\boldsymbol{x}_\pi,\boldsymbol{q}_\pi)\geqslant \min_{1\leqslant i\leqslant n}\left\{\frac{p_{\pi(i)}}{q_{\pi(i)}}\right\}\mathscr{J}_n(f,\boldsymbol{x}_\pi,\boldsymbol{q}_\pi).$$

Since $\mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{p})$ doesn't change if we simultaneously permute the components of $\boldsymbol{x}$ and $\boldsymbol{p}$, we have $\mathscr{J}_n(f,\boldsymbol{x}_\pi,\boldsymbol{p}_\pi)=\mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{p})$ and $\mathscr{J}_n(f,\boldsymbol{x}_\pi,\boldsymbol{q}_\pi)=\mathscr{J}_n(f,\boldsymbol{x},\boldsymbol{q})$. Also it is obvious that $\max_{1\leqslant i\leqslant n}\left\{\frac{p_{\pi(i)}}{q_{\pi(i)}}\right\}=\max_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\}$ and $\min_{1\leqslant i\leqslant n}\left\{\frac{p_{\pi(i)}}{q_{\pi(i)}}\right\}=\min_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\}$. Therefore, the proposed inequalities indeed hold. By Remark 1 the first and the last inequality can be strict. □

4. Integral versions

There are various integral versions of Jensen's inequality. Here we recall the simplest one for Riemann-Stieltjes integral [4, p. 58]: Suppose $x:[\alpha,\beta]\to(a,b)$, where $-\infty<\alpha<\beta<\infty$ and $-\infty\leqslant a<b\leqslant\infty$, is a continuous function and $f:(a,b)\to\mathbb{R}$ is a convex function. If $\lambda:[\alpha,\beta]\to\mathbb{R}$ is any nondecreasing function such that $\lambda(\beta)\neq\lambda(\alpha)$, then

$$f\left(\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta x(t)\mathrm{d}\lambda(t)\right)\leqslant\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f(x(t))\mathrm{d}\lambda(t). \tag{JI}$$

An integral analogue of (JS) was also proved by Steffensen, but here we consider a variant given by R. P. Boas [2] (see also [4, p. 59]):

THEOREM D. (Steffensen-Boas) *Let $x:[\alpha,\beta]\to(a,b)$ be a continuous and monotonic function (either nondecreasing or nonincreasing), where $-\infty<\alpha<\beta<\infty$ and $-\infty\leqslant a<b\leqslant\infty$, and let $f:(a,b)\to\mathbb{R}$ be a convex function. If $\lambda:[\alpha,\beta]\to\mathbb{R}$ is either continuous or of bounded variation satisfying*

$$\lambda(\alpha)\leqslant\lambda(t)\leqslant\lambda(\beta)\quad \textit{for all } t\in[\alpha,\beta],\qquad \lambda(\beta)-\lambda(\alpha)>0, \tag{4.1}$$

then (JI) holds.

The condition (4.1) on λ can be regarded as a very weak version of monotonicity, but the monotonicity condition on x is very restrictive. In the same paper [2] R. P. Boas proved that we can strengthen the hypothesis on λ and correspondingly weaken the hypothesis on x so that (JI) still holds:

THEOREM E. (Boas) *Let $\lambda:[\alpha,\beta]\to\mathbb{R}$ be either continuous or of bounded variation and such that there exist $k\geqslant 2$ points $\alpha=\gamma_0<\gamma_1<\ldots<\gamma_k=\beta$ so that*

$$\begin{gathered}\lambda(\alpha)\leqslant\lambda(t_1)\leqslant\lambda(\gamma_1)\leqslant\lambda(t_2)\leqslant\ldots\leqslant\lambda(\gamma_{k-1})\leqslant\lambda(t_k)\leqslant\lambda(\beta),\\ \textit{for all } t_i\in[\gamma_{i-1},\gamma_i],\qquad i=1,\ldots,k,\qquad \lambda(\beta)-\lambda(\alpha)>0.\end{gathered} \tag{4.2}$$

If $x : [\alpha, \beta] \to (a, b)$ is a continuous function and monotonic (either nondecreasing or nonincreasing) on each of the intervals $[\gamma_{i-1}, \gamma_i]$, $i = 1, \ldots, k$, then (JI) holds for any convex function $f : (a, b) \to \mathbb{R}$.

When (JI) is considered under assumptions of Theorem D we refer it as (JSI) and call it Jensen-Steffensen integral inequality. When (JI) is considered under assumptions of Theorem E we refer it as (JSBI) and call it Jensen-Steffensen-Boas integral inequality.

There is no loss in generality if we assume $\lambda(\beta) - \lambda(\alpha) = 1$ and consider the normalized Jensen functional

$$\mathscr{J}(f, x, \lambda) := \int_{\alpha}^{\beta} f(x(t)) \mathrm{d}\lambda(t) - f\left(\int_{\alpha}^{\beta} x(t) \mathrm{d}\lambda(t)\right).$$

Under appropriate assumptions on f, x and λ, either for (JI) or for (JSI) or for (JSBI) we always have $\mathscr{J}(f, x, \lambda) \geqslant 0$.

For $-\infty < \alpha < \beta < \infty$ let $\Lambda_{[\alpha,\beta]}$ denotes the class of all functions $\lambda : [\alpha, \beta] \to \mathbb{R}$ which are either continuous or of bounded variation and satisfy the condition

$$\lambda(\alpha) \leqslant \lambda(t) \leqslant \lambda(\beta) \quad \textit{for all } t \in [\alpha, \beta], \qquad \lambda(\beta) - \lambda(\alpha) = 1.$$

Note that any nondecreasing function $\lambda : [\alpha, \beta] \to \mathbb{R}$ with $\lambda(\beta) - \lambda(\alpha) = 1$ belongs to $\Lambda_{[\alpha,\beta]}$. Also let $\tilde{\Lambda}_{[\alpha,\beta]}$ denotes the subclass of $\Lambda_{[\alpha,\beta]}$ containing each $\lambda \in \Lambda_{[\alpha,\beta]}$ which satisfies the condition (4.2) defined in Theorem E.

Now we can state the integral analogues of the results from previous section. The first one is related to (JI).

THEOREM 4. *Let λ and μ be two functions from $\Lambda_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$. Let $x : [\alpha, \beta] \to (a, b)$, $-\infty \leqslant a < b \leqslant \infty$ be a continuous function and let $f : (a, b) \to \mathbb{R}$ be a convex function.*
a) If μ is nondecreasing and if $m \geqslant 0$ is a constant such that the function $\rho : [\alpha, \beta] \to \mathbb{R}$ defined by

$$\rho(t) := \lambda(t) - m\mu(t), \qquad t \in [\alpha, \beta] \tag{4.3}$$

is also nondecreasing, then

$$\mathscr{J}(f, x, \lambda) \geqslant m\mathscr{J}(f, x, \mu). \tag{4.4}$$

b) If λ is nondecreasing and if $M > 0$ is a constant such that the function $\sigma : [\alpha, \beta] \to \mathbb{R}$ defined by

$$\sigma(t) := M\mu(t) - \lambda(t), \qquad t \in [\alpha, \beta] \tag{4.5}$$

is also nondecreasing, then

$$M\mathscr{J}(f, x, \mu) \geqslant \mathscr{J}(f, x, \lambda). \tag{4.6}$$

Proof. a) Since μ and $\rho = \lambda - m\mu$ are assumed to be nondecreasing and $m \geqslant 0$, the function $\lambda = \rho + m\mu$ is nondecreasing too. Hence $\mathscr{J}(f, x, \lambda)$ and $\mathscr{J}(f, x, \mu)$ are well defined. First we consider the case $m \geqslant 1$. Since $\lambda(\beta)-\lambda(\alpha)=\mu(\beta)-\mu(\alpha)=1$, we have

$$\rho(\beta)-\rho(\alpha)=\mu(\beta)-\mu(\alpha)-m(\lambda(\beta)-\lambda(\alpha))=1-m \leqslant 0.$$

But ρ is nondecreasing by our assumption so that we must have $\rho(\beta)-\rho(\alpha) \geqslant 0$. Hence we must have $m=1$, $\rho(t)=\lambda(t)-\mu(t)$, $t \in [\alpha,\beta]$ and $\rho(\beta)=\rho(\alpha)$. This is possible only when ρ is a constant function that is $\lambda(t)=\mu(t)+c$, $t \in [\alpha,\beta]$, for some constant c. But then the integral with respect to λ coincides with the integral with respect to μ so that $\mathscr{J}(f,x,\lambda)=\mathscr{J}(f,x,\mu)$ and (4.4).holds with equality sign.

It remains to consider the case $m<1$. In this case we have $\rho(\beta)-\rho(\alpha)=1-m>0$ and ρ is nondecreasing by our assumption so that (JI) can be applied to ρ. Hence we first apply (JI) and then we apply (J) to obtain

$$\begin{aligned}
m&\left[f\left(\int_\alpha^\beta x(t)\mathrm{d}\mu(t)\right)-\int_\alpha^\beta f(x(t))\mathrm{d}\mu(t)\right]+\int_\alpha^\beta f(x(t))\mathrm{d}\lambda(t)\\
&=mf\left(\int_\alpha^\beta x(t)\mathrm{d}\mu(t)\right)+\int_\alpha^\beta f(x(t))\mathrm{d}\rho(t)\\
&\geqslant mf\left(\int_\alpha^\beta x(t)\mathrm{d}\mu(t)\right)+(\rho(\beta)-\rho(\alpha))\cdot f\left(\frac{\int_\alpha^\beta x(t)\mathrm{d}\rho(t)}{\rho(\beta)-\rho(\alpha)}\right)\\
&=mf\left(\int_\alpha^\beta x(t)\mathrm{d}\mu(t)\right)+(1-m)f\left(\frac{\int_\alpha^\beta x(t)\mathrm{d}\lambda(t)-m\int_\alpha^\beta x(t)\mathrm{d}\mu(t)}{1-m}\right)\\
&\geqslant f\left(m\int_\alpha^\beta x(t)\mathrm{d}\mu(t)+\int_\alpha^\beta x(t)\mathrm{d}\lambda(t)-m\int_\alpha^\beta x(t)\mathrm{d}\mu(t)\right)\\
&=f\left(\int_\alpha^\beta x(t)\mathrm{d}\lambda(t)\right),
\end{aligned}$$

that is

$$m\left[f\left(\int_\alpha^\beta x(t)\mathrm{d}\mu(t)\right)-\int_\alpha^\beta f(x(t))\mathrm{d}\mu(t)\right]+\int_\alpha^\beta f(x(t))\mathrm{d}\lambda(t)\geqslant f\left(\int_\alpha^\beta x(t)\mathrm{d}\lambda(t)\right)$$

and this is equivalent to (4.4).

b) Since λ and $\sigma=M\mu-\lambda$ are assumed to be nondecreasing and $M>0$, the function $\mu=\frac{1}{M}(\sigma+\lambda)$ is nondecreasing too. Hence $\mathscr{J}(f,x,\lambda)$ and $\mathscr{J}(f,x,\mu)$ are well defined. First we consider the case $M \leqslant 1$. Then we have

$$\sigma(\beta)-\sigma(\alpha)=M(\mu(\beta)-\mu(\alpha))-(\lambda(\beta)-\lambda(\alpha))=M-1\leqslant 0$$

and $\sigma(\beta)-\sigma(\alpha)\geqslant 0$, since σ is assumed to be nondecreasing. We conclude that it must be $M=1$, $\sigma(t)=\mu(t)-\lambda(t)$, $t\in[\alpha,\beta]$ and $\sigma(\beta)=\sigma(\alpha)$. This is possible only when σ is a constant function that is $\mu(t)=\lambda(t)+c$, $t\in[\alpha,\beta]$, for some constant c. By the same argument as in the previous case a) we conclude that $\mathscr{J}(f,x,\lambda)=\mathscr{J}(f,x,\mu)$ and (4.6).holds with equality sign.

It remains to consider the case $M>1$. In this case we have $\sigma(\beta)-\sigma(\alpha)=M-1>0$ and σ is nondecreasing by our assumption so that (JI) can be applied to σ. Hence we first apply (JI) and then we apply (JR) to obtain

$$\begin{aligned} M&\left[f\left(\int_\alpha^\beta x(t)\mathrm{d}\mu(t)\right)-\int_\alpha^\beta f(x(t))\mathrm{d}\mu(t)\right]+\int_\alpha^\beta f(x(t))\mathrm{d}\lambda(t)\\ &=Mf\left(\int_\alpha^\beta x(t)\mathrm{d}\mu(t)\right)-\int_\alpha^\beta f(x(t))\mathrm{d}\sigma(t)\\ &\leqslant Mf\left(\int_\alpha^\beta x(t)\mathrm{d}\mu(t)\right)-(\sigma(\beta)-\sigma(\alpha))\cdot f\left(\frac{\int_\alpha^\beta x(t)\mathrm{d}\sigma(t)}{\sigma(\beta)-\sigma(\alpha)}\right)\\ &=Mf\left(\int_\alpha^\beta x(t)\mathrm{d}\mu(t)\right)+(1-M)f\left(\frac{M\int_\alpha^\beta x(t)\mathrm{d}\mu(t)-\int_\alpha^\beta x(t)\mathrm{d}\lambda(t)}{M-1}\right)\\ &\leqslant f\left(M\int_\alpha^\beta x(t)\mathrm{d}\mu(t)-M\int_\alpha^\beta x(t)\mathrm{d}\mu(t)+\int_\alpha^\beta x(t)\mathrm{d}\lambda(t)\right)\\ &=f\left(\int_\alpha^\beta x(t)\mathrm{d}\lambda(t)\right), \end{aligned}$$

that is

$$M\left[f\left(\int_\alpha^\beta x(t)\mathrm{d}\mu(t)\right)-\int_\alpha^\beta f(x(t))\mathrm{d}\mu(t)\right]+\int_\alpha^\beta f(x(t))\mathrm{d}\lambda(t)\leqslant f\left(\int_\alpha^\beta x(t)\mathrm{d}\lambda(t)\right)$$

and this is equivalent to (4.6). □

COROLLARY 3. *Let λ and μ be two functions from $\Lambda_{[\alpha,\beta]}$, $-\infty<\alpha<\beta<\infty$. Let $x:[\alpha,\beta]\to(a,b)$, $-\infty\leqslant a<b\leqslant\infty$ be a continuous function and let $f:(a,b)\to\mathbb{R}$ be a convex function.*
a) Assume that μ is strictly increasing and define

$$\tilde{m}=\tilde{m}(\lambda,\mu):=\inf_{\alpha<t<\beta}\left\{\inf\left\{\frac{\lambda(t)-\lambda(s)}{\mu(t)-\mu(s)}:\alpha\leqslant s\leqslant\beta,\ s\neq t\right\}\right\}.$$

If $m\geqslant 0$, then

$$\mathscr{J}(f,x,\lambda)\geqslant\tilde{m}\,\mathscr{J}(f,x,\mu). \tag{4.7}$$

b) Assume that λ is nondecreasing and that μ is strictly increasing and define

$$\tilde{M}=\tilde{M}(\lambda,\mu):=\sup_{\alpha<t<\beta}\left\{\sup\left\{\frac{\lambda(t)-\lambda(s)}{\mu(t)-\mu(s)}:\alpha\leqslant s\leqslant\beta,\ s\neq t\right\}\right\}.$$

If $M<\infty$, then

$$\tilde{M}\,\mathscr{J}(f,x,\mu)\geqslant\mathscr{J}(f,x,\lambda). \tag{4.8}$$

Proof. a) Since μ is strictly increasing it is injective and $\tilde{m}$ is well defined quantity in $[-\infty,\infty)$. If $\tilde{m} \geqslant 0$, then $\rho = \lambda - \tilde{m}\mu$ is well defined function on $[\alpha,\beta]$, By the definition of $\tilde{m}$, if $t,s \in [\alpha,\beta]$ are such that $s < t$, then $(\lambda(t)-\lambda(s))/(\mu(t)-\mu(s)) \geqslant \tilde{m}$. Since $\mu(t)-\mu(s) > 0$, this is equivalent to

$$\rho(t)-\rho(s) = \lambda(t)-\lambda(s)-\tilde{m}(\mu(t)-\mu(s)) \geqslant 0,$$

which shows that ρ is nondecreasing. Therefore we can apply Theorem 4 a) with $m = \tilde{m}$ to get (4.7).

b) Since λ is nondecreasing and μ is strictly increasing and therefore injective, $\tilde{M}$ is well defined quantity in $(0,\infty]$. If additionally $\tilde{M} < \infty$, then we first observe that $\sigma = \tilde{M}\mu - \lambda$ is well defined function on $[\alpha,\beta]$, Next by the definition of $\tilde{M}$, if $t,s \in [\alpha,\beta]$ are such that $s < t$, then $(\lambda(t)-\lambda(s))/(\mu(t)-\mu(s)) \leqslant \tilde{M}$. Since $\mu(t)-\mu(s) > 0$, this is equivalent to

$$\sigma(t)-\sigma(s) = \tilde{M}(\mu(t)-\mu(s))-(\lambda(t)-\lambda(s)) \geqslant 0,$$

which shows that σ is nondecreasing. Therefore we can apply Theorem 4 b) with $M = \tilde{M}$ to get (4.8). □

REMARK 2. If $\tilde{m} = 0$, then the inequality (4.7) is trivially fulfilled. Similarly, if $\tilde{M} = \infty$, then the inequality (4.8) is trivially fulfilled. Two simple examples which illustrate such cases are as follows:

For $[\alpha,\beta] = [0,1]$ let $\lambda(t) = t$, $\mu(t) = \sqrt{t}$, $t \in [0,1]$. Then we have $\tilde{m} = 0$, $\tilde{M} = 2$.

For $[\alpha,\beta] = [0,1]$ let $\lambda(t) = \sqrt{t}$, $\mu(t) = t$, $t \in [0,1]$. Then we have $\tilde{m} = \frac{1}{2}$, $\tilde{M} = \infty$.

As in the discrete case we can consider the uniform distribution i.e. the function $\upsilon \in \Lambda_{[\alpha,\beta]}$ defined by

$$\upsilon(t) := \tfrac{1}{\beta-\alpha}t, \qquad t \in [\alpha,\beta]$$

and corresponding nonweighted integral Jensen functional

$$\mathscr{J}(f,x) := \mathscr{J}(f,x,\upsilon) = \tfrac{1}{\beta-\alpha}\int_\alpha^\beta f(x(t))\mathrm{d}t - f\left(\tfrac{1}{\beta-\alpha}\int_\alpha^\beta x(t)\mathrm{d}t\right).$$

Then we can state the following special case of Corollary 3:

COROLLARY 4. *Let λ and μ be two functions from $\Lambda_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$. Let $x : [\alpha,\beta] \to (a,b)$, $-\infty \leqslant a < b \leqslant \infty$ be a continuous function and let $f : (a,b) \to \mathbb{R}$ be a convex function.*
a) Define

$$\tilde{m}_0 := \tilde{m}(\lambda,\upsilon) = (\beta-\alpha)\cdot \inf_{\alpha<t<\beta}\left\{\inf\left\{\tfrac{\lambda(t)-\lambda(s)}{t-s} : \alpha \leqslant s \leqslant \beta,\ s \neq t\right\}\right\}.$$

If $\tilde{m}_0 \geqslant 0$, then

$$\mathscr{J}(f,x,\lambda) \geqslant \tilde{m}_0\mathscr{J}(f,x).$$

b) Assume that λ *is nondecreasing and define*

$$\tilde{M}_0 := \tilde{M}(\lambda, \upsilon) = (\beta - \alpha) \cdot \sup_{\alpha < t < \beta} \left\{ \sup \left\{ \tfrac{\lambda(t) - \lambda(s)}{t - s} : \alpha \leqslant s \leqslant \beta,\ s \neq t \right\} \right\}.$$

If $\tilde{M}_0 < \infty$*, then*

$$\tilde{M}_0 \mathscr{J}(f, x, \mu) \geqslant \mathscr{J}(f, x, \lambda).$$

Proof. a) We apply Corollary 3 a) with arbitrary $\lambda \in \Lambda_{[\alpha,\beta]}$ and $\mu = \upsilon$. In this case we have $\mathscr{J}(f, x, \mu) = \mathscr{J}(f, x)$ and $\tilde{m} = \tilde{m}_0$.

b) We apply Corollary 3 b) with arbitrary nondecreasing $\lambda \in \Lambda_{[\alpha,\beta]}$ and $\mu = \upsilon$. In this case we have $\mathscr{J}(f, x, \mu) = \mathscr{J}(f, x)$ and $\tilde{M} = \tilde{M}_0$. □

Next we give the results related to (JSI).

THEOREM 5. *Let* λ *and* μ *be two functions from* $\Lambda_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$, *either both continuous or both of bounded variation.. Let* $x : [\alpha, \beta] \to (a, b)$, $-\infty \leqslant a < b \leqslant \infty$ *be a monotonic function (either nondecreasing or nonincreasing) and let* $f : (a, b) \to \mathbb{R}$ *be a convex function.*
a) If $m \geqslant 0$ *is a constant such that for all* $\alpha < t < \beta$

$$\lambda(t) - \lambda(\alpha) - m(\mu(t) - \mu(\alpha)) \geqslant 0, \quad \lambda(\beta) - \lambda(t) - m(\mu(\beta) - \mu(t)) \geqslant 0, \tag{4.9}$$

then

$$\mathscr{J}(f, x, \lambda) \geqslant m \mathscr{J}(f, x, \mu). \tag{4.10}$$

b) If $M > 0$ *is a constant such that for all* $\alpha < t < \beta$

$$M(\mu(t) - \mu(\alpha)) - (\lambda(t) - \lambda(\alpha)) \geqslant 0, \quad M(\mu(\beta) - \mu(t)) - (\lambda(\beta) - \lambda(t)) \geqslant 0, \tag{4.11}$$

then

$$M \mathscr{J}(f, x, \mu) \geqslant \mathscr{J}(f, x, \lambda). \tag{4.12}$$

Proof. First note that under given assumptions on x, f, λ and μ, $\mathscr{J}(f, x, \lambda)$ and $\mathscr{J}(f, x, \mu)$ are well defined and nonnegative.

a) Suppose that $m \geqslant 1$. Since $\lambda, \mu \in \Lambda_{[\alpha,\beta]}$, we have $\lambda(\alpha) = \lambda(\beta) - 1$, $\mu(\alpha) = \mu(\beta) - 1$, $\mu(t) - \mu(\alpha) \geqslant 0$ and $\mu(\beta) - \mu(t) \geqslant 0$, so that from (4.9) we get for all $\alpha < t < \beta$

$$\begin{aligned} \lambda(t) - \lambda(\alpha) - (\mu(t) - \mu(\alpha)) &\geqslant \lambda(t) - \lambda(\alpha) - m(\mu(t) - \mu(\alpha)) \geqslant 0, \\ \mu(t) - \mu(\alpha) - (\lambda(t) - \lambda(\alpha)) &= \lambda(\beta) - \lambda(t) - (\mu(\beta) - \mu(t)) \\ &\geqslant \lambda(\beta) - \lambda(t) - m(\mu(\beta) - \mu(t)) \geqslant 0. \end{aligned}$$

This implies that $\lambda(t) - \lambda(\alpha) = \mu(t) - \mu(\alpha)$ for all $\alpha < t < \beta$. This, together with $\lambda(\beta) - \lambda(\alpha) = \mu(\beta) - \mu(\alpha) = 1$ implies that

$$\lambda(t) = \mu(t) + \lambda(\alpha) - \mu(\alpha) \quad \text{for all } \alpha \leqslant t \leqslant \beta. \tag{4.13}$$

But then the integral with respect to λ coincides with the integral with respect to μ so that $\mathscr{J}(f,x,\lambda) = \mathscr{J}(f,x,\mu)$. Moreover the condition (4.9) is possible only with $m = 1$ so that (4.10) obviously holds with equality sign.

It remains to consider the case $0 \leqslant m < 1$. To prove the inequality (4.10) we consider the function $\rho : [\alpha,\beta] \to \mathbb{R}$ defined by

$$\rho(t) := \lambda(t) - m\mu(t), \quad t \in [\alpha,\beta].$$

If λ and μ are both continuous (both of bounded variation), then ρ is continuous (of bounded variation) too. Further, using (4.9) we get that for all $\alpha < t < \beta$

$$\begin{aligned}\rho(t) - \rho(\alpha) &= \lambda(t) - \lambda(\alpha) - m(\mu(t) - \mu(\alpha)) \geqslant 0,\\ \rho(\beta) - \rho(t) &= \lambda(\beta) - \lambda(t) - m(\mu(\beta) - \mu(t)) \geqslant 0.\end{aligned}$$

Also, since $\lambda, \mu \in \Lambda_{[\alpha,\beta]}$

$$\rho(\beta) - \rho(\alpha) = \lambda(\beta) - \lambda(\alpha) - m(\mu(\beta) - \mu(\alpha)) = 1 - m > 0.$$

We conclude that the normalized function $\frac{1}{1-m}\rho$ belongs to the class $\Lambda_{[\alpha,\beta]}$. Now, we follow our proof of Theorem 4 a), but instead of using (JI) in the first step we use (JSI) and then we use (J). So we get

$$m\left[f\left(\int_\alpha^\beta x(t)\mathrm{d}\mu(t)\right) - \int_\alpha^\beta f(x(t))\mathrm{d}\mu(t)\right] + \int_\alpha^\beta f(x(t))\mathrm{d}\lambda(t) \geqslant f\left(\int_\alpha^\beta x(t)\mathrm{d}\lambda(t)\right)$$

and this is equivalent to the inequality (4.10).

b) First suppose that $M \leqslant 1$. In this case we use the fact that $\lambda(\alpha) = \lambda(\beta) - 1$, $\mu(\alpha) = \mu(\beta) - 1$, $\lambda(t) - \lambda(\alpha) \geqslant 0$ and $\lambda(\beta) - \lambda(t) \geqslant 0$, so that from (4.11) we get for all $\alpha < t < \beta$

$$\begin{aligned}\mu(t) - \mu(\alpha) - (\lambda(t) - \lambda(\alpha)) &\geqslant M(\mu(t) - \mu(\alpha)) - (\lambda(t) - \lambda(\alpha)) \geqslant 0,\\ \lambda(t) - \lambda(\alpha) - (\mu(t) - \mu(\alpha)) &= \mu(\beta) - \mu(t) - (\lambda(\beta) - \lambda(t))\\ &\geqslant M(\mu(\beta) - \mu(t)) - (\lambda(\beta) - \lambda(t)) \geqslant 0\end{aligned}$$

which implies that (4.13) holds again and by the same argument as in the previous case we conclude that $\mathscr{J}(f,x,\lambda) = \mathscr{J}(f,x,\mu)$. Moreover the condition (4.11) is possible only with $M = 1$, so that (4.12) holds with equality sign.

It remains to consider the case when $M > 1$. To prove the inequality (4.12) we consider the function $\sigma : [\alpha,\beta] \to \mathbb{R}$ defined by

$$\sigma(t) := M\mu(t) - \lambda(t), \quad t \in [\alpha,\beta].$$

If λ and μ are both continuous (both of bounded variation), then σ is continuous (of bounded variation) too. Further, using (4.11) we get that for all $\alpha < t < \beta$

$$\begin{aligned}\sigma(t) - \sigma(\alpha) &= M(\mu(t) - \mu(\alpha)) - (\lambda(t) - \lambda(\alpha)) \geqslant 0,\\ \sigma(\beta) - \sigma(t) &= M(\mu(\beta) - \mu(t)) - (\lambda(\beta) - \lambda(t)) \geqslant 0.\end{aligned}$$

Also, since $\lambda, \mu \in \Lambda_{[\alpha,\beta]}$

$$\sigma(\beta) - \sigma(\alpha) = M(\mu(\beta) - \mu(\alpha)) - (\lambda(\beta) - \lambda(\alpha)) = M - 1 > 0.$$

We conclude that the normalized function $\frac{1}{M-1}\sigma$ belongs to the class $\Lambda_{[\alpha,\beta]}$. Now we follow our proof of Theorem 4 b) but instead of using (JI) in the first step we use (JSI) and then we use (JR). So we get

$$M\left[f\left(\int_\alpha^\beta x(t)\mathrm{d}\mu(t)\right) - \int_\alpha^\beta f(x(t))\mathrm{d}\mu(t)\right] + \int_\alpha^\beta f(x(t))\mathrm{d}\lambda(t) \leqslant f\left(\int_\alpha^\beta x(t)\mathrm{d}\lambda(t)\right)$$

and this is equivalent to the inequality (4.12). □

COROLLARY 5. *Let λ and μ be two functions from $\Lambda_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$, either both continuous or both of bounded variation. Let $x : [\alpha,\beta] \to (a,b)$, $-\infty \leqslant a < b \leqslant \infty$ be a monotonic function (either nondecreasing or nonincreasing) and let $f : (a,b) \to \mathbb{R}$ be a convex function. Assume that*

$$\mu(\alpha) < \mu(t) < \mu(\beta) \quad \textit{for all} \ \ \alpha < t < \beta.$$

If $\tilde{m}$ and $\tilde{M}$ are defined by

$$\tilde{m} = \tilde{m}(\lambda,\mu) := \inf\left\{\frac{\lambda(t)-\lambda(\alpha)}{\mu(t)-\mu(\alpha)}, \frac{\lambda(\beta)-\lambda(t)}{\mu(\beta)-\mu(t)} : \alpha < t < \beta\right\}, \tag{4.14}$$

$$\tilde{M} = \tilde{M}(\lambda,\mu) := \sup\left\{\frac{\lambda(t)-\lambda(\alpha)}{\mu(t)-\mu(\alpha)}, \frac{\lambda(\beta)-\lambda(t)}{\mu(\beta)-\mu(t)} : \alpha < t < \beta\right\}, \tag{4.15}$$

then

$$\tilde{M}\mathscr{J}(f,x,\mu) \geqslant \mathscr{J}(f,x,\lambda) \geqslant \tilde{m}\mathscr{J}(f,x,\mu). \tag{4.16}$$

Proof. Since $\mu(\alpha) < \mu(t) < \mu(\beta)$ *for all* $\alpha < t < \beta$, $\tilde{m}$ and $\tilde{M}$ are well defined quantities and obviously $\tilde{m} \in [0,\infty)$ and $\tilde{M} \in (0,\infty]$. Therefore the right inequality in (4.16) follows from Theorem 5 a) with $m = \tilde{m}$. If $\tilde{M} = \infty$, then the left inequality in (4.16) holds trivially, while for $\tilde{M} < \infty$ it follows from Theorem 5 b) with $M = \tilde{M}$. □

As in the previous case we can consider the uniform distribution υ and corresponding nonweighted integral Jensen functional $\mathscr{J}(f,x)$ and state the following special case of Corollary 5:

COROLLARY 6. *Let λ be a function from $\Lambda_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$. Let $x : [\alpha,\beta] \to (a,b)$, $-\infty \leqslant a < b \leqslant \infty$ be a monotonic function (either nondecreasing or nonincreasing) and let $f : (a,b) \to \mathbb{R}$ be a convex function. If $\tilde{m}_0$ and $\tilde{M}_0$ are defined by*

$$\tilde{m}_0 := \tilde{m}(\lambda,\upsilon) := (\beta-\alpha)\cdot\inf\left\{\frac{\lambda(t)-\lambda(\alpha)}{t-\alpha}, \frac{\lambda(\beta)-\lambda(t)}{\beta-t} : \alpha < t < \beta\right\},$$

$$\tilde{M}_0 := \tilde{M}(\lambda,\upsilon) := (\beta-\alpha)\cdot\sup\left\{\frac{\lambda(t)-\lambda(\alpha)}{t-\alpha}, \frac{\lambda(\beta)-\lambda(t)}{\beta-t} : \alpha < t < \beta\right\},$$

then

$$\tilde{M}_0\mathscr{J}(f,x) \geqslant \mathscr{J}(f,x,\lambda) \geqslant \tilde{m}_0\mathscr{J}(f,x).$$

Proof. We apply Corollary 5 with arbitrary $\lambda \in \Lambda_{[\alpha,\beta]}$ and $\mu = \upsilon$. In this case we have $\mathscr{J}(f,x,\mu) = \mathscr{J}(f,x)$, $\tilde{m} = \tilde{m}_0$ and $\tilde{M} = \tilde{M}_0$. □

To complete this section we give the results related to (JSBI):

THEOREM 6. *Let λ and μ be two functions from $\tilde{\Lambda}_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$, either both continuous or both of bounded variation. Let $\alpha = \gamma_0 < \gamma_1 < \ldots < \gamma_k = \beta$, $k \geqslant 2$ be a points in $[\alpha,\beta]$. Assume that $x : [\alpha,\beta] \to (a,b)$, $-\infty \leqslant a < b \leqslant \infty$ is a continuous function which is monotonic (either nondecreasing or nonincreasing) on each of the intervals $[\gamma_{i-1},\gamma_i]$, $i = 1,\ldots,k$ and that $f : (a,b) \to \mathbb{R}$ is a convex function.*

a) If $m \geqslant 0$ is a constant such that for all $\alpha < t < \beta$

$$\lambda(t) - \lambda(\gamma_{i-1}) - m(\mu(t) - \mu(\gamma_{i-1})) \geqslant 0, \quad \lambda(\gamma_i) - \lambda(t) - m(\mu(\gamma_i) - \mu(t)) \geqslant 0, \\ \text{for} \quad t \in [\gamma_{i-1},\gamma_i], \quad i = 1,\ldots,k \tag{4.17}$$

and

$$\mu(\alpha) \leqslant \mu(t_1) \leqslant \mu(\gamma_1) \leqslant \mu(t_2) \leqslant \ldots \leqslant \mu(\gamma_{k-1}) \leqslant \mu(t_k) \leqslant \mu(\beta), \\ \text{for all } t_i \in [\gamma_{i-1},\gamma_i], \quad i = 1,\ldots,k, \tag{4.18}$$

then

$$\mathscr{J}(f,x,\lambda) \geqslant m\mathscr{J}(f,x,\mu). \tag{4.19}$$

b) If $M > 0$ is a constant such that for all $\alpha < t < \beta$

$$M(\mu(t) - \mu(\gamma_{i-1})) - (\lambda(t) - \lambda(\gamma_{i-1})) \geqslant 0, \; M(\mu(\gamma_i) - \mu(t)) - (\lambda(\gamma_i) - \lambda(t)) \geqslant 0, \\ \text{for} \quad t \in [\gamma_{i-1},\gamma_i], \quad i = 1,\ldots,k \tag{4.20}$$

and

$$\lambda(\alpha) \leqslant \lambda(t_1) \leqslant \lambda(\gamma_1) \leqslant \lambda(t_2) \leqslant \ldots \leqslant \lambda(\gamma_{k-1}) \leqslant \lambda(t_k) \leqslant \lambda(\beta), \\ \text{for all } t_i \in [\gamma_{i-1},\gamma_i], \quad i = 1,\ldots,k, \tag{4.21}$$

then

$$M\mathscr{J}(f,x,\mu) \geqslant \mathscr{J}(f,x,\lambda). \tag{4.22}$$

Proof. a) We consider the function $\rho : [\alpha,\beta] \to \mathbb{R}$ defined by

$$\rho(t) := \lambda(t) - m\mu(t), \quad t \in [\alpha,\beta].$$

If λ and μ are both continuous (both of bounded variation), then ρ is continuous (of bounded variation) too. Further, it is obvious that the condition (4.17) is equivalent to the following condition

$$\rho(\alpha) \leqslant \rho(t_1) \leqslant \rho(\gamma_1) \leqslant \rho(t_2) \leqslant \ldots \leqslant \rho(\gamma_{k-1}) \leqslant \rho(t_k) \leqslant \rho(\beta), \\ \text{for all } t_i \in [\gamma_{i-1},\gamma_i], \quad i = 1,\ldots,k. \tag{4.23}$$

Moreover, since $m \geqslant 0$ and $\lambda = \rho + m\mu$, from (4.23) and (4.18) it easily follows that λ satisfies the condition (4.21). Therefore both functions λ and μ satisfy Boas' monotonicity condition with the same prescribed points $\gamma_i, \quad i = 0, 1, \ldots, k$ so that $\mathscr{J}(f, x, \lambda)$ and $\mathscr{J}(f, x, \mu)$ are well defined and nonnegative (by Theorem E) under proposed assumptions on x and f. Further, for any fixed t, $\alpha < t < \beta$ there is exactly one $j \in \{1, \ldots, k\}$ such that $t \in [\gamma_{j-1}, \gamma_j]$. Using (4.17) we get

$$\begin{aligned}\lambda(t)-&\lambda(\alpha) - m(\mu(t) - \mu(\alpha))\\ &= \lambda(t) - \lambda(\gamma_{j-1}) - m(\mu(t) - \mu(\gamma_{j-1})) + \lambda(\gamma_{j-1}) - \lambda(\gamma_{j-2})\\ &\quad - m(\mu(\gamma_{j-1}) - \mu(\gamma_{j-2})) + \ldots + \lambda(\gamma_1) - \lambda(\alpha) - m(\mu(\gamma_1) - \mu(\alpha)) \geqslant 0\end{aligned}$$

and

$$\begin{aligned}\lambda(\beta)-&\lambda(t) - m(\mu(\beta) - \mu(t))\\ &= \lambda(\beta) - \lambda(\gamma_{k-1}) - m(\mu(\beta) - \mu(\gamma_{k-1})) + \lambda(\gamma_{k-1}) - \lambda(\gamma_{k-2})\\ &\quad - m(\mu(\gamma_{k-1}) - \mu(\gamma_{k-2})) + \ldots + \lambda(\gamma_j) - \lambda(t) - m(\mu(\gamma_j) - \mu(t)) \geqslant 0.\end{aligned}$$

We conclude that the condition (4.17) implies the condition (4.9) of Theorem 5 a). Therefore, the argument for the case when $m \geqslant 1$ is quite the same as the one given in the proof of Theorem 5 a). So in this case we conclude that $m = 1$ and (4.19) trivially holds with equality sign.

It remains to discuss the case when $m < 1$. Since $\lambda(\beta) - \lambda(\alpha) = \mu(\beta) - \mu(\alpha) = 1$, we get $\rho(\beta) - \rho(\alpha) = 1 - m > 0$ and ρ satisfies the condition (4.23) so that Theorem E can be applied for integral with respect to ρ and for x and f satisfying proposed assumptions. The rest of the argument is quite the same as the one given in the proof of Theorem 4 a) with only difference that in the first step we apply (JSBI) instead of (JI). Hence, (4.19) indeed holds.

b) We consider the function $\sigma : [\alpha, \beta] \to \mathbb{R}$ defined by

$$\sigma(t) := M\mu(t) - \lambda(t), \qquad t \in [\alpha, \beta].$$

If λ and μ are both continuous (both of bounded variation), then σ is continuous (of bounded variation) too. Further, it is obvious that the condition (4.20) is equivalent to the following condition

$$\begin{gathered}\sigma(\alpha) \leqslant \sigma(t_1) \leqslant \sigma(\gamma_1) \leqslant \sigma(t_2) \leqslant \ldots \leqslant \sigma(\gamma_{k-1}) \leqslant \sigma(t_k) \leqslant \sigma(\beta),\\ \textit{for all } \; t_i \in [\gamma_{i-1}, \gamma_i], \qquad i = 1, \ldots, k.\end{gathered} \tag{4.24}$$

Moreover, since $M > 0$ and $\mu = \frac{1}{M}(\sigma + \lambda)$, from (4.24) and (4.21) it easily follows that μ satisfies the condition (4.18). By the same argument as in the above proof for case a) we conclude that $\mathscr{J}(f, x, \lambda)$ and $\mathscr{J}(f, x, \mu)$ are well defined and nonnegative (by Theorem E) under proposed assumptions on x and f. Further, similarly as we proved that the condition (4.17) implies the condition (4.9), we can prove that the condition (4.20) implies the condition (4.11) of Theorem 5 b). Therefore, the argument for the case when $M \leqslant 1$ is quite the same as the one given in the proof of Theorem 5 b). So in this case we conclude that $M = 1$ and (4.22) trivially holds with equality sign.

It remains to discuss the case when $M > 1$. Since $\lambda(\beta) - \lambda(\alpha) = \mu(\beta) - \mu(\alpha) = 1$, we get $\sigma(\beta) - \sigma(\alpha) = M - 1 > 0$ and σ satisfies the condition (4.24) so that

Theorem E can be applied for integral with respect to σ and for x and f satisfying proposed assumptions. The rest of the argument is quite the same as the one given in the proof of Theorem 4 b) with only difference that in the first step we apply (JSBI) instead of (JI). Hence, (4.22) indeed holds. □

REMARK 3. We can omit the requirement that the condition (4.18) holds from the statement of Theorem 6. a) and similarly we can omit the requirement that the condition (4.21) holds from the statement of Theorem 6. b), but then in both cases we must require that the integrals $\int_\alpha^\beta x(t)\mathrm{d}\lambda(t)$ and $\int_\alpha^\beta x(t)\mathrm{d}\mu(t)$ are in the domain of f.

COROLLARY 7. *Let λ and μ be two functions from $\tilde{\Lambda}_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$, either both continuous or both of bounded variation. Let $\alpha = \gamma_0 < \gamma_1 < \ldots < \gamma_k = \beta$, $k \geqslant 2$ be a points in $[\alpha,\beta]$. Assume that $x : [\alpha,\beta] \to (a,b)$, $-\infty \leqslant a < b \leqslant \infty$ is a continuous function which is monotonic (either nondecreasing or nonincreasing) on each of the intervals $[\gamma_{i-1},\gamma_i]$, $i = 1,\ldots,k$ and that $f : (a,b) \to \mathbb{R}$ is a convex function. Assume that λ satisfies the condition (4.21), while μ satisfies the stronger condition than (4.18) that is*

$$\mu(\alpha) < \mu(t_1) < \mu(\gamma_1) < \mu(t_2) < \ldots < \mu(\gamma_{k-1}) < \mu(t_k) < \mu(\beta),$$
$$\text{for all } \gamma_{i-1} < t_i < \gamma_i, \quad i = 1,\ldots,k.$$

If $\tilde{m}$ and $\tilde{M}$ are defined by

$$\tilde{m} = \tilde{m}(\lambda,\mu) := \min_{i=1,\ldots,k}\left\{\inf\left\{\frac{\lambda(t)-\lambda(\gamma_{i-1})}{\mu(t)-\mu(\gamma_{i-1})}, \frac{\lambda(\gamma_i)-\lambda(t)}{\mu(\gamma_i)-\mu(t)} : \gamma_{i-1} < t < \gamma_i\right\}\right\}$$

and

$$\tilde{M} = \tilde{M}(\lambda,\mu) := \max_{i=1,\ldots,k}\left\{\sup\left\{\frac{\lambda(t)-\lambda(\gamma_{i-1})}{\mu(t)-\mu(\gamma_{i-1})}, \frac{\lambda(\gamma_i)-\lambda(t)}{\mu(\gamma_i)-\mu(t)} : \gamma_{i-1} < t < \gamma_i\right\}\right\},$$

then

$$\tilde{M}\mathscr{J}(f,x,\mu) \geqslant \mathscr{J}(f,x,\lambda) \geqslant \tilde{m}\mathscr{J}(f,x,\mu). \tag{4.25}$$

Proof. Under given assumptions, $\tilde{m}$ and $\tilde{M}$ are well defined quantities and obviously $\tilde{m} \in [0,\infty)$ and $\tilde{M} \in (0,\infty]$. Therefore the right inequality in (4.25) follows from Theorem 6. a) with $m = \tilde{m}$. If $\tilde{M} = \infty$, then the left inequality in (4.16) holds trivially, while for $\tilde{M} < \infty$ it follows from Theorem 6. b) with $M = \tilde{M}$. □

Again, we can consider the uniform distribution υ and corresponding nonweighted integral Jensen functional $\mathscr{J}(f,x)$ and state the following special case of Corollary 7:

COROLLARY 8. *Let λ be a function from $\tilde{\Lambda}_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$. Let $\alpha = \gamma_0 < \gamma_1 < \ldots < \gamma_k = \beta$, $k \geqslant 2$ be a points in $[\alpha,\beta]$. Assume that $x : [\alpha,\beta] \to (a,b)$, $-\infty \leqslant a < b \leqslant \infty$ is a continuous function which is monotonic (either nondecreasing or nonincreasing) on each of the intervals $[\gamma_{i-1},\gamma_i]$, $i = 1,\ldots,k$ and that $f : (a,b) \to \mathbb{R}$ is a convex function. Assume that λ satisfies the condition (4.21) and define*

$$\tilde{m}_0 = \tilde{m}(\lambda,\upsilon) := \min_{i=1,\ldots,k}\left\{(\gamma_i - \gamma_{i-1})\inf\left\{\frac{\lambda(t)-\lambda(\gamma_{i-1})}{t-\gamma_{i-1}}, \frac{\lambda(\gamma_i)-\lambda(t)}{\gamma_i - t} : \gamma_{i-1} < t < \gamma_i\right\}\right\},$$
$$\tilde{M}_0 = \tilde{M}(\lambda,\upsilon) := \max_{i=1,\ldots,k}\left\{(\gamma_i - \gamma_{i-1})\sup\left\{\frac{\lambda(t)-\lambda(\gamma_{i-1})}{t-\gamma_{i-1}}, \frac{\lambda(\gamma_i)-\lambda(t)}{\gamma_i - t} : \gamma_{i-1} < t < \gamma_i\right\}\right\}.$$

Then

$$\tilde{M}_0 \mathscr{J}(f,x) \geqslant \mathscr{J}(f,x,\lambda) \geqslant \tilde{m}_0 \mathscr{J}(f,x).$$

Proof. We apply Corollary 7 with $\mu = \upsilon$. □

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