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A NOTE ON SINGULAR INTEGRALS ASSOCIATED WITH A VARIABLE SURFACE OF REVOLUTION

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Abstract. We prove L^p boundedness of certain singular integral operators associated with a variable surface of revolution assuming a boundedness of related lower dimensional maximal operators. The singular integrals are defined by rough kernels satisfying certain size and cancellation conditions.

1. Introduction

Let $\Omega \in L^1(S^{n-1})$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$ $(n \geqslant 2)$ with the Lebesgue surface measure $d\sigma$. We assume that

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0. \tag{1.1}$$

Define a singular integral

$$Sf(x) = \text{p.v.} \int_{\mathbb{R}^n} \Omega(y')|y|^{-n} f(x-y)\, dy$$

for $f \in \mathscr{S}(\mathbb{R}^n)$ (the Schwartz space), where $y' = y/|y|$ for $y \in \mathbb{R}^n \setminus \{0\}$. Let $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi\rangle}\, dx$ be the Fourier transform of f, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$. Then it is known that $(Sf)\hat{}\,(\xi) = m(\xi')\hat{f}(\xi)$, where

$$m(\xi') = -\int_{S^{n-1}} \Omega(\theta) \left[i\frac{\pi}{2} \operatorname{sgn}(\langle \xi', \theta\rangle) + \log|\langle \xi', \theta\rangle| \right] d\sigma(\theta).$$

By using this expression of the Fourier transform of Sf, we can show that S extends to a bounded operator on $L^2(\mathbb{R}^n)$ if

$$\sup_{\xi' \in S^{n-1}} \int_{S^{n-1}} |\Omega(\theta)| \log\left(|\langle \xi', \theta\rangle|^{-1}\right) d\sigma(\theta) < \infty. \tag{1.2}$$

By Young's inequality, (1.2) follows from the condition $\Omega \in L\log L(S^{n-1})$, where $L\log L(S^{n-1})$ is the Zygmund class of all those functions Ω on S^{n-1} which satisfy

$$\int_{S^{n-1}} |\Omega(\theta)| \log(2 + |\Omega(\theta)|)\, d\sigma(\theta) < \infty.$$

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Furthermore, if $\Omega \in L\log L(S^{n-1})$, by the method of rotations of Calderón-Zygmund (see [1]) we can show that S extends to a bounded operator on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$ (see also [18] for the best possibility of the class $L\log L(S^{n-1})$).

In the article [10], Grafakos-Stefanov considered the following condition:

$$\sup_{\xi\in S^{n-1}} \int_{S^{n-1}} |\Omega(\theta)| \left(\log\left(|\langle \theta,\xi\rangle|^{-1}\right)\right)^{\beta} d\sigma(\theta) < \infty \quad \text{for } \beta > 1, \tag{1.3}$$

which is stronger than (1.2), and proved the L^p boundedness of S defined by Ω satisfying (1.3) for a certain range of p depending on β. It has been improved by Fan-Guo-Pan [4] as follows:

THEOREM A. *Suppose that* Ω *satisfies* (1.3) *for* $\beta > 1$. *Then* S *is bounded on* $L^p(\mathbb{R}^n)$ *for* $p \in (2\beta/(2\beta-1), 2\beta)$.

For $s \geqslant 1$, let Δ_s denote the collection of measurable functions $h(t)$ on $\mathbb{R}_+ = \{t \in \mathbb{R} : t \geqslant 0\}$ satisfying

$$\|h\|_{\Delta_s} = \sup_{u>0}\left(u^{-1}\int_0^u |h(t)|^s\,dt\right)^{1/s} < \infty.$$

For $b \in \Delta_1$ and $\Omega \in L^1(S^{n-1})$ satisfying (1.1), we define a singular integral of R. Fefferman type (see [9]):

$$S_b f(x) = \text{p.v.}\int_{\mathbb{R}^n} b(|y|)\Omega(y')|y|^{-n} f(x-y)\,dy.$$

In this note, we study singular integrals associated with a variable surface of revolution. We shall prove the L^p boundedness of the singular integrals under certain conditions and as an application we shall extend Theorem A to the case of S_b on $\mathbb{R}^2$.

We write $f(x,z)$, $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$ for functions on $\mathbb{R}^{n+m}$, $n \geqslant 2$, $m \geqslant 1$. Define a singular integral with a rough kernel by

$$Tf(x,z) = \text{p.v.}\int_{\mathbb{R}^n} b(|y|)\Omega(y')|y|^{-n} f(x-y, z-\gamma(|y|,z))\,dy, \tag{1.4}$$

initially for $f \in \mathscr{S}(\mathbb{R}^{n+m})$, where $b \in \Delta_1$ and

$$\gamma(t,z) = (\gamma_1(t,z), \gamma_2(t,z), \ldots, \gamma_m(t,z))$$

is a suitable continuous mapping from $\mathbb{R}_+ \times \mathbb{R}^m$ to $\mathbb{R}^m$ such that the singular integral (1.4) exists for all (x,z).

To study the mapping property of T, we consider two lower dimensional maximal functions associated with the function γ:

$$\mu_\gamma(g)(z) = \sup_{u>0} u^{-1}\int_0^u |g(z-\gamma(t,z))|\,dt, \tag{1.5}$$

$$M_\gamma(h)(s,z) = \sup_{u>0} u^{-1}\int_0^u |h(s-t, z-\gamma(t,z))|\,dt. \tag{1.6}$$

It is known that the L^r boundedness of M_γ implies that of μ_γ (see [13]).

Let $H^1(S^{n-1})$ denote the Hardy space on S^{n-1}. The space $L\log L(S^{n-1})$ is a proper subspace of $H^1(S^{n-1})$. The following result has been shown by [7]:

THEOREM B. *Suppose that $\Omega \in H^1(S^{n-1})$ satisfies (1.1), b is bounded and M_γ is bounded on $L^r(\mathbb{R}^{m+1})$ for all $r > 1$. Then T is bounded on $L^p(\mathbb{R}^{n+m})$ for all $p \in (1,2]$.*

See [13], [8], [6], [12], [14] for the case where $\gamma(t,z)$ is independent of the z variable. We consider the singular integral T in (1.4) defined by Ω satisfying a condition similar to (1.3) and prove L^p boundedness of it for a certain range of p contained in $(1,\infty)$. Let φ be a positive, continuous function on $(0,\infty)$ such that

(1) φ is increasing,

(2) $\varphi(s)/s$ is decreasing,

(3) $\varphi(st) \leqslant c(\varphi(s)+\varphi(t))$,

(4) $\varphi(2s) \leqslant c\varphi(s)$.

A prime example of $\varphi(t)$ is the function $(\log(a+t))^\beta$, where $\beta > 0$, $a \geqslant 2$ and a may depend on β. Define a function ψ by

$$\psi(t) = \sup_{s>0} \frac{\min(1,st)}{\varphi(s)} = \frac{1}{\varphi(1/t)}, \tag{1.7}$$

where the last equality follows from (1) and (2) of the properties of φ. We consider the following two conditions on Ω:

$$\sup_{\xi\in S^{n-1}} \int_{S^{n-1}} |\Omega(\theta)|\, \varphi\left(|\langle \theta, \xi\rangle|^{-1}\right) d\sigma(\theta) < \infty, \tag{1.8}$$

$$\sup_{\xi\in S^{n-1}} \iint_{S^{n-1}\times S^{n-1}} |\Omega(\theta)\Omega(\omega)| \varphi\left(|\langle \theta-\omega, \xi\rangle|^{-1}\right) d\sigma(\theta)\, d\sigma(\omega) < \infty. \tag{1.9}$$

For $\Omega \in L^1(S^{n-1})$, the condition (1.3) is equivalent to (1.8) with $\varphi(t) = (\log(a+t))^\beta$ ($a \geqslant 2$). In Section 2, we shall see that the condition (1.8) implies (1.9) when $n = 2$.

In this note, we shall prove the following:

THEOREM 1. *Let $0 < \alpha < \frac{1}{2}$, $\frac{2}{1-2\alpha} < s$. Suppose that $b \in \Delta_s$, $\sum_{j\geqslant 1} \varphi(2^j)^{-\alpha} < \infty$, Ω satisfies (1.9) and M_γ is bounded on $L^r(\mathbb{R}^{m+1})$ for all $r > 1$. Then T is bounded on $L^p(\mathbb{R}^{n+m})$ for $s/(s(1-\alpha)-1) < p \leqslant 2$.*

We shall see that the proof of the L^2 boundedness of Theorem 1 only uses the L^r boundedness of μ_γ. Thus we have the following:

COROLLARY 1. *Let the numbers α, s and the functions φ, b, Ω be as in Theorem 1. If μ_γ is bounded on $L^r(\mathbb{R}^m)$ for all $r > 1$, then T is bounded on $L^2(\mathbb{R}^{n+m})$.*

For examples of functions $\gamma(t,z)$ for which the maximal operator μ_γ is bounded on $L^r(\mathbb{R}^m)$, see [2]. When the function γ is independent of z, we have the following result:

THEOREM 2. *Let* $s \in (1,2]$, $0 < \alpha < 1/s'$, *where* $s' = s/(s-1)$. *Let* γ *be independent of the* z *variable. Suppose that* $\sum_{j\geqslant 1} \varphi(2^j)^{-\alpha} < \infty$, $b \in \Delta_s$, Ω *satisfies* (1.9) *and* M_γ *is bounded on* $L^r(\mathbb{R}^{m+1})$ *for all* $r > 1$. *Then* T *is bounded on* $L^p(\mathbb{R}^{n+m})$ *for* $2s/((3-2\alpha)s-2) < p < 2s/((2\alpha-1)s+2)$.

When $\gamma(t) = \mathscr{P}(t)$, where $\mathscr{P}(t)$ is a polynomial mapping from $\mathbb{R}_+$ to $\mathbb{R}^m$, the boundedness of M_γ can be found in [16, pp. 476-478]. Taking $f(x,z) = g(x)h(z)$ and $\gamma = 0$ in Theorem 2, we have the following:

COROLLARY 2. *Let the numbers* α, s *and the functions* φ, b, Ω *be as in Theorem* 2. *Then* S_b *is bounded on* $L^p(\mathbb{R}^n)$ *for the same range of* p *as in Theorem* 2.

By Lemma 1 in Section 2, the condition (1.9) follows from (1.8) when $n = 2$. Therefore, by taking $\varphi(t) = (\log(a+t))^\beta$ in Theorems 1 and 2, we have the following two results.

THEOREM 3. *Let* $n = 2$. *Let* $2 < \beta$, $2\beta/(\beta-2) < s$. *Suppose that* $b \in \Delta_s$, Ω *satisfies* (1.3) *and* M_γ *is bounded on* $L^r(\mathbb{R}^{m+1})$ *for all* $r > 1$. *Then* T *is bounded on* $L^p(\mathbb{R}^{n+m})$ *for* $s\beta/(s(\beta-1)-\beta) < p \leqslant 2$.

THEOREM 4. *Let* $n = 2$. *Suppose that* $s \in (1,2]$, $s' < \beta$, $b \in \Delta_s$. *Let* γ *be independent of the* z *variable. Suppose that* Ω *satisfies* (1.3) *and* M_γ *is bounded on* $L^r(\mathbb{R}^{m+1})$ *for all* $r > 1$. *Then* T *is bounded on* $L^p(\mathbb{R}^{n+m})$ *for* $2s\beta/((3\beta-2)s-2\beta) < p < 2s\beta/((2-\beta)s+2\beta)$.

Also, Theorem 4 implies the following:

COROLLARY 3. *Let* $n = 2$. *Let the numbers* β, s *and the functions* b, Ω *be as in Theorem* 4. *Then* S_b *is bounded on* $L^p(\mathbb{R}^n)$ *for the same range of* p *as in Theorem* 4.

In [10], it was shown that there exists a function Ω which satisfies the condition (1.3) for all $\beta > 0$ but does not belong to $H^1(S^{n-1})$ and there exists an $f \in H^1(S^{n-1})$ which does not satisfy (1.3) for any $\beta > 1$.

If we combine the proof of Theorem 1 with that of Theorem B in [7], we have the following:

THEOREM 5. *Suppose that* $\Omega \in H^1(S^{n-1})$ *satisfies* (1.1), $b \in \Delta_s$ *for some* $s > 2$ *and* M_γ *is bounded on* $L^r(\mathbb{R}^{m+1})$ *for all* $r > 1$. *Then* T *is bounded on* $L^p(\mathbb{R}^{n+m})$ *for* $s' < p \leqslant 2$.

In this note, C is used to denote non-negative constants which may be different in different occurrences. In section 2 we shall give several results which will be used to prove Theorems 1–4. We shall prove Theorems 1 and 2 in Section 3. Finally, in Section 4, we shall give a related result for the Marcinkiewicz integrals.

2. Results for the proofs of Theorems 1–4

The following lemma is for the case $n=2$, which is the reason we confine ourselves to the case $n=2$ in Theorems 3, 4 and Corollary 3.

LEMMA 1. *Let $n=2$. Let φ be as in Section 1 and let $\Psi \in L^1(S^{n-1})$, $\Psi \geqslant 0$. Suppose that*

$$\sup_{\xi \in S^{n-1}} \int_{S^{n-1}} \Psi(\theta)\varphi\left(|\langle \theta, \xi \rangle|^{-1}\right) d\sigma(\theta) < \infty.$$

Then

$$\sup_{\xi \in S^{n-1}} \iint_{S^{n-1}\times S^{n-1}} \Psi(\theta)\Psi(\omega)\varphi\left(|\langle \theta - \omega, \xi \rangle|^{-1}\right) d\sigma(\theta)\, d\sigma(\omega) < \infty.$$

Proof. Put $\tilde{\Psi}(t) = \Psi(\cos t, \sin t)$. Then the condition assumed for Ψ is equivalent to

$$\begin{aligned} &\sup_{u\in\mathbb{R}} \int_0^{2\pi} \tilde{\Psi}(t)\varphi\left(|\cos t \cos u + \sin t \sin u|^{-1}\right) dt \\ &\qquad = \sup_{u\in\mathbb{R}} \int_0^{2\pi} \tilde{\Psi}(t)\varphi\left(|\cos(t-u)|^{-1}\right) dt \\ &\qquad = \sup_{u\in\mathbb{R}} \int_0^{2\pi} \tilde{\Psi}(t)\varphi\left(|\sin(t-u)|^{-1}\right) dt < \infty. \end{aligned} \tag{2.1}$$

Also, the conclusion of the lemma is equivalent to

$$\begin{aligned} &\sup_{u\in\mathbb{R}} \int_0^{2\pi}\int_0^{2\pi} \tilde{\Psi}(t)\tilde{\Psi}(s)\varphi\left(|\cos(t-u) - \cos(s-u)|^{-1}\right) dt\, ds \\ &\quad = \sup_{u\in\mathbb{R}} \int_0^{2\pi}\int_0^{2\pi} \tilde{\Psi}(t)\tilde{\Psi}(s)\varphi\left(|2\sin((t+s-2u)/2)\sin((t-s)/2)|^{-1}\right) dt\, ds < \infty. \end{aligned}$$

Thus by (3) of the properties of φ, it suffices to show that

$$\sup_{u\in\mathbb{R}} \int_0^{2\pi} \tilde{\Psi}(t)\varphi\left(|\sin(t/2-u)|^{-1}\right) dt < \infty.$$

This can be shown by applying (1), (4) of the properties of φ and (2.1) as follows:

$$\begin{aligned}\sup_{u\in\mathbb{R}}\int_0^{2\pi}\tilde{\Psi}(t)\varphi\left(|\sin(t/2-u)|^{-1}\right)dt\\ &\leqslant C\sup_{u\in\mathbb{R}}\int_0^{2\pi}\tilde{\Psi}(t)\varphi\left(|2\sin((t-u)/2)\cos((t-u)/2)|^{-1}\right)dt\\ &= C\sup_{u\in\mathbb{R}}\int_0^{2\pi}\tilde{\Psi}(t)\varphi\left(|\sin(t-u)|^{-1}\right)dt<\infty.\end{aligned}$$

This completes the proof.

For $k\in\mathbb{Z}$, let $I_k=[2^k,2^{k+1})$, where $\mathbb{Z}$ denotes the set of all integers. Then

$$T(f)(x,z)=\sum_{k=-\infty}^{\infty}T_kf(x,z),$$

where

$$T_kf(x,z)=\int_{\mathbb{R}^n}\chi_{I_k}(|y|)b(|y|)|y|^{-n}\Omega(y')f(x-y,z-\gamma(|y|,z))\,dy.$$

Let $\mathscr{F}$ be the Fourier transform acting on the x variable. Then it is easy to see that

$$\mathscr{F}(T_kf)(\xi,z)=\int\chi_{I_k}(|y|)b(|y|)|y|^{-n}\Omega(y')\mathscr{F}(f)(\xi,z-\gamma(|y|,z))e^{-2\pi i\langle y,\xi\rangle}\,dy.$$

By applying the boundedness of the maximal operator μ_γ as in [7], we have Lemmas 2 and 3 below for the estimates of $\mathscr{F}(T_kf)$.

LEMMA 2. *Let $b\in\Delta_s$ for some $s>2$. Suppose that μ_γ is bounded on $L^r(\mathbb{R}^m)$ for all $r>1$. Then*

$$\|\mathscr{F}(T_kf)(\xi,\cdot)\|_{L^2(\mathbb{R}^m)}\leqslant C|2^k\xi|\|\mathscr{F}(f)(\xi,\cdot)\|_{L^2(\mathbb{R}^m)},$$

where C is a constant independent of $k\in\mathbb{Z}$ and $\xi\in\mathbb{R}^n$.

Proof. Note that

$$\mathscr{F}(T_kf)(\xi,z)=\int_{I_k}b(t)\mathscr{F}f(\xi,z-\gamma(t,z))\int_{S^{n-1}}\Omega(\theta)e^{-2\pi it\langle\xi,\theta\rangle}\,d\sigma(\theta)\,dt/t. \tag{2.2}$$

We define $F_\xi(z)=\mathscr{F}f(\xi,z)$. Then by (1.1) and Hölder's inequality we have

$$\begin{aligned}|\mathscr{F}(T_kf)(\xi,z)|&\leqslant\int_{I_k}\left|b(t)F_\xi(z-\gamma(t,z))\int_{S^{n-1}}\Omega(\theta)\left(e^{-2\pi it\langle\xi,\theta\rangle}-1\right)d\sigma(\theta)\right|dt/t\\ &\leqslant C\|\Omega\|_1\int_{I_k}\left|b(t)F_\xi(z-\gamma(t,z))\right||\xi|\,dt\\ &\leqslant C\|\Omega\|_1\|b\|_{\Delta_s}\left|2^k\xi\right|\left(\mu_\gamma(|F_\xi|^{s'})(z)\right)^{1/s'}.\end{aligned}$$

By the $L^{2/s'}$ boundedness of μ_γ, we have

$$\left(\int_{\mathbb{R}^m} |\mathscr{F}(T_k f)(\xi,z)|^2\,dz\right)^{1/2} \leqslant C\|\Omega\|_1\|b\|_{\Delta_s}\left|2^k\xi\right|\left\|\left(\mu_\gamma(|F_\xi|^{s'})\right)^{1/s'}\right\|_{L^2(\mathbb{R}^m)}$$
$$\leqslant C\|\Omega\|_1\|b\|_{\Delta_s}\left|2^k\xi\right|\,\|F_\xi\|_{L^2(\mathbb{R}^m)},$$

which completes the proof of Lemma 2.

LEMMA 3. *Let $b\in\Delta_s$ for some $s>2$ and let Ω satisfy (1.9). Suppose that μ_γ is bounded on $L^r(\mathbb{R}^m)$ for all $r>1$. Let $1<q<2s/(s+2)$. Then*

$$\|\mathscr{F}(T_k f)(\xi,\cdot)\|_{L^2(\mathbb{R}^m)} \leqslant C\varphi\left(|2^k\xi|\right)^{-1/q'}\|\mathscr{F}(f)(\xi,\cdot)\|_{L^2(\mathbb{R}^m)},$$

where C is a constant independent of $k\in\mathbb{Z}$ and $\xi\in\mathbb{R}^n$.

Proof. For q satisfying $1<q<2s/(s+2)$, by using Hölder's inequality in (2.2), we have

$$|\mathscr{F}(T_k f)(\xi,z)| \leqslant \left(\int_{I_k}\left|b(t)F_\xi(z-\gamma(t,z))\right|^q\,dt/t\right)^{1/q}\mathscr{L}_{q',k}$$
$$\leqslant C\|b\|_{\Delta_s}\left(\mu_\gamma(|F_\xi|^{sq/(s-q)})(z)\right)^{(s-q)/(sq)}\mathscr{L}_{q',k},$$

where F_ξ is as in the proof of Lemma 2 and

$$\mathscr{L}_{r,k} = \left(\int_{I_k}\left|\int_{S^{n-1}}\Omega(\theta)e^{-2\pi it\langle\xi,\theta\rangle}\,d\sigma(\theta)\right|^r dt/t\right)^{1/r}.$$

Note that

$$\mathscr{L}_{q',k} \leqslant \|\Omega\|_1^{(q'-2)/q'}\mathscr{L}_{2,k}^{2/q'}.$$

A direct computation shows

$$\mathscr{L}_{2,k}^2 = \iint_{S^{n-1}\times S^{n-1}}\Omega(\theta)\bar{\Omega}(\omega)\left[\int_{2^k}^{2^{k+1}}\exp(-2\pi it\langle\theta-\omega,\xi\rangle)\,dt/t\right]d\sigma(\theta)\,d\sigma(\omega)$$
$$\leqslant C\iint_{S^{n-1}\times S^{n-1}}|\Omega(\theta)\Omega(\omega)|\min\left(1,|2^k\langle\theta-\omega,\xi\rangle|^{-1}\right)d\sigma(\theta)\,d\sigma(\omega)$$
$$=: I.$$

By (1.7) we have

$$I \leqslant C\iint_{S^{n-1}\times S^{n-1}}|\Omega(\theta)\Omega(\omega)|\varphi\left(|\langle\theta-\omega,\xi'\rangle|^{-1}\right)\psi(2^{-k}|\xi|^{-1})\,d\sigma(\theta)\,d\sigma(\omega)$$
$$\leqslant C\varphi(2^k|\xi|)^{-1}\iint_{S^{n-1}\times S^{n-1}}|\Omega(\theta)\Omega(\omega)|\varphi\left(|\langle\theta-\omega,\xi'\rangle|^{-1}\right)d\sigma(\theta)\,d\sigma(\omega).$$

Since Ω satisfies (1.9), we have

$$\mathscr{L}_{2,k}^2 \leqslant C\varphi(2^k|\xi|)^{-1}. \tag{2.3}$$

Therefore we have

$$|\mathscr{F}(T_k f)(\xi,z)| \leqslant C\|b\|_{\Delta_s}\varphi(2^k|\xi|)^{-1/q'}\left(\mu_\gamma(|F_\xi|^{sq/(s-q)})(z)\right)^{(s-q)/(sq)},$$

which proves Lemma 3, since

$$\left\|(\mu_\gamma(|F_\xi|^{sq/(s-q)}))^{(s-q)/(sq)}\right\|_{L^2(\mathbb{R}^m)} \leqslant C\|F_\xi\|_{L^2(\mathbb{R}^m)}.$$

When the function γ is independent of the z variable, we have the following two lemmas.

LEMMA 4. *Suppose that the function γ is independent of the z variable: $\gamma(t,z)=\gamma(t)$. Then*

$$|(T_k f)\hat{}(\xi,\omega)| \leqslant C\|b\|_{\Delta_1}\|\Omega\|_1|2^k\xi||\hat{f}(\xi,\omega)|,$$

where C is a constant independent of $k\in\mathbb{Z}$, $\xi\in\mathbb{R}^n$ and $\omega\in\mathbb{R}^m$.

Proof. It is easy to see that

$$(T_k f)\hat{}(\xi,\omega) = \hat{f}(\xi,\omega)A_k(\xi,\omega),$$

where

$$A_k(\xi,\omega) = \int \chi_{I_k}(|y|)b(|y|)\Omega(y')|y|^{-n}e^{-2\pi i(\langle y,\xi\rangle+\langle\gamma(|y|),\omega\rangle)}\,dy.$$

By (1.1) we have

$$A_k(\xi,\omega) = \int \chi_{I_k}(|y|)b(|y|)\Omega(y')|y|^{-n}e^{-2\pi i\langle\gamma(|y|),\omega\rangle}\left(e^{-2\pi i\langle y,\xi\rangle}-1\right)dy,$$

from which the conclusion immediately follows.

LEMMA 5. *Suppose that the function γ is independent of the z variable. Let $b\in\Delta_s$ for some $s\in(1,2]$ and let Ω satisfy (1.9). Then*

$$|(T_k f)\hat{}(\xi,\omega)| \leqslant C\varphi\left(|2^k\xi|\right)^{-1/s'}|\hat{f}(\xi,\omega)|,$$

where C is a constant independent of $k\in\mathbb{Z}$, $\xi\in\mathbb{R}^n$ and $\omega\in\mathbb{R}^m$.

Proof. Let $A_k(\xi,\omega)$ be as in the proof of Lemma 4 and $\mathscr{L}_{r,k}$ as in the proof of Lemma 3. By Hölder's inequality we have

$$|A_k(\xi,\omega)| \leqslant C\|b\|_{\Delta_s}\mathscr{L}_{s',k} \leqslant C\|b\|_{\Delta_s}\mathscr{L}_{2,k}^{2/s'},$$

which combined with the estimate (2.3) completes the proof of Lemma 5.

3. Proofs of Theorems 1 and 2

Let $\Phi \in C_0^\infty(\mathbb{R}^n)$ be a radial function supported in $\{\xi : 1/2 < |\xi| \leqslant 2\}$. We assume that

$$\sum_{j=-\infty}^{\infty} \Phi(2^j\xi)^2 = 1 \quad \text{for all } \xi \neq 0.$$

Let T_k be as in Section 2. Decompose

$$Tf = \sum_j \left(\sum_k S_{j+k}(T_k(S_{j+k}f)) \right) = \sum_j U_j f,$$

where the operator S_j is defined by

$$\mathscr{F}(S_j f)(\xi, z) = \mathscr{F}(f)(\xi, z)\Phi(2^j\xi)$$

and

$$U_j f = \sum_k S_{j+k}(T_k(S_{j+k}f)).$$

By the Littlewood-Paley theory we have the following:

LEMMA 6. *Let* $p \in (1,\infty)$. *Then*

$$\|\sum_k S_k f_k\|_{L^p(\mathbb{R}^{n+m})} \leqslant C \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^{n+m})}, \tag{3.1}$$

where $f_k \in \mathscr{S}(\mathbb{R}^{n+m})$ *and* $f_k = 0$ *for all but a finite number of values of* k; *also*

$$\left\| \left(\sum_k |S_k f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^{n+m})} \leqslant C\|f\|_{L^p(\mathbb{R}^{n+m})}, \tag{3.2}$$

where $f \in \mathscr{S}(\mathbb{R}^{n+m})$.

Now, we prove Theorem 1. Let $D_j = \{\xi \in \mathbb{R}^n : 2^{-j-1} \leqslant |\xi| \leqslant 2^{-j+1}\}$. Then, by (3.1) with $p = 2$ and the Plancherel theorem we have

$$\begin{aligned} \|U_j f\|_{L^2(\mathbb{R}^{n+m})}^2 &\leqslant C \sum_k \int_{\mathbb{R}^{n+m}} \left| T_k(S_{j+k}f)(y,z) \right|^2 dy\,dz \\ &= C \sum_k \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} \left| \Phi(2^{j+k}\xi) \right|^2 |\mathscr{F}(T_k f)(\xi,z)|^2 \, d\xi \right) dz \\ &\leqslant C \sum_k \int_{D_{j+k}} \left(\int_{\mathbb{R}^m} |\mathscr{F}(T_k f)(\xi,z)|^2 \, dz \right) d\xi. \end{aligned} \tag{3.3}$$

Let $j \geqslant 0$. Since the L^r boundedness of M_γ implies the L^r boundedness of μ_γ (see [13]), by Lemma 2 and (3.3) we see that

$$\begin{aligned}\|U_j f\|^2_{L^2(\mathbb{R}^{n+m})} &\leqslant C\sum_k \int_{\mathbb{R}^m}\left(\int_{D_{j+k}} |\mathscr{F}(f)(\xi,z)|^2 |2^k\xi|^2 \, d\xi\right) dz \\ &\leqslant C2^{-2j}\int_{\mathbb{R}^m}\left(\sum_k \int_{D_{j+k}} |\mathscr{F}(f)(\xi,z)|^2 \, d\xi\right) dz \\ &\leqslant C2^{-2j}\int_{\mathbb{R}^{n+m}} |\mathscr{F}(f)(\xi,z)|^2 \, d\xi \, dz.\end{aligned}$$

Thus we have

$$\|U_j f\|_{L^2(\mathbb{R}^{n+m})} \leqslant C2^{-j}\|f\|_{L^2(\mathbb{R}^{n+m})} \quad \text{for } j \geqslant 0. \tag{3.4}$$

Next, let $j < 0$. Then, using Lemma 3 and (3.3), we have

$$\|U_j f\|_{L^2(\mathbb{R}^{n+m})} \leqslant C\varphi(2^{|j|})^{-1/q'}\|f\|_{L^2(\mathbb{R}^{n+m})} \quad \text{for } 1 < q < 2s/(s+2). \tag{3.5}$$

Let $s' < r \leqslant 2$. We shall prove

$$\|U_j f\|_{L^r(\mathbb{R}^{m+n})} \leqslant C\|f\|_{L^r(\mathbb{R}^{m+n})}. \tag{3.6}$$

By interpolating between the estimates (3.4), (3.6) and between (3.5), (3.6), we complete the proof of Theorem 1, since $\|T(f)\|_p \leqslant \sum_j \|U_j f\|_p$.

By Lemma 6, to prove (3.6) it suffices to show that

$$\left\|\left(\sum_{k=-\infty}^{\infty} |T_k f_k|^2\right)^{1/2}\right\|_{L^r(\mathbb{R}^{m+n})} \leqslant C\left\|\left(\sum_{k=-\infty}^{\infty} |f_k|^2\right)^{1/2}\right\|_{L^r(\mathbb{R}^{m+n})} \quad \text{for } r \in (s',2]. \tag{3.7}$$

LEMMA 7. *For $h \in \Delta_u$, $u > 1$, $\Omega \in L^1(S^{n-1})$, let*

$$Q_h(f)(x,z) = \sup_{k\in\mathbb{Z}} \int_{2^k \leqslant |y| < 2^{k+1}} |h(|y|)\Omega(y')||y|^{-n}|f(x-y, z-\gamma(|y|,z))| \, dy.$$

Suppose that M_γ is bounded on $L^r(\mathbb{R}^{m+1})$ for all $r > 1$. Then Q_h is bounded on $L^p(\mathbb{R}^{n+m})$ for $p > u'$.

Proof. The proof is similar to that of [8, Lemma 2.4]. By Hölder's inequality we have

$$Q_h(f)(x,z) \leqslant C\|h\|_{\Delta_u}\int_{S^{n-1}} |\Omega(\theta)|\left(M_{\theta,\gamma}(|f|^{u'})(x,z)\right)^{1/u'} d\sigma(\theta),$$

where

$$M_{\theta,\gamma}(F)(x,z) = \sup_{k\in\mathbb{Z}} 2^{-k}\int_{2^k}^{2^{k+1}} |F(x-t\theta, z-\gamma(t,z))| \, dt.$$

In [7], it was proved that the boundedness of M_γ implies

$$\left\|M_{\theta,\gamma}(g)\right\|_{L^p(\mathbb{R}^{n+m})} \leqslant C\|g\|_{L^p(\mathbb{R}^{n+m})} \quad \text{for all } p > 1,$$

with C independent of $\theta \in S^{n-1}$. Therefore, if $p > u'$,

$$\begin{aligned}\|Q_h(f)\|_{L^p(\mathbb{R}^{n+m})} &\leqslant C\|h\|_{\Delta_u} \left\| \int_{S^{n-1}} |\Omega(\theta)| \left(M_{\theta,\gamma}(|f|^{u'})(x,z) \right)^{1/u'} d\sigma(\theta) \right\|_{L^p(\mathbb{R}^{n+m})} \\ &\leqslant C\|h\|_{\Delta_u} \int_{S^{n-1}} |\Omega(\theta)| \left\| M_{\theta,\gamma}(|f|^{u'}) \right\|^{1/u'}_{L^{p/u'}(\mathbb{R}^{n+m})} d\sigma(\theta) \\ &\leqslant C\|h\|_{\Delta_u} \|\Omega\|_1 \|f\|_{L^p(\mathbb{R}^{n+m})}.\end{aligned}$$

This completes the proof.

Now, we can prove (3.7) by applying the method of [7]. We have the following estimates:

$$\left\| \sup_{k\in\mathbb{Z}} |T_k f_k| \right\|_{L^r(\mathbb{R}^{n+m})} \leqslant C \left\| \sup_{k\in\mathbb{Z}} |f_k| \right\|_{L^r(\mathbb{R}^{n+m})} \quad \text{for all } r > s', \tag{3.8}$$

$$\left\| \left(\sum_{k\in\mathbb{Z}} |T_k f_k|^r \right)^{1/r} \right\|_{L^r(\mathbb{R}^{n+m})} \leqslant C \left\| \left(\sum_{k\in\mathbb{Z}} |f_k|^r \right)^{1/r} \right\|_{L^r(\mathbb{R}^{n+m})} \quad \text{for all } r > s' \tag{3.9}$$

since $\sup_{k\in\mathbb{Z}} |T_k f_k| \leqslant Q_b(\sup_{k\in\mathbb{Z}} |f_k|)$, $|T_k f_k| \leqslant Q_b(|f_k|)$ and Q_b is bounded on $L^r(\mathbb{R}^{n+m})$ for $r > s'$ by Lemma 7, where Q_b is defined as in Lemma 7 by using the functions b, Ω and γ of Theorem 1. Thus, (3.7) follows by interpolating between the estimates (3.8) and (3.9). This completes the proof of (3.7) and hence the proof of Theorem 1.

Now, we turn to the proof of Theorem 2. Let V_j be defined in the same way as U_j in the proof of Theorem 1 by using the functions b, Ω and γ satisfying the conditions assumed in Theorem 2. Then, arguing as in the proof of Theorem 1 and using Lemmas 4 and 5, we can get estimates similar to (3.4) and (3.5); that is,

$$\|V_j f\|_{L^2(\mathbb{R}^{n+m})} \leqslant C 2^{-j} \|f\|_{L^2(\mathbb{R}^{n+m})} \quad \text{for } j \geqslant 0, \tag{3.10}$$

$$\|V_j f\|_{L^2(\mathbb{R}^{n+m})} \leqslant C \varphi(2^{|j|})^{-1/s'} \|f\|_{L^2(\mathbb{R}^{n+m})} \quad \text{for } j < 0. \tag{3.11}$$

Also we have

$$\|V_j f\|_{L^r(\mathbb{R}^{m+n})} \leqslant C \|f\|_{L^r(\mathbb{R}^{m+n})} \quad \text{for } |1/r - 1/2| < 1/s'. \tag{3.12}$$

This follows from Lemma 6 and the estimate

$$\left\| \left(\sum_k |T_k f_k|^2 \right)^{1/2} \right\|_{L^r(\mathbb{R}^{n+m})} \leqslant C \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_{L^r(\mathbb{R}^{n+m})}$$

for $|1/r - 1/2| < 1/s'$, where T_k is as in the proof of Theorem 1, with everything adapted for the present case. The proof of this estimate is essentially the same as that of (3.14) of [8] (see also [5, Theorem 7.5]). Applying interpolation to the estimates (3.10)–(3.12) and noting $\|T(f)\|_p \leqslant \sum_j \|V_j f\|_p$, we complete the proof of Theorem 2.

4. Marcinkiewicz integral analog

We can also consider the Marcinkiewicz integral

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left| \int_{|y|\leqslant t} \Omega(y')|y|^{-n+1} b(|y|) f(x-y)\, dy \right|^2 t^{-3}\, dt \right)^{1/2}.$$

The following theorem related to the above operator is from Theorem 1 in [11].

THEOREM C. ([11]) *Let* $\{K_t^j\}_{-\infty<j\leqslant 0}$ *be a sequence of functions on* $\mathbb{R}^n \times (0,\infty)$ *such that* $\|K_t^j\|_1 \leqslant C2^j$, $\operatorname{supp}(K_t^j) \subset \{t2^{j-1} \leqslant |x| \leqslant t2^{j+1}\}$ *for all* $t>0$ *and* $-\infty < j \leqslant 0$, *and such that the operator*

$$\mu(f)(x) = \left(\int_0^\infty \left| \sum_{j=-\infty}^{0} K_t^j * f(x) \right|^2 t^{-1}\, dt \right)^{1/2}$$

is bounded on $L^2(\mathbb{R}^n)$. *Suppose that*

(a) *for each non-positive integer* j, *the maximal operator* $\sup_{t>0} \left| \left| K_t^j \right| * f(x) \right|$ *is bounded on* $L^p(\mathbb{R}^n)$ *with bound* $C(n,p)2^j$ *for all* $1<p<\infty$;

(b) *there exists some positive constant* $\beta > 1/2$ *such that for any non-positive integer* j *and* $t>0$

$$\left| \hat{K}_t^j(\xi) \right| \leqslant C2^j \left(\log(2+|t2^j\xi|) \right)^{-\beta}.$$

Then the operator μ *is bounded on* $L^p(\mathbb{R}^n)$ *for* $4\beta/(4\beta-1) < p < 4\beta$.

Now we write

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left| \sum_{j=-\infty}^{0} K_{\Omega,t}^j * f(x) \right|^2 t^{-1}\, dt \right)^{1/2}$$

with

$$K_{\Omega,t}^j(x) = t^{-1}|x|^{-n+1} b(|x|)\Omega(x')\chi_{\{t2^{j-1}<|x|\leqslant t2^j\}}(x).$$

For simplicity in the discussion, we only study the case $b \in \Delta_2$.

THEOREM 6. *Let* $n=2$, $b\in\Delta_2$. *If a function* Ω *is homogeneous of degree* 0 *on* $\mathbb{R}^n$ *and satisfies* (1.1) *and* (1.3) *with* $\beta > 1$, *then* μ_Ω *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $2 \leqslant p < 2\beta$. *If we further assume that* b *is bounded, then* μ_Ω *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $2\beta/(2\beta-1) < p < 2\beta$.

Proof. First, it is easy to see that

$$\|K^j_{\Omega,t}\|_1 \leqslant C2^j\|b\|_{\Delta_1}\|\Omega\|_1 \leqslant C2^j\|b\|_{\Delta_2}\|\Omega\|_1.$$

Second, using the argument in Lemmas 3 and 5, we find that

$$\left|\hat{K}^j_{\Omega,t}(\xi)\right| \leqslant C2^{j/2}\|b\|_{\Delta_2}L_{2,j,t},$$

where

$$(L_{2,j,t})^2 = \iint_{S^{n-1}\times S^{n-1}} \Omega(\theta)\bar{\Omega}(\omega)\left[t^{-1}\int_{t2^{j-1}}^{t2^j}\exp\left(-2\pi ir\langle\theta-\omega,\xi\rangle\right)dr\right]d\sigma(\theta)\,d\sigma(\omega).$$

Thus using $t2^j$ to replace 2^k in (2.3), we obtain that

$$\left|\hat{K}^j_{\Omega,t}(\xi)\right| \leqslant C2^j\left(\log(2+|t2^j\xi|)\right)^{-\beta/2}.$$

The L^2 boundedness of μ_Ω is obvious from this estimate and (1.1). We omit the detail. Therefore, by combining the results and checking the proof of Theorem C, we can get the first part of the conclusion of Theorem 6 (the boundedness of the maximal operator $\sup_{t>0}\left|\left|K^j_t\right|*f(x)\right|$ is not needed for this case). It is easy to see that the maximal operator $\sup_{t>0}\left|\left|K^j_t\right|*f(x)\right|$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C(n,p)2^j$ for all $1<p<\infty$ if b is bounded. Thus the second part of the conclusion of Theorem 6 follows from Theorem C.

We refer to [17] for related results on the Marcinkiewicz integrals (see also [15]).

REFERENCES

[1] A. P. CALDERÓ N AND A. ZYGMUND, *On singular integrals*, Amer. J. Math., **78** (1956), 289–309.

[2] A. CARBERY, A. SEEGER, S. WAINGER AND J. WRIGHT, *Classes of singular integral operators along variable lines*, J. Geometric Analysis, **9** (1999), 583–605.

[3] J. DUOANDIKOETXEA AND J. L. RUBIO DE FRANCIA, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math., **84** (1986), 541–561.

[4] D. FAN, K. GUO AND Y. PAN, *A note of a rough singular integral operator*, Math. Inequal. Appl., **2** (1999), 73–81.

[5] D. FAN AND Y. PAN, *Singular integral operators with rough kernels supported by subvarieties*, Amer. J. Math., **119** (1997), 799–839.

[6] D. FAN AND S. SATO, *Remarks on Littlewood-Paley functions and singular integrals*, J. Math. Soc. Japan, **54** (2002), 565–585.

[7] D. FAN AND S. SATO, *Singular and fractional integrals along variable surfaces*, Hokkaido Math. J., **35** (2006), 61-85.

[8] D. FAN AND Q. ZHENG, *Maximal singular integral operators along surfaces*, J. Math. Anal. Appl., **267** (2002), 746–759.

[9] R. FEFFERMAN, *A note on singular integrals*, Proc. Amer. Math. Soc., **74** (1979), 266–270.

[10] L. GRAFAKOS AND A. STEFANOV, *L^p bounds for singular integrals and maximal singular integrals with rough kernels*, Indiana Univ. Math. J., **47** (1998), 455–469.

[11] G. HU, *$L^p(\mathbb{R}^n)$ boundedness for the Marcinkiewicz integral*, Approx. Theory Appl. (N. S.), **18**, 4 (2002), 93–100.

[12] W. KIM, S. WAINGER, J. WRIGHT AND S. ZIESLER, *Singular integrals and maximal functions associated to surfaces of revolution*, Bull. London Math. Soc., **28** (1996), 291–296.

[13] S. LU, Y. PAN AND D. YANG, *Rough singular integrals associated to surfaces of revolution*, Proc. Amer. Math. Soc., **129** (2001), 2931–2940.

[14] S. SATO, *Estimates for singular integrals along surfaces of revolution*, arXiv:0809.3315v1 [math.CA], to appear in J. Aust. Math. Soc.

[15] S. SATO, *Estimates for Littlewood-Paley functions and extrapolation*, Integr. equ. oper. theory, **62** (2008), 429–440.

[16] E. M. STEIN, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993.

[17] T. WALSH, *On the function of Marcinkiewicz*, Studia Math., **44** (1972), 203–217.

[18] M. WEISS AND A. ZYGMUND, *An example in the theory of singular integrals*, Studia Math., **26** (1965), 101–111.

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