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AN EXTENSION OF THE LUPAŞ THEOREM FOR HERMITE–HADAMARD FUNCTIONALS

IOAN GAVREA

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Abstract. In this paper we investigate a representation theorem of Lupaş type for Hermite-Hadamard functionals. We use the obtained results by considering integral mean values with respect to certain signed measures.

1. Introduction

The well known Hermite-Hadamard inequality [7] is

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{1}$$

which holds for a convex function $f:[a,b]\to\mathbb{R}$. A weighted form of this inequality was proved by L. Fejér [3].

More precisely, if $f:[a,b]\to\mathbb{R}$ is a convex function and $p:[a,b]\to\mathbb{R}$ is a nonnegative integrable function, symmetric with respect to the point $(a+b)/2$ then

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx \leqslant \int_a^b f(x)p(x)dx \leqslant \frac{f(a)+f(b)}{2}\int_a^b f(x)dx. \tag{2}$$

A. M. Fink [4] made an important remark concerning the generality of (1), by noting the possibility of considering certain signed measures.

Let $A: C[a,b]\to\mathbb{R}$ be a linear bounded functional such that $A(e_0)=1$, where $e_i:[a,b]\to\mathbb{R}$, $e_i(x)=x^i$, $i\in\mathbb{N}$.

It is well known that for a functional A as above there exists a real Borel measure μ on $[a,b]$ such that

$$A(f)=\int_a^b f(x)d\mu(x), \tag{3}$$

for every $f\in C[a,b]$ and $\displaystyle\int_a^b d\mu(x)=1$.

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In the following we denote by

$$a_k = A(e_k), \quad k \in \mathbb{N}.$$

We say that, A is a linear Hermite-Hadamard functional if for every convex function $f : [a,b] \to \mathbb{R}$, the following inequalities hold:

$$A(f) \geqslant f(a_1) \tag{4}$$

$$A(f) \leqslant \frac{b-a_1}{b-a} f(a) + \frac{a_1-a}{b-a} f(b) \tag{5}$$

A. Florea and C. P. Niculescu [5], called the Borel measure μ corresponding to the Hermite-Hadamard functional A a Hermite-Hadamard measure.

A. M. Fink [4] gave a complete characterization of the real-valued Borel measure μ for which (4) works.

In the same paper A. M. Fink proved a sufficient condition for the measure from (3) for which (5) is true.

A. Florea and C. P. Niculescu [5] gave a complete characterization of the measures for which (5) works for every convex function $f : [a,b] \to \mathbb{R}$.

The following theorem shows that the characterization of the functionals A which satisfy (5) follows by the characterization of the functionals which satisfy (4) for every convex function.

THEOREM 1.1. *Let A be a linear functional, $A(e_0) = 1$. The functional A is a Hermite-Hadamard functional if and only if the functional A and the functional B defined by*

$$B(f) = -A(f) + \frac{b-a_1}{b-a} f(a) + \frac{a_1-a}{b-a} f(b) + f(a_1) \tag{6}$$

satisfy inequality (4).

Proof. First, suppose that the functionals A and B satisfy inequality (4) for every convex function f. Then, we have

$$B(e_1) = a_1 \tag{7}$$

From (7) and from the inequality

$$B(f) \geqslant f(a_1)$$

follows inequality (5).

Now, let us suppose that (5) works for every convex function f.

Inequality (5) can be written as:

$$\frac{b-a_1}{b-a} f(a) + \frac{a_1-a}{b-a} f(b) + f(a_1) - A(f) \geqslant f(a_1)$$

or

$$B(f) \geqslant f(B(e_1)).$$

P. Czinder and Z. Páles ([1]) extended the Hermite-Hadamard inequality in another direction, by considering functions that mix the up and down convexity.

THEOREM 1.2. ([1]) *Suppose that I is an interval and $f: I \to \mathbb{R}$ is a symmetric function with respect to an element $m \in I$, that is:*

$$f(x)+f(2m-x)=2f(m), \tag{8}$$

for all $x \in I \cap (-\infty, m]$.

If f is convex over the interval $I \cap (-\infty, m]$ and concave over the interval $I \cap [m, \infty)$, then, for every interval $[a,b] \subset I$ with $(a+b)/2 \geqslant m$ the following inequalities hold true:

$$f\left(\frac{a+b}{2}\right) \geqslant \frac{1}{b-a}\int_a^b f(x)dx \geqslant \frac{f(a)+f(b)}{2}. \tag{9}$$

If $(a+b)/2 \leqslant m$ then the inequalities (9) should be reversed.

An extension of Theorem 2.1 for absolutely continuous measures with positive weights was obtained by P. Czinder ([2]).

A. Florea and C. P. Niculescu ([5]) proved that Theorem 1.2 works in the general framework of Hermite-Hadamard measures. □

THEOREM 1.3. ([5]) *Suppose that $f: I \to \mathbb{R}$ verifies condition (8) and is convex over the interval $I \cap (-\infty, m]$ and concave over the interval $I \cap [m, \infty)$.*

If $(a+b)/2 \geqslant m$ and μ is a Hermite-Hadamard measure on each of the intervals $[a, 2m-a]$ and $[2m-a, b]$ and is invariant with respect to the map $T(x) = 2m-x$ on $[a, 2m-a]$ then

$$f(x_\mu) \geqslant \frac{1}{\mu([a,b])}\int_a^b f(x)d\mu(x) \geqslant \frac{b-x_\mu}{b-a}f(a)+\frac{x_\mu-a}{b-a}f(b) \tag{10}$$

where $\mu([a,b]) > 0$ and $x_\mu = \dfrac{1}{\mu([a,b])}\displaystyle\int_a^b x d\mu(x)$.

If $(a+b)/2 \leqslant m$, then the inequalities (10) work in a reverse way, provided μ is a Hermite-Hadamard measure on each of the intervals $[a, 2m-b]$ and $[2m-b, b]$ and is invariant with respect to the map $T(x) = 2m-x$ on $[2m-b, b]$.

The divided difference, $[x_1, x_2, ..., x_{n+1}; f]$ of a function $f \in \mathbb{R}^{[a,b]}$ on the distinct knots $x_1, x_2, ..., x_{n+1}$ is defined by:

$$[x_1, x_2, ..., x_{n+1}; f] := \sum_{i=1}^{n+1} \frac{f(x_i)}{l'(x_i)}.$$

A function $f \in \mathbb{R}^{[a,b]}$ is a convex function iff:

$$[x_1, x_2, x_3; f] \geqslant 0,$$

for every distinct knots $x_1, x_2, x_3 \in [a,b]$. A function f, $f: [a,b] \to \mathbb{R}$ is a convex function of order n if all divided differences of the function f on $n+2$ distinct points from the interval $[a,b]$ are positive.

A. Lupaş ([6]) proved the following representation theorem for linear positive functionals:

THEOREM 1.4. ([6]) *Let* $A : C[a,b] \to \mathbb{R}$ *be a linear positive functional with* $A(e_0) = 1$. *Then, for any* $f \in C[a,b]$ *there exist distinct points* $\theta_1, \theta_2, \theta_3 \in [a,b]$, *depending on* f, *such that*

$$A(f) = f(a_1) + (a_2 - a_1^2)[\theta_1, \theta_2, \theta_3; f]. \tag{11}$$

The aim of this paper is to prove that Theorem 1.4 is true in the general framework of Hermite-Hadamard functionals and to improve the results from Theorems 1.2 and 1.3.

The important tool for the demonstration of the main result is given by T. Popoviciu ([8]) in connection with P_n-simple functionals. In order to recall it here we need a preparation.

Let n be an integer number, $n \geqslant -1$. A linear functional A, $A : C[a,b] \to \mathbb{R}$ has the degree of exactness n if

$$A(e_i) = 0, \quad i = 0, 1, \dots, n$$

and

$$A(e_{n+1}) \neq 0.$$

A functional A is a P_n-simple functional if A has the degree of exactness n and for every $f \in C[a,b]$ there exist $n+2$ distinct points $\theta_i = \theta_i(f) \in [a,b]$, $i = 1, 2, \dots, n+2$ such that

$$A(f) = A(e_{n+1})[\theta_1, \theta_2, \dots, \theta_{n+2}; f]$$

where $[\theta_1, \theta_2, \dots, \theta_{n+2}; f]$ is divided difference of the function f on the points $\theta_1, \theta_2, \dots,$ θ_{n+2}.

THEOREM 1.5. ([8]) *If* $A : C[a,b] \to \mathbb{R}$ *is a linear bounded functional of degree of exactness* n *then the functional* A *is* P_n*-simple if and only if*

$$A(e_{n+1})A(\psi_{t,n}) \geqslant 0, \quad t \in [a,b] \tag{12}$$

where

$$\psi_{t,n}(x) := (x-t)_+^n := \left(\frac{x - t + |x-t|}{2} \right)^n.$$

2. The extension of the Lupaş theorem

Let $A : C[a,b] \to \mathbb{R}$ be a linear functional. The following theorem shows that Lupaş's theorem works for Hermite-Hadamard functionals.

THEOREM 2.1. *If* A *is a linear bounded functional such that* $A(e_0) = 1$ *and* A *satisfies the inequality (4) for every convex function* f, *then for every* $f \in C[a,b]$ *there exist points* $\theta_i = \theta_i(f) \in [a,b]$, $i = 1, 2, 3$, *such that*

$$A(f) = f(a_1) + (a_2 - a_1^2)[\theta_1, \theta_2, \theta_3; f]. \tag{13}$$

Proof. If $A=\delta_{x_1}$, $x_1\in[a,b]$, then (13) is true for every selection of the distinct points $\theta_1,\theta_2,\theta_3\in[a,b]$.

Now, we suppose that A is not a Dirac functional. Let B be the linear bounded functional, $B:C[a,b]\to\mathbb{R}$, defined by

$$B(f)=A(f)-f(a_1).$$

We have

$$B(e_0)=B(e_1)=0. \tag{14}$$

By (4), for every convex function, we have

$$B(f)\geqslant 0.$$

In particular

$$B(e_2)\geqslant 0. \tag{15}$$

Let us prove that in (15) the inequality is strictly satisfied.

In the contrary case, from (14) we get:

$$B(e_1-a)^2=B(e_2)=0.$$

Let k be a fixed natural number, $k\geqslant 3$. If inequality (4) holds for every convex function f, then for a concave function inequality (4) reverses sign.

Let us consider the function g defined by

$$g(x)=-(x-a)^{k+1}+\frac{k(k+1)}{2}(b-a)^{k-1}(a_1-a)(x-a).$$

The function g is a concave function on $[a,b]$ and we have:

$$\begin{aligned}A(g)&=-A((e_1-a)^{k+1})+\frac{k(k+1)}{2}(b-a)^{k-1}(a_1-a)^2\\&=-A((e_1-a)^{k+1})+\frac{k(k+1)}{2}(b-a)^{k-1}A((e_1-a)^2)\end{aligned} \tag{16}$$

Because the function g is a concave function we get

$$A(g)\leqslant-(a_1-a)^{k+1}+\frac{k(k+1)}{2}(b-a)^{k-1}(a_1-a)^2 \tag{17}$$

Let h be the function defined by

$$h(x)=-(x-a)^{k+1}+\frac{k(k+1)}{2}(b-a)^{k-1}(x-a)^2,\quad x\in[a,b].$$

h is a convex function, and

$$A(h)=A(g) \tag{18}$$

h being a convex function we obtain

$$A(h)\geqslant-(a_1-a)^{k+1}+\frac{k(k+1)}{2}(b-a)^{k-1}(a_1-a)^2. \tag{19}$$

From (16), (17), (18) and (19) we get

$$A((e_1-a)^{k+1}) = (a_1-a)^{k+1}, \quad k \in \mathbb{N}. \tag{20}$$

The equalities (20) imply that

$$A(P) = P(a_1) \tag{21}$$

for every polynomial P.

The functional A is bounded and therefore A is a continuous functional.

Since the set of all polynomials is dense in $C[a,b]$, from (21) we get

$$A(f) = f(a_1).$$

This is a contradiction with the fact that A is different from the Dirac functional. So,

$$B(e_2) > 0.$$

This means that the functional A has the degree of exactness 1. Because the functional A satisfies (4) we have

$$B(e_2)B(\psi_{t,1}) \geqslant 0.$$

Since the conditions of Theorem 1.5 are satisfied for the functional B, it follows that B is P_1-simple.

This means that, for $f \in C[a,b]$ there exist the distinct points $\theta_1, \theta_2, \theta_3 \in [a,b]$, depending on the function f such that

$$B(f) = B(e_2)[\theta_1, \theta_2, \theta_3; f]$$

or

$$A(f) = f(a_1) + (a_2 - a_1^2)[\theta_1, \theta_2, \theta_3; f]. \qquad \square$$

COROLLARY 2.2. *A linear bounded functional A, $A : C[a,b] \to \mathbb{R}$ for which $A(e_0) = 1$ is a Hermite-Hadamard functional if and only if for every $t \in [a,b]$ the following inequalities hold:*

$$A(|e_1 - t|) \geqslant |a_1 - t| \tag{22}$$

$$A(|e_1 - t|) \leqslant \frac{b-a_1}{b-a}(t-a) + \frac{a_1-a}{b-a}(b-t). \tag{23}$$

Proof. In the case when A is Dirac functional, the relations (22) and (23) are straightforward to obtain. In what follows we assume that A is not a Dirac functional. In this case Theorem 2.1 shows that A is a Hermite-Hadamard functional if and only if the functionals B_1, $B_2 : C[a,b] \to \mathbb{R}$ defined by

$$B_1(f) = A(f) - f(a_1)$$
$$B_2(f) = \frac{b-a_1}{b-a} f(a) + \frac{a_1-a}{b-a} f(b) - A(f)$$

are P_1–simple functionals and

$$B_1(e_2) > 0$$
$$B_2(e_2) > 0.$$

From Theorem 1.5, B_1 and B_2 are P_1–simple functionals if and only if

$$B_1(\Psi_{t,1}) \geqslant 0$$
$$B_2(\Psi_{t,1}) \geqslant 0.,$$

inequalities that are the same with (22) and (23). □

REMARK 2.3. In the case when μ is a Borel measure on $[a,b]$ with $\mu([a,b]) > 0$ and

$$A(f) = \frac{1}{\mu([a,b])} \int_a^b f(x) d\mu(x),$$

then (22) is equivalent with the conditions:

i) $\int_a^t (t-x) d\mu(x) \geqslant 0$ for every $t \in [a,a_1]$,

ii) $\int_t^b (x-t) d\mu(x) \geqslant 0$ for every $t \in [a_1,b]$.

Let us prove i). We can assume that $\mu([a,b]) = 1$. If $t \in [a,a_1]$ inequality (22) can be written in the following form:

$$\int_a^b |x-t| d\mu(x) \geqslant a_1 - t$$

or

$$\int_a^t (t-x) d\mu(x) + \int_t^b (x-t) d\mu(x) \geqslant a_1 - t$$

⇔

$$\int_a^t (t-x) d\mu(x) + \int_a^b (x-t) d\mu(x) + \int_a^t (t-x) d\mu(x) \geqslant a_1 - t$$

⇔

$$2\int_a^t (t-x) d\mu(x) \geqslant 0.$$

For $t \in [a,b]$ inequality (22) becomes:

$$\int_a^t (t-x) d\mu(x) + \int_t^b (x-t) d\mu(x) \geqslant t - a_1$$

or

$$\int_a^b (t-x) d\mu(x) + 2\int_t^b (x-t) d\mu(x) \geqslant \int_a^b (t-x) d\mu(x).$$

The last inequality is the same with ii).

A. M. Fink ([4]) proved that if i) and ii) are true for every $t \in [a,b]$, then the functional

$$A(f) = \frac{1}{\mu([a,b])} \int_a^b f(x) d\mu(x)$$

satisfies inequality (4) for every convex function f. Inequality (23) becomes inequality (10) obtained by A. Florea and C. P. Niculescu in [5]. The following theorem is an extension of Theorem 2.1.

THEOREM 2.3. *Let A be a linear bounded functional defined on $C[a,b]$, different from the null functional.*

If

$$A(f) \geqslant 0 \tag{24}$$

for every convex function f of order n, then A is a P_n-simple functional.

Proof. The functions $e_i, -e_i$, $i = 0,1,2,\ldots,n$ are simultaneously convex and concave of order n. So, from (24) it follows that

$$A(e_i) = 0, \quad i = 0,1,2,\ldots,n. \tag{25}$$

To complete the proof it is necessary to show that

$$A(e_{n+1}) > 0. \tag{26}$$

Let suppose the contrary, $A(e_{n+1}) = 0$.

Let m be a natural number, $m \geqslant 1$. The function g, $g : [a,b] \to \mathbb{R}$ defined by

$$g(x) = -(x-a)^{n+m+1} + \frac{(n+m+1)!}{(n+1)!m!}(b-a)^m(x-a)^{n+1}$$

is a convex function of order n.

From (24), it follows that:

$$A(g) \geqslant 0.$$

On the other hand, we have:

$$A(g) = -A((e_1-a)^{m+n+1}) \tag{27}$$

and therefore

$$A((e_1-a)^{m+n+1}) \leqslant 0. \tag{28}$$

The function h defined by

$$h(x) = -(x-a)^{m+n+1}$$

is a n-concave function and so

$$A(h) \leqslant 0$$

or

$$A((e_1-a)^{m+n+1}) \geqslant 0. \tag{29}$$

From (28) and (29) we obtain

$$A((e_1-a)^{m+n+1}) = 0 \tag{30}$$

for every $m = 0,1,2,\ldots$

Relations (25) and (30) imply that

$$A(P) = 0 \tag{31}$$

for every polynomial P.

The functional A being a continuous functional, (31) imply that A is the null functional. This concludes the proof of the result. $\square$

COROLLARY 2.4. *Let A be a non-zero linear bounded functional defined on $C[a,b]$. The inequality*

$$A(f) \geqslant 0$$

holds for every convex function of order n if and only if

$$A(\psi_{t,n}) \geqslant 0 \tag{32}$$

for every $t \in [a,b]$, where $\psi_{t,n}$ is the function defined by

$$\psi_{t,n}(x) = \left(\frac{x-t+|x-t|}{2}\right)^n.$$

Proof. The proof follows from Theorem 2.3 and by Popoviciu's Theorem 1.5. □

REMARK 2.5. Corollary 2.4 generalizes a result obtained by A. M. Fink ([4]) (Theorem 4, pp. 227).

3. An application of Hermite-Hadamard measure to convex-concave symmetrical functions

In what follows we consider a function $f: I \to \mathbb{R}$ which verifies the symmetry condition (8).

The following theorems improve the results obtained by A. Florea and C. P. Niculescu ([5]).

THEOREM 3.1. *Let $f: I \to \mathbb{R}$ be a function which verifies the symmetry condition (8) and is convex over the interval $I \cap (-\infty, m]$ and concave over the interval $I \cap [m, \infty)$.*

If $(a+b)/2 \geqslant m$ and μ is a Hermite-Hadamard measure on each of the intervals $[a, 2m-a]$ and $[2m-a, b]$, and is invariant with respect to the map $T(x) = 2m - x$ on $[a, 2m-a]$, then there exist the distinct points $\theta_i = \theta_i(f) \in [2m-a, b]$, $i = 1,2,3$ such that the following equality holds:

$$\begin{aligned}
&\frac{1}{\mu([a,b])}\int_a^b f(x)d\mu(x) - f(x_\mu) \\
&\quad = \frac{\mu([a,2m-a])\mu([2m-a,b])}{\mu^2([a,b])}\left[m, x_\mu, \frac{\int_{2m-a}^b x d\mu(x)}{\mu([2m-a,b])}; f\right] \\
&\qquad + \frac{1}{\mu([a,b])}\left(\int_{2m-a}^b x^2 d\mu(x) - \frac{1}{\mu([2m-a,b])}\left(\int_{2m-a}^b x d\mu(x)\right)^2\right)[\theta_1, \theta_2, \theta_3; f],
\end{aligned} \tag{33}$$

where $K = \left(m - \frac{1}{\mu([2m-a,b])}\int_{2m-a}^a x d\mu(x)\right)^2.$

Proof. The following equality follows from the invariance properties of f and μ (see [5]):

$$\begin{aligned}
&\frac{1}{\mu([a,b])}\int_a^b f(x)d\mu(x)\\
&\quad= f(m)\frac{\mu([a,2m-a])}{\mu([a,b])}+\frac{\mu([2m-a,b])}{\mu([a,b])}f\left(\frac{1}{\mu([2m-a,b])}\int_{2m-a}^b xd\mu(x)\right)\\
&\qquad+\frac{\mu([2m-a,b])}{\mu([a,b])}\left(\frac{1}{\mu([2m-a,b])}\int_{2m-a}^b f(x)d\mu(x)\right.\\
&\qquad\left.-f\left(\frac{1}{\mu([2m-a,b])}\int_{2m-a}^b xd\mu(x)\right)\right).
\end{aligned}\tag{34}$$

The Borel measure μ being a Hermite-Hadamard measure on the interval $[2m-a,b]$, we obtain from Theorem 2.1 the existence of the distinct points $\theta_i(f)\in[2m-a,b]$, $i=1,2,3$ such that

$$\begin{aligned}
&\frac{1}{\mu([2m-a,b])}\int_{2m-a}^b f(x)d\mu(x)-f\left(\frac{1}{\mu([2m-a,b])}\int_{2m-a}^b xd\mu(x)\right)\\
&=\left(\frac{1}{\mu([2m-a,b])}\int_{2m-a}^b x^2d\mu(x)-\left(\frac{1}{\mu([2m-a,b])}\int_{2m-a}^b xd\mu(x)\right)^2\right)[\theta_1,\theta_2,\theta_3;f].
\end{aligned}\tag{35}$$

Let $\alpha\in[0,1]$ and $x,y\in[a,b]$. The following equality holds

$$(1-\alpha)f(x)+\alpha f(y)-f((1-\alpha)x+\alpha y)=\alpha(1-\alpha)(x-y)^2[x,(1-\alpha)x+\alpha y,y;f].\tag{36}$$

From (35) we get

$$\begin{aligned}
&f(m)\frac{\mu([a,2m-a])}{\mu([a,b])}+\frac{\mu([2m-a,b])}{\mu([a,b])}f\left(\frac{1}{\mu([2m-a,b])}\int_{2m-a}^b xd\mu(x)\right)-f(x_\mu)\\
&\quad=\frac{\mu([a,2m-a])}{\mu([a,b])}\cdot\frac{\mu([2m-a,b])}{\mu([a,b])}\left[m,x_\mu,\frac{1}{\mu([2m-a,b])}\int_{2m-a}^b xd\mu(x);f\right].
\end{aligned}\tag{37}$$

From (34), (35) and (37) we obtain (33). □

THEOREM 3.2. *Let $f:I\to\mathbb{R}$ be a function which verifies the symmetry condition (8) and is convex over the interval $I\cap(-\infty,m]$ and concave over the interval $I\cap[m,\infty)$.*

If $(a+b)/2\geqslant m$ and μ is a Hermite-Hadamard measure on each of the intervals $[a,2m-a]$ and $[2m-a,b]$, $\mu([a,b])=1$, and is invariant with respect to the map $T(x)=2m-x$ on $[a,2m-a]$, then there exist distinct points $\eta_i(f)\in[2m-a,b]$, $i=1,2,3$, such that the following equality is true:

$$\begin{aligned}
&\int_a^b f(x)d\mu(x)-\frac{x_\mu-a}{b-a}f(b)-\frac{b-x_\mu}{b-a}f(a)\\
&\quad=(1-\lambda)(a_2-x_\mu'^2-(x_\mu'-2m+a)(b-x_\mu'))[\eta_1,\eta_2,\eta_3;f]\\
&\qquad-\frac{((2-\lambda)b+\lambda a-2x_\mu)(m-a)(b-m)}{(b-a)}[m,2m-a,b;f]
\end{aligned}\tag{38}$$

where

$$\lambda = \mu([a,2m-a]), \quad x'_\mu = \frac{\int_{2m-a}^{b} x d\mu(x)}{1-\lambda}, \quad a_2 = \frac{\int_{2m-a}^{b} x^2 d\mu(x)}{1-\lambda}.$$

Proof. We have

$$\int_a^b f(x)d\mu(x) = f(m)\lambda + \int_{2m-a}^{b} f(x)d\mu(x).$$

The last equality can be written in the following form:

$$\int_a^b f(x)d\mu(x) = \lambda f(m) + (1-\lambda)\left(\frac{x'_\mu - 2m + a}{a+b-2m} f(b) + \frac{b - x'_\mu}{a+b-2m} f(2m-a)\right)$$

$$+(1-\lambda)\left(\frac{\int_{2m-a}^{b} f(x)d\mu(x)}{1-\lambda} - \frac{x'_\mu - 2m + a}{a+b-2m} f(b) - \frac{b - x'_\mu}{a+b-2m} f(2m-a)\right) \quad (39)$$

μ being a Hermite-Hadamard measure on $[2m-a,b]$.

From Theorem 1.1 and Theorem 2.1, it follows that there exist the distinct points $\eta_i(f) \in [2m-a,b]$, $i = 1,2,3$ such that

$$\frac{\int_{2m-a}^{b} f(x)d\mu(x)}{1-\lambda} - \frac{x'_\mu - 2m + a}{a+b-2m} f(b) - \frac{b - x'_\mu}{a+b-2m} f(2m-a)$$
$$= (a_2 - x'^2_\mu - (x'_\mu - 2m + a)(b - x'_\mu))[\eta_1,\eta_2,\eta_3; f]. \quad (40)$$

To complete the proof, we see from (39) and (40) that it is sufficient to show the following equality

$$\lambda f(m) + (1-\lambda)\left(\frac{x'_\mu - 2m + a}{a+b-2m} f(b) + \frac{b - x'_\mu}{a+b-2m} f(2m-a)\right) - \frac{x_\mu - a}{b-a} f(b) - \frac{b - x_\mu}{b-a} f(a)$$
$$= -\frac{((2-\lambda)b + \lambda a - 2x_\mu)(m-a)(b+a-2m)}{(b-a)(b+a-2m)(b-m)}[m, 2m-a, b; f]. \quad (41)$$

We have

$$\begin{aligned} x'_\mu &= \frac{1}{1-\lambda}\int_{2m-a}^{b} x d\mu(x) \\ &= \frac{1}{1-\lambda}\left(\int_a^b x d\mu(x) - \int_a^{2m-a} x d\mu(x)\right) \\ &= \frac{x_\mu - m\lambda}{1-\lambda}. \end{aligned}$$

Since

$$f(a) = 2f(m) - f(2m-a)$$

we obtain

$$\begin{aligned}
&\lambda f(m)+(1-\lambda)\left(\frac{x'_\mu-2m+a}{a+b-2m}f(b)+\frac{b-x'_\mu}{a+b-2m}f(2m-a)\right)-\frac{x_\mu-a}{b-a}f(b)-\frac{b-x_\mu}{b-a}f(a)\\
&=\frac{(b-m)((2-\lambda)b+\lambda a-x_\mu)}{(b-a)(b+a-2m)}f(2m-a)-\frac{(2-\lambda)b+\lambda a-2x_\mu}{b-a}f(m)\\
&\quad-\frac{(m-a)((2-\lambda)b+a\lambda-2x_\mu)}{(b-a)(b+a-2m)}f(b)\\
&=\frac{(b-m)((2-\lambda)b+\lambda a-2x_\mu)}{(b-a)(b+a-2m)}\left(f(2m-a)-\frac{b+a-2m}{b-m}f(m)-\frac{m-a}{b-m}f(b)\right)\\
&=-\frac{((2-\lambda)b+\lambda a-2x_\mu)(m-a)(b-m)}{(b-a)}[m,2m-a,b;f]. \qquad \square
\end{aligned}$$

REMARK. From Theorem 3.1 and 3.2 we get the result obtained by A. Florea and C. P. Niculescu in ([5]).

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Ioan Gavrea
Technical University of Cluj-Napoca
Department of Mathematics
Str. C. Daicoviciu 15
3400 Cluj-Napoca
Romania
e-mail: Ioan.Gavrea@math.utcluj.ro