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LOWER AND UPPER ESTIMATES OF NUMERICAL SUMS

L. LEINDLER

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Abstract. Some useful inequalities pertaining to lower and upper estimates of numerical series are improved. Moreover necessary and sufficient conditions are given that certain estimates be valid. Finally an application of the new inequalities is presented.

1. Introduction

One of the aims of the present paper is to extend some useful inequalities pertaining to numerical sums.

In [2] (Lemma) we proved that *for any positive sequence* $\boldsymbol{\gamma} := \{\gamma_n\}$ *the inequalities*

$$H_m := \sum_{n=1}^{m} \gamma_n \leqq K\gamma_m, \tag{1.1}$$

or

$$T_m := \sum_{n=m}^{\infty} \gamma_n \leqq K\gamma_m \tag{1.2}$$

hold if and only if the sequence $\boldsymbol{\gamma}$ *is a quasi geometrically increasing or decreasing, respectively.*

A sequence $\boldsymbol{\gamma}$ of positive terms is *quasi geometrically increasing* (*decreasing*) if there exist a natural number μ and a constant $K = K(\boldsymbol{\gamma}) \geqq 1$ such that

$$\gamma_{n+\mu} \geqq 2\gamma_n \quad \text{and} \quad \gamma_n \leqq K\gamma_{n+1} \left(\gamma_{n+\mu} \leqq \frac{1}{2}\gamma_n \quad \text{and} \quad \gamma_{n+1} \leqq K\gamma_n\right)$$

hold for all natural numbers n.

Hereafter K, K_i, $K(.)$ will designate either an absolute constant, or a constant depending on the indicated parameters, not necessarily the same at each occurrence.

The letters H and T refer to the initials of the words “Heads and Tails of the sums”.

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In [4] (Lemma 4), we verified that (1.1) *and* (1.2) *imply that there exists a small positive* ε *such that*

$$\sum_{n=1}^{m} \gamma_n 2^{-n\varepsilon} \leqq K\gamma_m 2^{-m\varepsilon} \tag{1.3}$$

and

$$\sum_{n=m}^{\infty} \gamma_n 2^{n\varepsilon} \leqq K\gamma_m 2^{m\varepsilon} \tag{1.4}$$

also hold, respectively.

It is clear that (1.1) with $\gamma_n = q^n c_n$, $q > 1$ and $c_n > 0$; and (1.2) with $\gamma_n = q^n c_n$, $0 < q < 1$, imply that

$$\sum_{n=1}^{m} q^n c_n 2^{-n\varepsilon} \leqq K q^m c_m 2^{-m\varepsilon}, \tag{1.5}$$

and

$$\sum_{n=m}^{\infty} q^n c_n 2^{n\varepsilon} \leqq K q^m c_m 2^{m\varepsilon} \tag{1.6}$$

also hold.

Newly we encountered a similar phenomenon at the class AM_p of weight functions (see e.g. the new attractive book of A. Kufner, L. Maligranda and L. E. Persson [1], Lemma 2(a) on p. 94), where it is proved that if $w \in AM_p$, then $w \in AM_{p-\varepsilon}$ for some $\varepsilon > 0$ also holds.

The definition of AM_p is the following: $w \in AM_p$ if

$$\int_y^{\infty} t^{-p} w(t)dt \leqq K y^{-p} \int_0^{y} w(t)dt \quad \text{for all} \quad y > 0. \tag{1.7}$$

This fact brought my inspiration into action to extend the inequalities (1.5) and (1.6) such that our new inequalities are going to imply the statement $w \in AM_p \Longrightarrow w \in AM_{p-\varepsilon}$, too.

We shall sharpen some further inequalities likewise.

2. Theorems

First we extend the inequalities (1.5) and (1.6)

THEOREM 1. *The inequalities*

$$\sum_{n=0}^{m} q^n c_n \leqq K q^m \sum_{n=m}^{\infty} c_n, \quad q > 1,\ m = 1, 2, \ldots \tag{2.1}$$

imply

$$\sum_{n=0}^{m} q^n 2^{-n\varepsilon} c_n \leqq K(\varepsilon) q^m 2^{-m\varepsilon} \sum_{n=m}^{\infty} c_n \tag{2.2}$$

for some $\varepsilon > 0$*; furthermore*

$$\sum_{n=m}^{\infty} q^n c_n \leqq K q^m \sum_{n=0}^{m} c_n, \quad 0 < q < 1,\ m = 1,2,\dots \tag{2.3}$$

imply

$$\sum_{n=m}^{\infty} q^n c_n 2^{n\varepsilon} \leqq K(\varepsilon) q^m 2^{m\varepsilon} \sum_{n=0}^{m} c_n. \tag{2.4}$$

Before recalling two further useful inequalities, we present two definitions.

A sequence $\boldsymbol{\gamma} := \{\gamma_n\}$ of positive terms is *quasi* β*-power-monotone increasing* (*decreasing*) if there exists a constant $K = K(\beta, \boldsymbol{\gamma}) \geqq 1$ such that

$$K n^{\beta} \gamma_n \geqq m^{\beta} \gamma_m \quad (n^{\beta} \gamma_n \leqq K m^{\beta} \gamma_m)$$

holds for any $n \geqq m$.

These sequences have very strong connection with the quasi geometrically increasing and decreasing sequences. See e.g. in [4] Corollary 1.

A sequence $\boldsymbol{\gamma}$ is called *bounded by blocks* if the inequalities

$$K_1 \min(\gamma_{2^k}, \gamma_{2^{k+1}}) \leqq \gamma_n \leqq K_2 \max(\gamma_{2^k}, \gamma_{2^{k+1}}) \quad 0 < K_1 \leqq K_2 < \infty$$

hold for any $2^k \leqq n \leqq 2^{k+1}$, $k = 1,2,\dots$.

In [4] (Corollary 2) we proved the following result:

A positive sequence $\boldsymbol{\gamma}$ *bounded by blocks is quasi* β*-power-monotone increasing* (*decreasing*) *with a certain negative* (*positive*) *exponent* β *if and only if the inequalities*

$$\widetilde{H}_m := \sum_{n=1}^{m} \gamma_n n^{-1} \leqq K \gamma_m \quad \left(\widetilde{T}_m := \sum_{n=m}^{\infty} \gamma_n n^{-1} \leqq K \gamma_m\right) \tag{2.5}$$

hold for all natural number m*, respectively.*

Next we improve the inequalities (2.5) as (1.1) and (1.2) were extended by (1.3) and (1.4), respectively.

THEOREM 2. *If a positive sequence* $\boldsymbol{\gamma}$ *satisfies* (2.5)*, then there exists a positive* ε *such that*

$$\sum_{n=1}^{m} \gamma_n n^{-1-\varepsilon} \leqq K(\varepsilon) \gamma_m m^{-\varepsilon}, \tag{2.6}$$

and

$$\sum_{n=m}^{\infty} \gamma_n n^{-1+\varepsilon} \leqq K(\varepsilon) \gamma_m m^{\varepsilon} \tag{2.7}$$

hold, respectively.

Recently we have tried to prove the dual of (2.5), that is, to give necessary and sufficient conditions pertaining to $\boldsymbol{\gamma}$ such that

$$\gamma_m \leqq K \widetilde{H}_m \quad \text{and} \quad \gamma_m \leqq K \widetilde{T}_m \tag{2.8}$$

should hold; but we could solve this problem only if the estimates

$$K_1\gamma_m \leqq \widetilde{H}_m \leqq K_2\gamma_m \quad \text{and} \quad K_1\gamma_m \leqq \widetilde{T}_m \leqq K_2\gamma_m$$

hold simultaneously. (See [3], Theorem 2.1.)

Now we give necessary and sufficient conditions for (2.8), but our conditions pertaining to the sums $\widetilde{H}_m$ and $\widetilde{T}_m$. Finally, with $\widetilde{H}_m$ and $\widetilde{T}_m$, we give necessary and sufficient conditions for the inequalities (2.5), too.

These assertions are collected in the following theorem.

THEOREM 3. (i) *The inequalities* (2.8) *hold if and only if there exists a positive* β *such that the sequence* $\{n^{-\beta}\widetilde{H}_m\}$ *is nonincreasing (* $\{n^{\beta}\widetilde{T}_m\}$ *is nondecreasing) for all sufficiently large* n.

(ii) *The inequalities* (2.5) *hold if and only if for some* $\beta > 0$ $\{n^{-\beta}\widetilde{H}_m\}$ *is nondecreasing (* $\{n^{\beta}\widetilde{T}_n\}$ *is nonincreasing) for all sufficiently large* n.

The lower estimates given in (2.8) show a further strange feature. Namely we have intended to improve the inequalities (2.8) as we did it in the case of the upper estimates by (2.6) and (2.7), but our attempt was aborted. Finally we have constructed a simple counterexample; this clearly proves that the generalization intended is impossible universally. Now we present a sequence $\boldsymbol{\gamma}$ which fulfills the inequalities (2.8), but the stricter inequalities

$$\gamma_m m^{\varepsilon} \leqq K\sum_{n=1}^{m}\gamma_n n^{-1+\varepsilon} \quad \text{and} \quad \gamma_m m^{-\varepsilon} \leqq K\sum_{n=m}^{\infty}\gamma_n n^{-1-\varepsilon} \tag{2.9}$$

do not hold for all m at any $\varepsilon > 0$. We do believe that everyone can easily check our assertions if the sequence $\boldsymbol{\gamma}$ is defined as follows:

$$\gamma_n := \begin{cases} \log n, & \text{if} \quad n \neq 2^k, \\ \log^2 n, & \text{if} \quad n = 2^k, \end{cases} \qquad k = 1, 2, \dots\,.$$

Naturally the inequalities (2.9) fail at $m = 2^k$.

3. Proofs

Proof of Theorem 1. First we verify the implication (2.1)$\Longrightarrow$(2.2). For the time being let ε be a small positive number satisfying the condition $2^{\varepsilon} < q$. Furthermore let $E := (2^{-\varepsilon}q - 1)^{-1}$ and $V := (1 - 2^{-\varepsilon})^{-1}$.

By (2.1) we have that

$$\sum_{m=0}^{n} 2^{-m\varepsilon}\sum_{k=0}^{m} q^k c_k \leqq K\sum_{m=0}^{n} 2^{-m\varepsilon} q^m \sum_{k=m}^{\infty} c_k.$$

Now, changing the order of summation in both sides, we get that

$$\sum_{k=0}^{n} q^k c_k \sum_{m=k}^{n} 2^{-m\varepsilon} \leqq K\left\{\sum_{k=0}^{n} c_k \sum_{m=0}^{k} 2^{-m\varepsilon} q^m + \sum_{k=n+1}^{\infty} c_k \sum_{m=0}^{n} 2^{-m\varepsilon} q^m\right\}.$$

Hence we obtain that

$$V\sum_{k=0}^{n} q^k c_k (2^{-k\varepsilon} - 2^{-(n+1)\varepsilon})$$
$$\leqq KE\left\{\sum_{k=0}^{n} c_k (2^{-(k+1)\varepsilon} q^{k+1} - 1) + \sum_{k=n+1}^{\infty} c_k (2^{-(n+1)\varepsilon} q^{n+1} - 1)\right\},$$

thus

$$(V - KE2^{-\varepsilon}q)\sum_{k=0}^{n} q^k c_k 2^{-k\varepsilon} \leqq V2^{-(n+1)\varepsilon}\sum_{k=0}^{n} q^k c_k + KEq2^{-\varepsilon}2^{-n\varepsilon}q^n \sum_{k=n}^{\infty} c_k - KE\sum_{k=0}^{\infty} c_k.$$

Applying again (2.1) we arrive at the inequalities

$$(V - KE2^{-\varepsilon}q)\sum_{k=0}^{n} q^k c_k 2^{-k\varepsilon} \leqq (VK2^{-\varepsilon} + KEq2^{-\varepsilon})2^{-n\varepsilon}q^n \sum_{k=n}^{\infty} c_k - KE\sum_{k=0}^{\infty} c_k. \quad (3.1)$$

Finally letting $\varepsilon \to 0$, it is easy to see that $V \to \infty$, whence, by virtue of (3.1) we see that if ε is small enough, then (2.2) holds with some constant $K(\varepsilon)$.

A similar method can be used to prove the implication (2.3)$\Longrightarrow$(2.4). Now let $E_1 := 2^\varepsilon/(2^\varepsilon - 1)$ and $V_1 := 1/(1 - q2^\varepsilon)$ assuming that $q2^\varepsilon < 1$.

Then (2.3) yields

$$\sum_{n=m}^{\infty} 2^{n\varepsilon} \sum_{k=n}^{\infty} q^k c_k \leqq K\sum_{n=m}^{\infty} 2^{n\varepsilon} q^n \sum_{k=0}^{n} c_k.$$

Changing the order of summations we obtain that

$$\sum_{k=m}^{\infty} q^k c_k \sum_{n=m}^{k} 2^{n\varepsilon} \leqq K\left\{\sum_{k=0}^{m} c_k \sum_{n=m}^{\infty} 2^{n\varepsilon} q^n + \sum_{k=m+1}^{\infty} c_k \sum_{n=k}^{\infty} 2^{n\varepsilon} q^n\right\}.$$

Consequently

$$E_1\sum_{k=m}^{\infty} q^k c_k (2^{k\varepsilon} - 2^{m\varepsilon}) \leqq K\left\{\sum_{k=0}^{m} c_k V_1 2^{m\varepsilon} q^m + \sum_{k=m+1}^{\infty} c_k V_1 2^{k\varepsilon} q^k\right\},$$

whence, using again (2.3), we get

$$(E_1 - KV_1)\sum_{k=m}^{\infty} q^k c_k 2^{k\varepsilon} \leqq E_1 K2^{m\varepsilon} q^m \sum_{k=0}^{m} c_k + KV_1 q^m 2^{m\varepsilon} \sum_{k=0}^{m} c_k.$$

Since $E_1 \to \infty$ if $\varepsilon \to 0$, thus the inequality above clearly conveys (2.4).

The proof is complete. □

Proof of Theorem 2. The proof is carried out analogously to the proof of Theorem 1. Let ε be again a small positive number satisfying the condition $K\varepsilon < 1$, where K denotes the constant given in (2.5).

Then, by (2.5),

$$\sum_{n=1}^{m} n^{-\varepsilon-1} \sum_{k=1}^{n} \gamma_k k^{-1} \leqq K \sum_{n=1}^{m} n^{-\varepsilon-1} \gamma_n \tag{3.2}$$

and

$$\begin{aligned}\sum_{k=1}^{m} \gamma_k k^{-1} \sum_{n=k}^{m} n^{-\varepsilon-1} &\geqq \frac{1}{\varepsilon} \sum_{k=1}^{m} \gamma_k k^{-1} (k^{-\varepsilon} - m^{-\varepsilon}) \\ &= \frac{1}{\varepsilon} \sum_{k=1}^{m} \gamma_k k^{-1-\varepsilon} - \frac{m^{-\varepsilon}}{\varepsilon} \sum_{k=1}^{m} \gamma_k k^{-1},\end{aligned} \tag{3.3}$$

thus, by (2.5), (3.2) and (3.3) we get that

$$\left(\frac{1}{\varepsilon} - K\right) \sum_{k=1}^{m} \gamma_k k^{-1-\varepsilon} \leqq \frac{m^{-\varepsilon}}{\varepsilon} \sum_{k=1}^{m} \gamma_k k^{-1} \leqq \frac{K}{\varepsilon} m^{-\varepsilon} \gamma_m,$$

whence (2.6) plainly follows.

The proof of (2.7) runs analogously with the change that we start as follows:

$$\sum_{n=m}^{\infty} n^{\varepsilon-1} \sum_{k=n}^{\infty} \gamma_k k^{-1} \leqq K \sum_{n=m}^{\infty} n^{\varepsilon-1} \gamma_n,$$

and we change the order of summation on the left-hand side.

Omitting the details, the theorem is proved. □

Proof of Theorem 3. First we prove that if

$$\gamma_m \leqq K \widetilde{H}_m \tag{3.4}$$

holds, then there exists a $\beta > 0$ such that the sequence $\{n^{-\beta} \widetilde{H}_n\}$ is nonincreasing. Namely (3.4) implies that

$$\gamma_{m+1} \leqq N \widetilde{H}_{m+1} = N \left(\widetilde{H}_m + \frac{\gamma_{m+1}}{m+1} \right), \tag{3.5}$$

where $N (\geqq K)$ is an integer. By (3.5)

$$\gamma_{m+1} \left(1 - \frac{N}{m+1} \right) \leqq N \widetilde{H}_m$$

and

$$\gamma_{m+1} \leqq N \widetilde{H}_m \frac{m+1}{m+1-N}.$$

This and (3.5) yield

$$\widetilde{H}_{m+1} \leqq \widetilde{H}_m \left(1 + \frac{N}{m+1-N} \right) = \widetilde{H}_m \frac{m+1}{m+1-N}.$$

Hence

$$\frac{\widetilde{H}_{m+1}}{(m+1)^\beta} \leqq \frac{\widetilde{H}_m}{(m+1)^\beta}\frac{m+1}{m+1-N} \leqq \frac{\widetilde{H}_m}{m^\beta}$$

holds if

$$\frac{m+1}{m+1-N} \leqq \left(\frac{m+1}{m}\right)^\beta \tag{3.6}$$

is satisfied. But an elementary consideration, using e.g. Bernoulli inequality, shows that (3.6) holds if $\beta = 2N$ and $m \geqq 3N$.

Similarly if

$$\gamma_m \leqq N\widetilde{T}_m = N\left(\frac{\gamma_m}{m} + \widetilde{T}_{m+1}\right),$$

then

$$\gamma_m \leqq N\widetilde{T}_{m+1}\frac{m}{m-N},$$

whence

$$\widetilde{T}_m \leqq \widetilde{T}_{m+1}\frac{m}{m-N}$$

holds. Thus it suffices to show that

$$m^\beta\widetilde{T}_m \leqq m^\beta\widetilde{T}_{m+1}\frac{m}{m-N} \leqq \widetilde{T}_{m+1}(m+1)^\beta \tag{3.7}$$

uphold. As before, an easy calculation shows that if $\beta = 2N$ and $m \geqq 2N$, then (3.7) maintains.

Herewith the necessary part of the statement (i) is verified.

The proof of the sufficience of (i) is shorter. Namely if $\{\widetilde{H}_m m^{-\beta}\}$ is nonincreasing then

$$\begin{aligned}\frac{\gamma_m}{m} &= \widetilde{H}_m - \widetilde{H}_{m-1} = m^\beta\widetilde{H}_m m^{-\beta} - (m-1)^\beta\widetilde{H}_{m-1}(m-1)^{-\beta}\\ &\leqq \widetilde{H}_m m^{-\beta}(m^\beta - (m-1)^\beta) \leqq K\widetilde{H}_m m^{-1}.\end{aligned} \tag{3.8}$$

Analogously, if $\{\widetilde{T}_m m^\beta\}$ is nondecreasing, then

$$\frac{\gamma_m}{m} = \widetilde{T}_m - \widetilde{T}_{m+1} \leqq m^\beta\widetilde{T}_m(m^{-\beta} - (m+1)^{-\beta}) \leqq K\widetilde{T}_m m^{-1}. \tag{3.9}$$

The statement (i) is proved.

The statements (ii) can be proved by similar arguments.

If

$$\widetilde{H}_m \leqq K\gamma_m \leqq N\gamma_m,$$

then

$$\gamma_m \geqq \frac{\widetilde{H}_m}{N} = \frac{\widetilde{H}_{m-1}}{N} + \frac{\gamma_m}{Nm}, \tag{3.10}$$

thus

$$\gamma_m \geqq \widetilde{H}_{m-1} \frac{m}{Nm-1},$$

whence, by (3.10),

$$\widetilde{H}_m \geqq \widetilde{H}_{m-1} \frac{Nm}{Nm-1}, \tag{3.11}$$

furthermore, by (3.11),

$$m^{-\beta} \widetilde{H}_m \geqq \widetilde{H}_{m-1} (m-1)^{-\beta} \tag{3.12}$$

holds if $\beta = (N+1)^{-1}$ and $m \geqq N$.

If

$$\widetilde{T}_m \leqq N\gamma_m,$$

then, by

$$\gamma_m \geqq \frac{\widetilde{T}_m}{N} = \frac{\gamma_m}{Nm} + \frac{\widetilde{T}_{m+1}}{N}$$

we arrive to

$$\widetilde{T}_m \geqq \widetilde{T}_{m+1} \frac{Nm}{Nm-1},$$

and hence

$$m^{\beta} \widetilde{T}_m \geqq (m+1)^{\beta} \widetilde{T}_{m+1} \tag{3.13}$$

follows if $\beta = N^{-1}$ and $m \geqq 2$.

If (3.12) and (3.13) hold, then in the estimates (3.8) and (3.9) the inequality signs "$\leqq$" turn to "$\geqq$", herewith the sufficiency of (3.12) and (3.13) in the case of estimates (2.5) is also proved.

The proof of Theorem 3 is complete. □

4. Application

Finally we present an application of the second part of Theorem 1, that is, we show that the implication (2.3)$\Longrightarrow$(2.4) with

$$q := 2^{-p},\ p > 0 \quad \text{and} \quad c_0 := x^{-p} \int_0^x w(t)dt,\ c_n := x^{-p} \int_{2^{n-1}x}^{2^n x} w(t)dt,\ n \geqq 1; \tag{4.1}$$

proves that if $w \in AM_p$ then $w \in AM_{p-\varepsilon}$ also holds for some $\varepsilon > 0$.

Namely if $w \in AM_p$, then by (1.7), assuming that $y := 2^m x,\ x > 0,\ m = 0,1,\ldots$ and $0 < p < \infty$, we obtain that

$$I := \int_{2^m x}^{\infty} t^{-p} w(t)dt \leqq K(2^m x)^{-p} \int_0^{2^m x} w(t)dt. \tag{4.2}$$

Since

$$I = \sum_{n=m+1}^{\infty} \int_{2^{n-1}x}^{2^n x} t^{-p} w(t)dt \geqq \sum_{n=m+1}^{\infty} 2^{-pn} x^{-p} \int_{2^{n-1}x}^{2^n x} w(t)dt, \tag{4.3}$$

thus, (4.1), (4.2) and (4.3) imply that

$$\sum_{n=m+1}^{\infty} 2^{-pn} c_n \leqq K 2^{-pm} x^{-p} \int_0^{2^m x} w(t)dt = K 2^{-mp} \sum_{n=0}^{m} c_n. \tag{4.4}$$

By (4.4) it is clear that then (2.3) also holds, e.g. with $2K$ in place of K. Consequently we can utilize the implication (2.3)$\Longrightarrow$(2.4), and this conveys the inequality

$$\sum_{n=m+1}^{\infty} 2^{n(\varepsilon-p)} c_n \leqq K(\varepsilon) 2^{m(\varepsilon-p)} \sum_{n=0}^{m} c_n. \tag{4.5}$$

Multiplying both sides of (4.5) by x^{ε} and substituting the definitions of c_n, we get that

$$\sum_{n=m+1}^{\infty} 2^{n(\varepsilon-p)} x^{\varepsilon-p} \int_{2^{n-1}x}^{2^n x} w(t)dt \leqq K(\varepsilon) 2^{m(\varepsilon-p)} \left\{ x^{\varepsilon-p} \left(\int_0^x w(t)dt + \sum_{n=1}^{m} \int_{2^{n-1}x}^{2^n x} w(t)dt \right) \right\},$$

that is,

$$2^{(\varepsilon-p)} \int_{2^m x}^{\infty} t^{\varepsilon-p} w(t)dt \leqq K(\varepsilon)(2^m x)^{\varepsilon-p} \int_0^{2^m x} w(t)dt,$$

and this inequality means that $w \in AM_{p-\varepsilon}$, as stated.

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László Leindler
Bolyai Intézet
Aradi vértanúk tere 1
Szeged
Hungary H-6720
e-mail: leindler@math.u-szeged.hu