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SOME NEW HARDY TYPE INEQUALITIES WITH GENERAL KERNELS

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Abstract. We state and prove some new weighted Hardy type inequalities with an integral operator A_k defined by

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y) d\mu_2(y),$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is a general nonnegative kernel, (Ω_1, μ_1) and (Ω_2, μ_2) are measure spaces and

$$K(x) := \int_{\Omega_2} k(x,y) d\mu_2(y), x \in \Omega_1.$$

In particular, the obtained results unify and generalize most of the results of this type (including the classical ones by Hardy, Hilbert and Godunova).

1. Introduction

The classical Hardy inequality reads:

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^p dx \leqslant \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x)\, dx, \qquad p > 1, \tag{1.1}$$

where f is nonnegative function such that $f \in L^p(\mathbb{R}_+)$ and $\mathbb{R}_+ = (0, \infty)$. The almost dramatic period of research in at least 10 years until G. H. Hardy [5] stated and proved (1.1) was recently described in details in [8].

Another important inequality is the following:
If $p > 1$ and f is a nonnegative function such that $f \in L^p(\mathbb{R}_+)$, then

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy \leqslant \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right)^p \int_0^\infty f^p(y)\, dy. \tag{1.2}$$

It was early known that these inequalities are in fact equivalent. Moreover, (1.2) is sometimes called Hilbert's inequality even if Hilbert himself only considered the case

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$p=2$ (L^p spaces were not defined at that time). Sometimes, it is also referred to as Hardy–Hilbert's inequality.

We also note that (1.1) can be interpreted as the Hardy operator $H : Hf(x) := \frac{1}{x}\int_0^x f(t)\,dt$, maps L^p into L^p with the operator norm $p' = \frac{p}{p-1}$ (since, it is known that $\left(\frac{p}{p-1}\right)^p$ is the sharp constant in (1.1)). Similarly, (1.2) may be interpreted as also the operator A: $Af(y) := \int_0^\infty \frac{f(x)}{x+y}\,dx$ maps L^p into L^p with the operator norm $(\pi/\sin\pi/p)^p$.

It is now natural to generalize the operators above to the following ones:

$$H_k : H_kf(x) := \frac{1}{K(x)}\int_0^x f(t)k(x,t)\,dt, \tag{1.3}$$

where

$$K(x) := \int_0^x k(x,t)\,dt < \infty$$

and (more generally)

$$A_k : A_kf(x) := \frac{1}{K(x)}\int_0^\infty f(t)k(x,t)\,dt, \tag{1.4}$$

where now

$$K(x) := \int_0^\infty k(x,t)\,dt < \infty.$$

Here $k(x,y)$ is a general measurable and nonnegative function, a so called kernel.

One important question in the theory of Hardy type inequalities is to find necessary and sufficient conditions, when such mappings are continuous from one weighted L^p space to another weighted L^q space (so that the corresponding inequalities hold). See for example the recent books [10] and [9] and the references given there. However, without any further restrictions on the kernel the final solution is not known even if we study the simplest case $p=q$ (see Chapter 2 of the book [10]).

Another recent fundamental observation is the following one:
By putting $f(t) = g(t^{\frac{p-1}{p}})t^{\frac{-1}{p}}$ and making some obvious substitutions we find that (1.1) is equivalent to that

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(t)\,dt\right)^p \frac{dx}{x} \leqslant \int_0^\infty g^p(x)\frac{dx}{x}. \tag{1.5}$$

Obviously, the proof of (1.5) consists of a standard application of Jensen's inequality and the Fubini theorem. Note that (1.5) holds also for $p=1$ (with equality) while (1.1) has no meaning for $p=1$. Of course this proof also shows that the following more general inequality

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} \leqslant \int_0^\infty \Phi(f(x))\,\frac{dx}{x} \tag{1.6}$$

holds for each convex function Φ on the interval I with $\operatorname{Im} f \subseteq I$. This observation can be found in the recent papers [7] by Kaijser et al. but was known even before, see e.g. Godunova [3].

Guided by these results we will in this paper prove some new results for the Hardy type operator A_k with a general kernel k. Our results are only sufficient but we study both the case $p = q$ and also some cases when $p \neq q$. We point out that our results unify and generalize most results of this type in the literature we know (including the classical ones by Hardy, Hilbert and Godunova).

This paper is organized as follows: Some previous recent results we compare with are presented in Section 2, together with some other preliminaries. The main results are presented, discussed and proved in Section 3 and Section 4 is reserved for some concluding remarks and examples.

2. Preliminaries

Throughout this paper, all measures are assumed to be positive, all functions are assumed to be positive and measurable and expressions of the form $0 \cdot \infty$, $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are taken to be equal to zero. Moreover, by a weight $u = u(x)$ we mean a nonnegative measurable function on the actual interval or more general set.

The following results was recently proved by Kaijser et al. [6]:

THEOREM 2.1. *Let u be a weight function on $(0,b)$, $0 < b \leqslant \infty$, and let $k(x,y) \geqslant 0$ on $(0,b) \times (0,b)$. Assume that $\frac{k(x,y)u(x)}{xK(x)}$ is locally integrable on $(0,b)$ for each fixed $y \in (0,b)$ and define v by*

$$v(y) = y \int_y^b \frac{k(x,y)}{K(x)} u(x) \frac{dx}{x} < \infty, y \in (0,b).$$

If Φ is a positive and convex function on (a,c), $-\infty \leqslant a < c \leqslant \infty$, then

$$\int_0^b \Phi(H_k f(x)) u(x) \frac{dx}{x} \leqslant \int_0^b \Phi(f(x)) v(x) \frac{dx}{x}, \tag{2.1}$$

for all f with $a < f(x) < c$, $0 \leqslant x \leqslant b$, where H_k is defined by (1.3).

In the same paper the dual operator $H_{\bar{k}}$, defined by

$$H_{\bar{k}} f(x) := \frac{1}{\bar{K}(x)} \int_x^\infty k(x,y) f(y) dy \tag{2.2}$$

where $\bar{K}(x) = \int_x^\infty k(x,y)dy < \infty$, was studied and the following result was proved:

THEOREM 2.2. *For $0 \leqslant b < \infty$, let u be a weight function such that $\frac{k(x,y)u(x)}{x\bar{K}(x)}$ is locally integrable on (b,∞) for every fixed $y \in (b,\infty)$. Let the function v be defined by*

$$v(y) = y \int_b^y \frac{k(x,y)}{\bar{K}(x)} u(x) \frac{dx}{x} < \infty, y \in (b,\infty).$$

If Φ *is a positive and convex function on* (a,c), $-\infty \leqslant a < c \leqslant \infty$, *then*

$$\int_b^{\infty} \Phi(H_{\bar{k}} f(x)) u(x) \frac{dx}{x} \leqslant \int_b^{\infty} \Phi(f(x)) v(x) \frac{dx}{x}, \tag{2.3}$$

for all f *with* $a < f(x) < c$, $0 \leqslant x \leqslant b$, *where* $H_{\bar{k}}$ *is defined by* (2.2).

The most general result so far for the operator H_k (which also involves cases $p \neq q$ mentioned in the introduction) is the following by Kaijser et. al [6, Theorem 4.4]:

THEOREM 2.3. *Let* $1 < p \leqslant q < \infty$, $0 < b \leqslant \infty$, $s \in (1,p)$, *let* Φ *be a convex and strictly monotone function on* $I = (a,c)$, $-\infty \leqslant a < c \leqslant \infty$, *let* H_k *be defined by* (1.3) *and let* $u(x)$ *and* $v(x)$ *be weight functions on* $[0,b]$. *Then the inequality*

$$\left(\int_0^b [\Phi(H_k f(x))]^q u(x) \frac{dx}{x} \right)^{\frac{1}{q}} \leqslant C \left(\int_0^b \Phi^p(f(x)) v(x) \frac{dx}{x} \right)^{\frac{1}{p}} \tag{2.4}$$

holds for some finite constant C *and all functions* f *such that* $\operatorname{Im} f \subseteq I$ *if*

$$A(s) := \sup_{0<t\leqslant b} \left(\int_t^b \left(\frac{k(x,t)}{K(x)} \right)^q u(x) V(x)^{\frac{q(p-s)}{p}} \frac{dx}{x} \right)^{\frac{1}{q}} V(t)^{\frac{s-1}{p}} < \infty$$

holds, where $V(t) := \int_0^t v^{1-p'}(x) x^{p'-1} dx$. *Moreover, if* C *is the best constant in* (2.4), then

$$C \leqslant \inf_{1<s<p} \left(\frac{p-1}{p-s} \right)^{\frac{1}{p'}} A(s).$$

We also mention the following early multidimensional result in this direction by Godunova [4] (see also [15, Chapter VIII, p. 233]):

THEOREM 2.4. *Let* $K(\mathbf{t})$ *be defined on* $V_{\mathbf{t}} = \{\mathbf{t} = (t_1,...,t_n) : 0 < t_i < \infty, i = 1,...,n\}$ *with* $\int_{V_{\mathbf{t}}} K(\mathbf{t}) dV_{\mathbf{t}} = 1$, *and let* $V_{\mathbf{x}}$ *and* $V_{\mathbf{y}}$ *be defined similarly. Let* $\Phi(u)$ *be a nonnegative convex function for* $u \geqslant 0$ *and* f *be such that* $f(\mathbf{y}) \geqslant 0$ *for* $\mathbf{y} \in V_{\mathbf{y}}$, $f \neq 0$, *and* $\Phi(f(x))/(x_1...x_n)$ *is integrable on* $V_{\mathbf{x}}$. *Then*

$$\int_{V_{\mathbf{x}}} \frac{1}{x_1 \cdots x_n} \Phi\left(\frac{1}{x_1 \cdots x_n} \int_{V_{\mathbf{y}}} K\left(\frac{y_1}{x_1}, ..., \frac{y_n}{x_n} \right) f(y_1,...,y_n) dV_{\mathbf{y}} \right) dV_{\mathbf{x}}$$
$$\leqslant \int_{V_{\mathbf{x}}} \frac{\Phi(f(\mathbf{x}))}{x_1 \cdots x_n} dV_{\mathbf{x}}. \tag{2.5}$$

For our further discussions we also mention the following recent result by Oguntuase et. al [12]:

THEOREM 2.5. *Let* $\mathbf{b} \in (0,\infty]$, $-\infty \leqslant a < c \leqslant \infty$ *and let* Φ *be a positive function on* $[a,c]$. *Suppose that the weight function* u *defined on* $(\mathbf{0},\mathbf{b})$ *is nonnegative such that* $\frac{u(x_1,\dots,x_n)}{x_1^2\cdots x_n^2}$ *is locally integrable on* $(\mathbf{0},\mathbf{b})$ *and the weight function* v *is defined by*

$$v(t_1,\dots,t_n) = t_1\cdots t_n \int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n} \frac{u(x_1,\dots,x_n)}{x_1^2\cdots x_n^2} dx_1\cdots dx_n, \mathbf{t} \in (\mathbf{0},\mathbf{b}).$$

(i) If Φ *is convex, then*

$$\int_0^{b_1}\cdots\int_0^{b_n} u(x_1,\dots,x_n)\Phi\left(\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\dots,t_n)dt_1\cdots dt_n\right)\frac{dx_1\cdots dx_n}{x_1\cdots x_n}$$
$$\leqslant \int_0^{b_1}\cdots\int_0^{b_n} v(x_1,\dots,x_n)\Phi\left(f(x_1,\dots,x_n)\right)\frac{dx_1\cdots dx_n}{x_1\cdots x_n}$$

holds for every function f *on* $(\mathbf{0},\mathbf{b})$ *such that* $a < f(x_1,\dots,x_n) < c$.

(ii) If Φ *is concave, then*

$$\int_0^{b_1}\cdots\int_0^{b_n} u(x_1,\dots,x_n)\Phi\left(\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\dots,t_n)dt_1\cdots dt_n\right)\frac{dx_1\cdots dx_n}{x_1\cdots x_n}$$
$$\geqslant \int_0^{b_1}\cdots\int_0^{b_n} v(x_1,\dots,x_n)\Phi\left(f(x_1,\dots,x_n)\right)\frac{dx_1\cdots dx_n}{x_1\cdots x_n}$$

holds for every function f *on* $(\mathbf{0},\mathbf{b})$ *such that* $a < f(x_1,\dots,x_n) < c$.

REMARK 2.1. Also the obvious dual result was formulated and proved in [12]. For further developments in this directions even with a general kernel see [13] and [14]. See also our final Remark 4.3 in this paper.

3. The main results

In the sequel let $(\Omega_1,\Sigma_1,\mu_1)$, $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces and let A_k from (1.4) be generalized as follows:

$$A_k f(x) := \frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)d\mu_2(y), \tag{3.1}$$

where $f:\Omega_2 \to \mathbb{R}$ is measurable, $k:\Omega_1\times\Omega_2 \to \mathbb{R}$ is a measurable and nonnegative kernel and

$$K(x) := \int_{\Omega_2} k(x,y)d\mu_2(y) < \infty, x \in \Omega_1. \tag{3.2}$$

Our first result reads:

THEOREM 3.1. *Let* u *be a weight function,* $k(x,y) \geqslant 0$. *Assume that* $\frac{k(x,y)}{K(x)}u(x)$ *is locally integrable on* Ω_1 *for each fixed* $y \in \Omega_2$. *Define* v *by*

$$v(y) := \int_{\Omega_1} \frac{k(x,y)}{K(x)}u(x)d\mu_1(x) < \infty.$$

If Φ is a convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} \Phi(A_k f(x))u(x)d\mu_1(x) \leqslant \int_{\Omega_2} \Phi(f(y))v(y)d\mu_2(y) \tag{3.3}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $\operatorname{Im} f \subseteq I$, *where A_k is defined by* (3.1)–(3.2).

Proof. We must first prove that $A_k f(x) \in I$, for all $x \in \Omega_1$. The motivation for this is that $A_k f(x)$ is simply a generalized mean and since $f(y) \in I$ for all $y \in \Omega_2$ (by assumption) also the mean $A_k f(x) \in I$. We also include a more formal proof of this fact:
Assume that there exists $x \in \Omega_1$ such that $A_k f(x) \notin I$. Considering that I is an interval in $\mathbb{R}$ and $f(y) \in I$ we have $A_k f(x) - f(y) > 0$ or $A_k f(x) - f(y) < 0$ for all $y \in \Omega_2$. Now, define $h : \Omega_2 \to \mathbb{R}$, $h(z) = A_k f(x) - f(z)$. Then $h > 0$ or $h < 0$, that is h is strictly positive or strictly negative, so is $k(x,z)h(z) > 0$ for μ_2-a.e. $z \in \Omega_2$ or $k(x,z)h(z) < 0$ for μ_2-a.e. $z \in \Omega_2$. Hence, multiplying $h(z)$ by $k(x,z)$, then integrating it over Ω_2 we get that

$$L := \int_{\Omega_2} A_k f(x)k(x,z)d\mu_2(z) - \int_{\Omega_2} k(x,z)f(z)d\mu_2(z) \neq 0.$$

On the other hand, by (3.1) we see that $L = 0$ and this contradiction shows that $A_k f(x) \in I$, for all $x \in \Omega_1$. Note that if $A_k f(x)$ is an endpoint of I for some $x \in \Omega_1$ (in case when I is not an open interval), then h (or $-h$) will be a nonnegative function whose integral over Ω_2, with respect to measure μ_2, is equal to 0. Therefore, $h = 0$, that is, $f(y) = A_k f(x)$ holds for μ_2 -a.e. $y \in \Omega_2$.
Now, let us prove the inequality (3.3). By using Jensen's inequality and the Fubini theorem we find that

$$\begin{aligned}
\int_{\Omega_1} \Phi\left(A_k f(x)\right) u(x)d\mu_1(x) &= \int_{\Omega_1} \Phi\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)d\mu_2(y)\right) u(x)d\mu_1(x) \\
&\leqslant \int_{\Omega_1} \frac{1}{K(x)}\left(\int_{\Omega_2} k(x,y)\Phi(f(y))d\mu_2(y)\right) u(x)d\mu_1(x) \\
&= \int_{\Omega_2} \Phi(f(y))\left(\int_{\Omega_1} \frac{k(x,y)}{K(x)}u(x)d\mu_1(x)\right) d\mu_2(y) \\
&= \int_{\Omega_2} \Phi(f(y))v(y)d\mu_2(y)
\end{aligned}$$

and the proof is complete. □

EXAMPLE 3.1. By applying Theorem 3.1 with $\Omega_1 = \Omega_2 = (0,\infty)$ and $k(x,y) = 1$, $0 \leqslant y \leqslant x$, $k(x,y) = 0$, $y > x$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$ and $u(x) = \frac{1}{x}$ (so that $v(y) = \frac{1}{y}$), then we obtain (1.6) which, in its turn, is equivalent to the original Hardy inequality (1.1) when $\Phi(u) = u^p, p > 1$. □

EXAMPLE 3.2. Let $\Omega_1 = \Omega_2 = (0,\infty)$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesque measures dx and dy, respectively, let $k(x,y) = \frac{(\frac{y}{x})^{-1/p}}{x+y}$, $p > 1$ and $u(x) = \frac{1}{x}$. Then $K(x) = K = \frac{\pi}{\sin(\pi/p)}$ and $v(y) = \frac{1}{y}$. Let $\Phi(u) = u^p$ and inequality (3.3) reads:

$$K^{-p}\int_0^\infty \left(\int_0^\infty \left(\frac{y}{x}\right)^{-1/p} \frac{f(y)}{x+y} dy\right)^p \frac{dx}{x} = K^{-p}\int_0^\infty \left(\int_0^\infty \frac{f(y)}{x+y} y^{-1/p} dy\right)^p dx$$
$$\leqslant \int_0^\infty f^p(y)\frac{dy}{y}$$

Replace $f(t)t^{-1/p}$ with $f(t)$ and we get Hilbert's inequality (1.2). □

EXAMPLE 3.3. Let $\Omega_1 = \Omega_2 = (0,b)$, $0 < b \leqslant \infty$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesque measures dx and dy, respectively, and let $k(x,y) = 0$ for $x < y \leqslant b$. Then A_k coincides with the operator H_k defined by (1.3) and if also $u(x)$ is replaced by $u(x)/x$ and $v(x)$ by $v(x)/x$, then (3.3) coincides with (2.1) and we see that Theorem 2.1 is a special case of Theorem 3.1. □

EXAMPLE 3.4. By arguing as in Example 3.3 but $\Omega_1 = \Omega_2 = (b,\infty)$, $0 \leqslant b < \infty$ and with kernels such that $k(x,y) = 0$ for $b \leqslant y < x$ we find that now (3.3) coincides with (2.3) so that also Theorem 2.2 is a special case of Theorem 3.1. □

Next we shall point out that also Theorem 2.4 is a special case of Theorem 3.1. We use notation $\frac{\mathbf{y}}{\mathbf{x}} = \left(\frac{y_1}{x_n}, \dots, \frac{y_n}{x_n}\right)$ and consider the case with the kernel k of the type $k(\mathbf{x},\mathbf{y}) = K\left(\frac{\mathbf{y}}{\mathbf{x}}\right)$.

COROLLARY 3.1. *Let $V_{\mathbf{x}}$ be defined as in Theorem 2.4 and $v(\mathbf{x})$, $u(\mathbf{x})$ be weights such that $\frac{K\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{K_0(\mathbf{x})}u(\mathbf{x})$ is locally integrable on $V_{\mathbf{x}}$, where*

$$K_0(\mathbf{x}) := x_1 \cdots x_n \int_{V_{\mathbf{t}}} K(\mathbf{t})dV_{\mathbf{t}}$$

and

$$v(\mathbf{y}) := \int_{V_{\mathbf{x}}} \frac{K\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{K_0(\mathbf{x})} u(\mathbf{x})dV_{\mathbf{x}} < \infty.$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{V_{\mathbf{x}}} \Phi\left(\frac{1}{K_0(\mathbf{x})}\int_{V_{\mathbf{y}}} K\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{y})dV_{\mathbf{y}}\right) u(\mathbf{x})dV_{\mathbf{x}} \leqslant \int_{V_{\mathbf{y}}} \Phi(f(\mathbf{y}))v(\mathbf{y}))dV_{\mathbf{y}} \qquad (3.4)$$

holds for all measurable functions $f : V_{\mathbf{y}} \to \mathbb{R}$ such that $\operatorname{Im} f \subseteq I$.

Proof. First we note

$$\int_{V_{\mathbf{y}}} K\left(\frac{\mathbf{y}}{\mathbf{x}}\right) dV_{\mathbf{y}} = [y_1 = t_1x_1, \dots, y_n = t_nx_n] = K_0(\mathbf{x}).$$

Hence, by just applying (3.3) with $\Omega_1 = V_{\mathbf{x}}, \Omega_2 = V_{\mathbf{y}}$ in the current situation we find that (3.4) holds and the proof is complete. □

EXAMPLE 3.5. Let $1=\int_{V_{\mathbf{t}}} K(\mathbf{t})dV_{\mathbf{t}}$. By using Corollary 3.1 with $I=\mathbb{R}_+$, $u(\mathbf{x})=1/(x_1\cdots x_n)$ we find that

$$v(\mathbf{y})=\int_{V_{\mathbf{x}}}\frac{K\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{x_1^2\cdots x_n^2}dV_{\mathbf{x}}=\left[x_1=\frac{y_1}{t_1},\dots,x_n=\frac{y_n}{t_n}\right]$$
$$=\int_{V_{\mathbf{t}}}\frac{1}{y_1\cdots y_n}K(\mathbf{t})dV_{\mathbf{t}}=\frac{1}{y_1\cdots y_n},$$

which shows that Corollary 3.1 is a genuine generalization of Theorem 2.4. □

We shall continue by stating a somewhat more general theorem, which is of a type described in Theorem 2.3 but for general measures. More exactly, we state the following generalization of Theorem 3.1:

THEOREM 3.2. *Let* $0<p\leqslant q<\infty$ *and let the assumptions in Theorem* 3.1 *be satisfied but now with*

$$v(y):=\left(\int_{\Omega_1}\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}u(x)d\mu_1(x)\right)^{\frac{p}{q}}<\infty. \tag{3.5}$$

If Φ *is a positive convex function on the interval* $I\subseteq\mathbb{R}$*, then the inequality*

$$\left(\int_{\Omega_1}[\Phi(A_kf(x))]^{\frac{q}{p}}u(x)d\mu_1(x)\right)^{\frac{1}{q}}\leqslant\left(\int_{\Omega_2}\Phi(f(y))v(y)d\mu_2(y)\right)^{\frac{1}{p}} \tag{3.6}$$

holds for all measurable functions $f:\Omega_2\to R$*, such that* $\operatorname{Im} f\subseteq I$.

Proof. As in the proof of Theorem 3.1 we first note that $A_kf(x)\in I$, for all $x\in\Omega_1$. Moreover, by using Jensen's inequality and then Minkowski's general integral inequality we find that

$$\left(\int_{\Omega_1}[\Phi(A_kf(x))]^{\frac{q}{p}}u(x)d\mu_1(x)\right)^{\frac{1}{q}}$$
$$=\left(\int_{\Omega_1}\left[\Phi\left(\frac{1}{K(x)}\int_{\Omega_2}k(x,y)f(y)d\mu_2(y)\right)\right]^{\frac{q}{p}}u(x)d\mu_1(x)\right)^{\frac{1}{q}}$$
$$\leqslant\left(\int_{\Omega_1}\left[\frac{1}{K(x)}\int_{\Omega_2}k(x,y)\Phi(f(y))d\mu_2(y)\right]^{\frac{q}{p}}u(x)d\mu_1(x)\right)^{\frac{1}{q}}$$
$$\leqslant\left(\int_{\Omega_2}\Phi(f(y))\left(\int_{\Omega_1}\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}u(x)d\mu_1(x)\right)^{\frac{p}{q}}d\mu_2(y)\right)^{\frac{1}{p}}$$
$$=\left(\int_{\Omega_2}\Phi(f(y))v(y)d\mu_2(y))\right)^{\frac{1}{p}}$$

and the proof is complete. □

For the case $p=q$ we obtain Theorem 3.1 and as expected by applying Theorem 3.2 we obtain the following further generalization of the Godunova result:

COROLLARY 3.2. *Let* $0<p\leqslant q<\infty$ *and let the assumptions in Corollary* 3.1 *be satisfied with* v *defined by*

$$v(\mathbf{y})=\left(\int_{V_{\mathbf{x}}}\left(\frac{K\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{K_0(\mathbf{x})}\right)^{\frac{q}{p}}u(\mathbf{x})dV_{\mathbf{x}}\right)^{\frac{p}{q}}.$$

If Φ *is a convex function on an interval* $I\subseteq\mathbb{R}$, *then the inequality*

$$\left(\int_{V_{\mathbf{x}}}\left[\Phi\left(\frac{1}{K(\mathbf{x})}\int_{V_{\mathbf{y}}}K\left(\frac{\mathbf{y}}{\mathbf{x}}\right)f(\mathbf{y})dV_{\mathbf{y}}\right)\right]^{\frac{q}{p}}u(\mathbf{x})dV_{\mathbf{x}}\right)^{\frac{1}{q}}\leqslant\left(\int_{V_{\mathbf{y}}}\Phi(f(\mathbf{y}))v(\mathbf{y}))dV_{\mathbf{y}}\right)^{\frac{1}{p}} \tag{3.7}$$

holds for all measurable functions holds for all measurable functions $f:V_{\mathbf{y}}\to\mathbb{R}$ *such that* $\operatorname{Im}f\subseteq I$.

Proof. The proof only consists of obvious modifications in the proof of Corollary 3.1 so we omit the details. □

EXAMPLE 3.6. By using Theorem 3.2 with $\Omega_1=\Omega_2=(0,b)$, $0<b\leqslant\infty$, $k(x,y)=0$ for $x<y<b$, $u(x)$ replaced by $u(x)/x$ and $v(y)$ replaced by $v(y)/y$ we obtain the inequality

$$\left(\int_0^b[\Phi(H_kf(x))]^{\frac{q}{p}}u(x)\frac{d\mu_1(x)}{x}\right)^{\frac{1}{q}}\leqslant\left(\int_0^b\Phi(f(y))v(y)\frac{d\mu_2(y)}{y}\right)^{\frac{1}{p}},$$

where $v(y)$ is defined by (3.5). For Φ replaced by Φ^p, $1<p\leqslant q<\infty$ (Φ^p is convex function) this inequality is similar to (2.4). However, these results are not comparable but we conjecture that Theorem 2.3 can be generalized also to the case with general measures even to a multidimensional setting. □

We finish this Section by stating the following useful fact:

REMARK 3.1. Let the assumptions of Theorem 3.2 be satisfied. By applying Theorem 3.2 with $\Phi(x)=x$ we get the following inequality:

$$\left(\int_{\Omega_1}[A_kf(x)]^{\frac{q}{p}}u(x)d\mu_1(x)\right)^{\frac{1}{q}}\leqslant\left(\int_{\Omega_2}f(y)v(y)d\mu_2(y)\right)^{\frac{1}{p}}. \tag{3.8}$$

Now replace $f(x)$ with $\Phi(f(x))$ and we get that

$$\left(\int_{\Omega_1}[A_k\Phi(f(x))]^{\frac{q}{p}}u(x)d\mu_1(x)\right)^{\frac{1}{q}}\leqslant\left(\int_{\Omega_2}\Phi(f(y))v(y)d\mu_2(y)\right)^{\frac{1}{p}}. \tag{3.9}$$

On the other hand, applying Jensen's inequality to the left side of inequality (3.9) we obtain that

$$\begin{aligned}
&\left(\int_{\Omega_1}[A_k\Phi(f(x))]^{\frac{q}{p}}u(x)d\mu_1(x)\right)^{\frac{1}{q}}\\
&=\left(\int_{\Omega_1}\left[\frac{1}{K(x)}\int_{\Omega_2}k(x,y)\Phi(f(y))d\mu_2(y)\right]^{\frac{q}{p}}u(x)d\mu_1(x)\right)^{\frac{1}{q}}\\
&\geqslant\left(\int_{\Omega_1}\left[\Phi\left(\frac{1}{K(x)}\int_{\Omega_2}k(x,y)f(y)d\mu_2(y)\right)\right]^{\frac{q}{p}}u(x)d\mu_1(x)\right)^{\frac{1}{q}}\\
&=\left(\int_{\Omega_1}[\Phi(A_kf(x))]^{\frac{q}{p}}u(x)d\mu_1(x)\right)^{\frac{1}{q}},
\end{aligned}$$

i.e., by (3.9) that (3.6) holds. We conclude that if the assumptions of Theorem 3.2 hold, then each of (3.6), (3.8) and (3.9) holds and are equivalent.

4. Concluding remarks and examples

REMARK 4.1. By applying the results in this paper for special cases e.g. for kernels with additional homogeneity properties, $\Phi(u)=u^p$, $p>1$, and making some obvious variable transformations we obtain what in the literature is usually called Hilbert type or Hardy–Hilbert type inequalities, see e.g. Example 3.2 for the original case.

However, by keeping our convex functions we obtain further generalizations of Hilbert type inequalities. Here we only give two simple examples.

EXAMPLE 4.1. Let $\Omega_1=\Omega_2=(0,\infty)$. For $k(x,y)=(x+y)^{-s}$, $s>1$ we have $K(x)=\frac{x^{1-s}}{s-1}$ and

$$v(y)=(s-1)\int_0^\infty(x+y)^{-s}x^{s-1}u(x)dx.$$

Let $u(x)=x^{1-t-s}$, $t\in(1-s,1)$.
Then we have

$$v(y)=(s-1)\int_0^\infty(x+y)^{-s}x^{s-1}x^{1-t-s}dx=(s-1)y^{1-t-s}B(1-t,s+t-1),$$

where $B(.,.)$ is the usual Beta function.

By applying Theorem 3.1 we get the following inequality:

$$\int_0^\infty x^{1-t-s}\Phi(A_Kf(x))dx\leqslant(s-1)B(1-t,s+t-1)\int_0^\infty y^{1-t-s}\Phi(f(y))dy,$$

where Φ is a convex function and $A_kf(x)$ is defined by (3.1). □

EXAMPLE 4.2. Let $\Omega_1 = \Omega_2 = (0,\infty)$,

$$u(x) = x^{-2\alpha} \text{ and } k(x,y) = \frac{\ln y - \ln x}{y-x}\left(\frac{y}{x}\right)^{-\alpha}, \quad \alpha \in (0,1).$$

Evidently, it is homogeneous of degree -1, $K(x)$ converges for all $\alpha \in (0,1)$, and we have

$$\begin{aligned} K(x) &= \int_0^\infty \frac{\ln y - \ln x}{y-x}\left(\frac{y}{x}\right)^{-\alpha} dy = \int_0^\infty \frac{\ln u}{u-1} u^{-\alpha} du \\ &= \int_{-\infty}^\infty \frac{te^{(1-\alpha)t}}{e^t - 1} dt = \Psi'(\alpha) + \Psi'(1-\alpha) = \frac{\pi^2}{\sin^2 \pi\alpha}, \end{aligned}$$

where $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, x > 0,$ is the Digamma function and we used the identity

$$\Psi(1-x) = \Psi(x) + \pi \cot \pi x, \quad x \in (0,1)$$

(for details on Ψ see [1]). Then we have

$$\begin{aligned} v(y) &= \frac{\sin^2 \pi\alpha}{\pi^2} \int_0^\infty \frac{\ln x - \ln y}{x-y}\left(\frac{y}{x}\right)^{\alpha} y^{-2\alpha} dx \\ &= \frac{\sin^2 \pi\alpha}{\pi^2} y^{-2\alpha} \int_0^\infty \frac{\ln u}{u-1} u^{-\alpha} du = y^{-2\alpha}, \ (x = yu) \end{aligned}$$

Therefore, by applying (3.3) we get the following inequality:

$$\int_0^\infty \Phi\left(\frac{\sin^2 \pi\alpha}{\pi^2} \int_0^\infty \frac{\ln y - \ln x}{y-x}\left(\frac{y}{x}\right)^{-\alpha} f(y) dy\right) x^{-2\alpha} dx \leqslant \int_0^\infty y^{-2\alpha} \Phi(f(y)) dy,$$

where Φ is a convex function. □

Moreover, by applying our result with the convex function $\Phi(x) = e^x$ and making some suitable variable transformations we obtain what in the literature is called Pólya–Knopp type inequalities. We give the following example:

EXAMPLE 4.3. Let the assumptions in Theorem 3.1 be satisfied. Then, by applying (3.3) with $\Phi(x) = e^x$, and f replaced by $\ln f^p$, $p > 0$ we obtain that

$$\begin{aligned} \int_{\Omega_1} \left[\exp\left(\frac{1}{K(x)} \int_{\Omega_2} k(x,y) \ln f(y) d\mu_2(y)\right)\right]^p u(x) d\mu_1(x) \\ \leqslant \int_{\Omega_2} f^p(y) v(y) d\mu_2(y), \end{aligned} \tag{4.1}$$

where $k(x,y), K(x), u(x)$ and $v(y)$ are defined as in Theorem 3.1. In particular, if $p = 1, \Omega_1 = \Omega_2 = (0,\infty), k(x,y) = 1, 0 < y < x, k(x,y) = 0, y \geqslant x.$ (so that $K(x) = x$), $d\mu_1(x) = dx, d\mu_2(y) = dy, u(x) = 1/x$ (so that $v(x) = 1/x$) replacing $f(x)/x$ by $f(x)$ and making a simple calculation we find that (4.1) is equal to

$$\int_0^\infty \exp\left(\frac{1}{x} \int_0^x \ln f(y)\, dy\right) dx \leqslant e \int_0^\infty f(y)\, dy,$$

which is the classical form of Pólya–Knopp's inequality. □

REMARK 4.2. It is important to remark that obviously inequality (3.3) in Theorem 3.1 holds in the reversed direction if Φ is concave. Hence, we can also obtain a number of complements of the examples we have pointed out.

Our last important remark concerns the multidimensional case:

REMARK 4.3. As we have seen our results can be used to obtain also multidimensional Hardy-type inequalities. See e.g. our generalizations in Corollaries 3.1 and 3.2 of Godunova's multidimensional result in Theorem 2.4. It is also obvious that Theorem 2.5 follows by using our Theorem 3.1 in a similar way as we argued in our Example 3.3. In a forthcoming paper we aim to further develop the ideas in this paper to a multidimensional setting (see also Example 3.6). Some further results in this direction can also be found in the recent papers [13] and [14].

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REFERENCES

[1] M. ABRAMOWITZ AND I.A. STEGUN (EDS.), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, 9^{th} printing, Dover, New York, 1972.

[2] A. ČIŽMEŠIJA, J.E. PEČARIĆ AND L.E. PERSSON, *On strengthened Hardy and Pólya–Knopp's inequalities*, J. Approx. Theory, **125** (2003), 74–84.

[3] E. K. GODUNOVA, *Inequalities based on convex functions*, Izv.Vysh. Uchebn. Zaved. Matematika, **47** 4 (1965), 45–53; English transl. in Amer. Math. Soc., Transl., II Ser, **88** (1970), 57–66.

[4] E. K. GODUNOVA, *Integral'nye neravenstva s vypuklymi funkcijimi*, Materialy šestoi mež. fiz.-mat. nauč. konf. Dal'nego Vostoka. Diff. i int. uravnenija., Tom **3**, Habarovsk, 1967, 65–69.

[5] G. H. HARDY, *Notes on some points in the integral calculus LX: An inequality between integrals* (60), Messenger of Math., **54** (1925), 150–156.

[6] S. KAIJSER, L. NIKOLOVA, L.E. PERSSON AND A. WEDESTIG, *Hardy-Type Inequalities via Convexity*, Math. Inequal. Appl., **8** (2005), 403–417.

[7] S. KAIJSER, L.-E. PERSSON, AND A. ÖBERG, *On Carleman and Knopp's Inequalities*, J. Approx. Theory, **117** (2002), 140–151.

[8] A. KUFNER, L. MALIGRANDA, AND L.-E. PERSSON, *The prehistory of the Hardy inequality*, Amer. Math. Monthly, **113** 8 (2006), 715–732.

[9] A. KUFNER, L. MALIGRANDA, AND L.-E. PERSSON, *The Hardy Inequality. About its History and Some Related Results*, Vydavatelsky Servis Publishing House, Pilsen, 2007.

[10] A. KUFNER AND L.-E. PERSSON, *Weighted Inequalities of Hardy Type*, World Scientific Publishing Co, Singapore/ New Jersey/ London/ Hong Kong, 2003.

[11] C. NICULESCU AND L.-E. PERSSON, *Convex functions and their applications. A contemporary approach*, CMC Books in Mathematics, Springer, New York, 2006.

[12] J.A. OGUNTUASE, C.A. OKPOTI, L.E. PERSSON AND F.K.A. ALLOTEY, *Multidimensional Hardy Type Inequalities for* $p<0$ *and* $0<p<1$, J. Math. Inequal., **1** (2007), 1–11.

[13] J. A. OGUNTUASE, C.A. OKPOTI, L.E. PERSSON AND F.K.A. ALLOTEY, *Weighted multidimensional Hardy type inequalities via Jensen's inequality*, Proc. A. Razmadze Math. Inst., **144** (2007), 91–105.

[14] J. A. OGUNTUASE, L.E. PERSSON AND E.K. ESSEL, *Multidimensional Hardy type inequalities with general kernels*, J. Math. Anal. Appl., **348** 1 (2008), 411–418.

[15] J. E. PEČARIĆ, F. PROSCHAN, AND Y. L. TONG, *Convex functions, Partial Orderings and Statistical Applications*, Academic Press, San Diego, 1992.

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