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ITERATIVE APPROXIMATIONS FOR MULTIVALUED NONEXPANSIVE MAPPINGS IN REFLEXIVE BANACH SPACES

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Abstract. In this paper, we established the strong convergence of Browder type iteration $\{x_t\}$ for the multivalued nonexpansive nonself-mapping T satisfying the weakly inwardness condition in a reflexive and strictly convex Banach space E with a uniformly Gâteaux differentiable norm or in a reflexive Banach space with weakly sequentially continuous duality mapping. Furthermore, we also obtained the strong convergent results for the Halpern type iteration $\{x_n\}$ for multivalued nonexpansive nonself-mapping T.

1. Introduction

Let E be a Banach space and K a nonempty subset of E. We shall denote by 2^E the family of all subsets of E, $CB(E)$ the family of nonempty closed and bounded subsets of E and denote $C(E)$ by the family of nonempty compact subsets of E and $CC(E)$ stands for the family of nonempty compact convex subsets of E. Let H be the extended Hausdorff metric on the nonempty closed subsets of E, that is,

$$H(A,B) = \max\{\rho \geqslant 0 : A \subseteq N_\rho(B) \text{ and } B \subseteq N_\rho(A)\},$$

where $N_\rho(S) = \{u \in E : \inf_{x\in S} \|u-x\| \leqslant \rho\}$. It is well known that, if $A, B \in CB(E)$, then H is the Hausdorff metric as usual. For more detail, see Kirk[8] and Xu[25].

A mapping $T : K \to 2^E$ is called *nonexpansive* (resp., *contractive*) if, for any $x, y \in K$,

$$H(Tx,Ty)\| \leqslant \|x-y\|,$$

$$(\text{resp., } H(Tx,Ty) \leqslant k\|x-y\| \quad \text{for some } k \in (0,1)).$$

Since Banach's Contraction Mapping Principle was extended nicely to multivalued mappings by Nadler [13] in 1969, many authors have studied the fixed point theory for

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multivalued mappings (e.g., see [1–3, 6, 8–11, 16, 25, 27]). For example, Downing and Kirk [3] proved the following result in 1977.

THEOREM DK. ([3]) *Let K be a nonempty closed convex subset of a Banach space E and $T : K \to C(E)$ be a contraction. If $T(x) \in cl(I_K(x))$ for each $x \in K$, then T has a fixed point.*

Recently, another results for multivalued contractive mappings were obtained by Kirk [8] via the transfinite induction arguments and the extended Hausdorff metric on the nonempty closed subsets of E.

THEOREM K. ([8]) *Let D be a nonempty closed subset of a Banach space E and $T : D \to 2^X \setminus \emptyset$ be a multivalued contraction with closed values which is weakly inward on D. Then T has a fixed point.*

The above result (Theorem K) was proved by Xu [25] in 2001 for the mapping T satisfying the condition that each $x \in E$ has a nearest point in Tx.

The following theorem for multivalued nonexpansive mappings was given by Xu [25] also.

THEOREM X. *If C is a compact convex subset of a Banach space E and $T : C \to CC(E)$ is a nonexpansive mapping satisfying the boundary condition:*

$$Tx \cap I_C(x) \neq \emptyset, \quad \forall x \in C,$$

then T has a fixed point.

Let K b e a nonempty closed convex subset of a Banach space E and, for all $u \in K$ and $t \in (0,1)$, a nonexpansive mapping $T : K \to C(E)$ be weakly inward on K. Then we can define a contraction $G_t : K \to C(E)$ by $G_t x := (1-t)Tx + tu$ for all $x \in K$. Theorem DK or Theorem K assures that there exists $x_t \in K$ (non-unique, in general, see [13]) such that

$$x_t \in (1-t)Tx_t + tu. \tag{1.1}$$

For a single valued nonexpansive self- or nonself- mapping T, the strong convergence of $\{x_t\}$ as $t \to 0$ was studied in Hilbert space or certain Banach spaces by many authors (see [6, 11, 17, 21–24]). However, a simple example given by Pietramala [15] shows that the sequence $\{x_t\}$ doesn't converge strongly as $t \to 0$ for multivalued nonexpansive mappings even if E is Euclidean (also see [7]).

Now, a natural question arises whether the strongly convergent results of $\{x_t\}$ or $\{x_n\}$ defined by (1.2) for single valued nonexpansive mapping T can be extended to the multivalued case:

$$x_{n+1} \in (1-\alpha_n)Tx_n + \alpha_n u. \tag{1.2}$$

In 1995, G. Acedo and Xu [1] gave the strong convergence of $\{x_t\}$ under the restriction $F(T) = z$ in Hilbert space. Recently, Sahu [16] also studied the multivalued case in a uniformly convex Banach space with a uniformly Gáteaux differentiable norm. Very recently, Jung [7] obtained strong convergence theorems for $\{x_t\}$ of multivalued nonexpansive nonself-mappings in the frame of uniformly convex Banach spaces with a uniformly Gáteaux differentiable norm.

In this paper, we establish the strong convergence of $\{x_t\}$ defined by (1.1) for the multivalued nonexpansive nonself-mapping T satisfying the inwardness condition in a reflexive and strictly convex Banach space E with a uniformly Gáteaux differentiable norm. We also study the strong convergence of $\{x_t\}$ in a reflexive Banach space with weakly sequentially continuous duality mapping. Our results improve and extend the results in [6, 11, 17, 21–24] to the multivalued case and give the extensions and complements of the results of Jung [7], Acedo and Xu [1] and other existent literatures. Furthermore, we obtain the strong convergent results for the explicit iteration $\{x_n\}$ defined by (1.2) for multivalued nonexpansive nonself-mapping T.

2. Preliminaries

Let E be a real Banach space and J denote the *normalized duality mapping* from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x,f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}, \ \forall x \in E,$$

where E^* is the dual space of E and $\langle \cdot,\cdot\rangle$ denotes the generalized duality pairing. In the sequel, we denote the single-valued duality mapping by j and denote $F(T) = \{x \in D(T) : x \in Tx\}$, the fixed point set of T, where $D(T)$ is domain of T. If $K \subset E$, then $cl(K)$, $int(K)$ and $\partial(K)$ will stand for the closure, interior and boundary of K, respectively. We denote the weak convergence of the sequence $\{x_n\}$ to x as $x_n \rightharpoonup x$ and the strong convergence of the sequence x_n as $x_n \to x$, respectively.

For all $x \in K$, we define the *inward set* $I_K(x)$ as follows ([8, 15, 16, 21, 22, 25]):

$$I_K(x) = \{y \in E : y = x + \lambda(z - x), \ \ z \in K, \lambda \geqslant 0\}.$$

We say that a mapping $T : K \to 2^E$ satisfies the *inward condition* if $Tx \subset I_C(x)$ for all $x \in K$ and the mapping T satisfies the *weakly inward condition* if, for each $x \in K$, $Tx \subset cl(I_K(x))$. Clearly, $K \subset I_K(x)$ and it is not hard to show that $I_K(x)$ is a convex set as K does.

If Banach space E admits sequentially continuous duality mapping J from weak topology to weak star topology, then, by [5, Lemma 1], we know that the duality mapping J is single-valued. In this case, the duality mapping J is also said to be *weakly sequentially continuous*, that is, if $\{x_n\}$ is a subset of E with $x_n \rightharpoonup x$, then $J(x_n) \overset{*}{\rightharpoonup} J(x)$.

A Banach space E is said to be satisfy *Opial's condition* [14] if, for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ $(n \to \infty)$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \ \forall y \in E \ (x \neq y).$$

We know that Hilbert spaces and $l^p(l < p < \infty)$ satisfy Opial's condition and Banach spaces with weakly sequentially continuous duality mappings satisfy Opial's condition [5, 27].

Recall that the norm of a Banach space E is said to be *Gâteaux differentiable* (or E is said to be *smooth*) if the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for all x,y on the unit sphere $S(E)$ of E. Moreover, if, for all $y \in S(E)$, the limit defined by (2.1) is uniformly attained for each $x \in S(E)$, then we say that the norm of E is *uniformly Gâteaux differentiable*. The norm of E is said to be *Fréchet differentiable* if, for all $x \in S(E)$, the limit (2.1) is attained uniformly for each $y \in S(E)$. The norm of E is said to be *uniformly Fréchet differentiable* (or E is said to be *uniformly smooth*) if the limit (2.1) is attained uniformly for all $(x,y) \in S(E) \times S(E)$.

A Banach space E is said to *strictly convex* if

$$\|x\| = \|y\| = 1,\ x \neq y \ \text{implies} \ \frac{\|x+y\|}{2} < 1.$$

A Banach space E is said to *uniformly convex* if $\delta_E(\varepsilon) > 0$ for all $\varepsilon > 0$, where $\delta_E(\varepsilon)$ is *modulus of convexity* of E defined by

$$\delta_E(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| \leqslant 1, \|y\| \leqslant 1, \|x-y\| \geqslant \varepsilon\},\ \forall \varepsilon \in [0,2].$$

The following results are well known (see [12, 20]):

(i) The duality mapping J in a smooth Banach space E is single valued and strong-weak* continuous ([20, Lemma 4.3.3]).

(ii) If E is a Banach space with a uniformly Gâteaux differentiable norm, then the mapping $J : E \longrightarrow E^*$ is single-valued and norm to weak star uniformly continuous on bounded sets of E ([20, Theorem 4.3.6]).

(iii) A uniformly convex Banach space E is reflexive and strictly convex ([20, Theorem 4.1.6, Theorem 4.1.2]).

If C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a mapping $P : C \to D$ is called a *retraction* from C to D if P is continuous with $F(P) = D$. A mapping $P : C \to D$ is said to be *sunny* if

$$P(Px + t(x - Px)) = Px,\ \forall x \in C,$$

whenever $Px + t(x - Px) \in C$ and $t > 0$. A subset D of C is called a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D. For some more details, see [5, 20].

The following lemma is well known [5, 7, 20]:

LEMMA 2.1. *Let C be a nonempty convex subset of a smooth Banach space E, $D \subset C$, $J : E \to E^*$ be the (normalized) duality mapping of E and $P : C \to D$ be a retraction. Then the following are equivalent:*

(1) $\langle x - Px, j(y - Px) \rangle \leqslant 0$ *for all $x \in C$ and $y \in D$.*

(2) *P is both sunny and nonexpansive.*

In the sequel, we also need the following lemma that can be found in the existing literature [23, 24]:

LEMMA 2.2. *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the property:*

$$a_{n+1} \leqslant (1-\gamma_n)a_n + \gamma_n\beta_n, \ \forall n \geqslant 0,$$

where $\{\gamma_n\} \subset (0,1)$ *and* $\{\beta_n\} \subset \mathbb{R}$ *such that*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty} \beta_n \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n\beta_n| < +\infty$.

Then $\{a_n\}$ *converges to zero as* $n \to \infty$.

Let μ be a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on N if and only if

$$\inf\{a_n : n \in N\} \leqslant \mu(a) \leqslant \sup\{a_n : n \in N\}$$

for all $a = (a_1, a_2, \cdots) \in l^\infty$. Sometime, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on N is called a *Banach limit* if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for all $a = (a_1, a_2, \cdots) \in l^\infty$.

Furthermore, we know the following result [19, Lemma 1] and [20, Lemma 4.5.4]:

LEMMA 2.3. ([19, Lemma 1]) *Let* C *be a nonempty closed convex subset of a Banach space* E *with a uniformly Gâteaux differentiable norm. Let* $\{x_n\}$ *be a bounded sequence of* E *and* μ *be a mean on* N. *let* $z \in C$. *Then*

$$\mu_n\|x_n - z\|^2 = \min_{y\in C} \mu_n\|x_n - y\|^2$$

if and only if

$$\mu_n\langle y - z, j(x_n - z)\rangle \leqslant 0, \ \forall y \in C.$$

3. The strongly convergent theorems of $\{x_t\}$

PROPOSITION 3.1. *Let* K *be a nonempty convex subset of a Banach space* E. *Suppose that* $T : K \to 2^E \setminus \emptyset$ *is a nonexpansive mapping with closed values which is weakly inward on* K. *Then we have the following:*

(1) *For any* $t \in (0,1)$ *and* $u \in K$, *there exists* $x_t \in K$ *such that*

$$x_t \in tu + (1-t)Tx_t. \tag{3.1}$$

In addition, suppose that $F(T) \neq \emptyset$ *satisfying* $T(y) = \{y\}$ *for any* $y \in F(T)$.

(2) *For any fixed* $y \in F(T)$, $\|x_t - y\|^2 \leqslant \langle u - y, j(x_t - y)\rangle$.

(3) $\{x_t\}$ *is bounded and, moreover,* $\lim_{t\to 0} d(x_t, Tx_t) = 0$.

(4) *For any* $y \in F(T)$, $\langle x_t - u, j(x_t - y)\rangle \leqslant 0$.

Proof. For any given $t \in (0,1)$, we can define a multivalued contraction $G_t : K \to 2^E \setminus \emptyset$ by $G_t = tu + (1-t)T$ for $u \in K$. It is easily proved by the weak inwardness of T and the convexity of K that G_t is weak inward on K. An application of Theorem K yields a fixed point x_t of G_t for each $t \in (0,1)$, that is, there exists $x_t \in K$ such that $x_t \in tu + (1-t)Tx_t$. Thus (1) is proved.

Notice that the assumptions that $F(T) \neq \emptyset$ and $T(y) = \{y\}$ for any fixed point $y \in F(T)$ guarantee (2) and (3). In fact, for any given x_t, there exists $y_t \in Tx_t$ such that

$$x_t = tu + (1-t)y_t. \tag{3.2}$$

For any given $y \in F(T)$, we have

$$\begin{aligned}
\|x_t - y\|^2 &= t\langle u - y, j(x_t - y)\rangle + (1-t)\langle y_t - y, j(x_t - y)\rangle \\
&\leqslant t\langle u - y, j(x_t - y)\rangle + (1-t)d(y_t, Ty)\|j(x_t - y)\| \\
&\leqslant t\langle u - y, j(x_t - y)\rangle + (1-t)H(Tx_t, Ty)\|x_t - y\| \\
&\leqslant t\langle u - y, j(x_t - y)\rangle + (1-t)\|x_t - y\|^2
\end{aligned}$$

and so

$$\|x_t - y\|^2 \leqslant \langle u - y, j(x_t - y)\rangle \leqslant \|u - y\|\|x_t - y\|. \tag{3.3}$$

If $\|x_t - y\| = 0$, then the result is obvious. Let $\|x_t - y\| > 0$. Then it follows from (3.3) that

$$\|x_t - y\| \leqslant \|u - y\|.$$

This shows the boundedness of the net $\{x_t\}$. From (3.2), we obtain

$$\|y_t\| = \frac{\|x_t - tu\|}{1-t} \leqslant \frac{\|x_t\| + t\|u\|}{1-t}.$$

Therefore, y_t is also bounded (as $t \to 0$). Hence, as $t \to 0$,

$$d(x_t, Tx_t) \leqslant \|x_t - y_t\| = t\|u - y_t\| \to 0,$$

that is,

$$\lim_{t \to 0} d(x_t, Tx_t) = 0. \tag{3.4}$$

This shows (2) and (3).

Finally, we prove (4). It follows from (2) that

$$\begin{aligned}
\langle x_t - u, j(x_t - y)\rangle &= \langle x_t - y, j(x_t - y)\rangle + \langle y - u, j(x_t - y)\rangle \\
&= \|x_t - y\|^2 - \langle u - y, j(x_t - y)\rangle \leqslant 0.
\end{aligned}$$

This completes the proof. □

Subsequently, we show the strongly convergent theorems of x_t as $t \to 0$. Recall that a set A of M is a *Chebyshev set* if, for all $x \in M$, there exists a unique element $y \in A$ such that $d(x,y) = d(x,A)$, where (M,d) is a metric space and $d(x,A) = \inf_{y \in A} d(x,y)$.

THEOREM 3.2. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, K be a nonempty closed convex subset of E and $T : K \to C(E)$ be a nonexpansive mapping which is weakly inward on K. Suppose that $F(T) \neq \emptyset$ satisfying $T(y) = \{y\}$ for any fixed point $y \in F(T)$. Then, as $t \to 0$, the net $\{x_t\}$ defined by (3.1) converges strongly to a fixed point of T.*

Proof. It follows from Proposition 3.1 (3) that the net $\{x_t\}$ is bounded. We claim that the set $\{x_t : t \in (0,1)\}$ is sequentially compact. As the matter of fact, suppose that $x_n := x_{t_n}$ and

$$g(x) = \mu_n \|x_n - x\|^2, \ \forall x \in K,$$

where $\{t_n\}$ be a sequence in $(0,1)$ that converges to 0 $(n \to \infty)$ and μ_n is a Banach limit. Define the set

$$K_1 = \{x \in K; g(x) = \inf_{y \in K} g(y)\}.$$

Since E be a reflexive Banach space, K_1 is a nonempty bounded closed convex subset of $K \subset E$ (see [20, Theorem 1.3.11]). For all $x \in K_1$, the compactness of Tx implies that there exists $z_n \in Tx$ such that $\|x_n - z_n\| = d(x_n, Tx)$ and $z_n \to z \in Tx$. Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ by Proposition 3.1 (3), we have

$$\begin{aligned} g(z) &= \mu_n \|x_n - z\|^2 \leqslant \mu_n (\|x_n - z_n\| + \|z_n - z\|)^2 \\ &= \mu_n d(x_n, Tx)^2 \leqslant \mu_n (d(x_n, Tx_n) + H(Tx_n, Tx))^2 \\ &\leqslant \mu_n \|x_n - x\|^2 = g(x). \end{aligned}$$

Hence $z \in Tx \cap K_1$, that is, $Tx \cap K_1 \neq \emptyset$ for all $x \in K_1$. Since $F(T) \neq \emptyset$, let $y \in F(T)$. Since every nonempty closed convex subset of a strictly convex and reflexive Banach space E is a Chebyshev set (see [12, Corollary 5.1.19]), there exists a unique element $q \in K_1$ such that

$$\|y - q\| = \inf_{x \in K_1} \|y - x\|.$$

By $Tq \cap K_1 \neq \emptyset$, taking $z^* \in Tq \cap K_1$ and using $Ty = \{y\}$, then we have

$$\|y - z^*\| = d(Ty, z^*) \leqslant H(Ty, Tq) \leqslant \|y - q\|.$$

Hence $q = z^* \in Tq$ by the uniqueness of q in K_1. Using Lemma 2.3 and the definition of K_1, we get

$$\mu_n \langle x - q, j(x_n - q) \rangle \leqslant 0, \ \forall x \in K.$$

By Proposition 3.1 (2), taking $x = u \in K$, then we have

$$\mu_n \|x_n - q\|^2 \leqslant \mu_n \langle u - q, j(x_n - q) \rangle \leqslant 0,$$

that is,

$$\mu_n \|x_n - q\|^2 = 0.$$

Therefore, $\{x_n\}$ exists a subsequence which still denotes $\{x_n\}$ strongly converge to $q \in F(T)$.

Next, we show that $x_t \to q$ as $t \to 0$. Since the net $\{x_t\}$ is bounded and the duality mapping J is single-valued and norm to weak* uniformly continuous on bounded sets of a Banach space E with uniformly Gâteaux differentiable norm, we have that, for any $y \in F(T)$, as $x_n \to q$ and $n \to \infty$,

$$
\begin{aligned}
&|\langle x_n - u, j(x_n - y)\rangle - \langle q - u, j(q - y)\rangle| \\
&\quad = |\langle x_n - q, j(x_n - y)\rangle + \langle q - u, j(x_n - y) - j(q - y)\rangle| \\
&\quad \leqslant \|x_n - q\|\|x_n - y\| + |\langle q - u, j(x_n - y) - j(q - y)\rangle| \to 0.
\end{aligned} \tag{3.5}
$$

Therefore, from Proposition 3.1 (4), for any $y \in F(T)$,

$$\langle q - u, j(q - y)\rangle = \lim_{n\to\infty} \langle x_n - u, j(x_n - y)\rangle \leqslant 0.$$

To prove that the entire net $\{x_t\}$ converges to q, suppose that there exists another sequence $\{x_{s_k}\} \subset \{x_t\}$ such that $x_{s_k} \to p$ as $s_k \to 0$. Then we also have $p \in F(T)$ and $\langle q - u, j(q - y)\rangle \leqslant 0$. Now, interchanging y and p or q, then we obtain

$$\langle q - u, j(q - p)\rangle \leqslant 0, \quad \langle p - u, j(p - q)\rangle \leqslant 0.$$

Thus we have

$$\|p - q\|^2 = \langle p - q, j(p - q)\rangle \leqslant 0.$$

That is, $p = q$. Therefore, we have proved that the set $\{x_t\}$ is sequentially compact and each cluster point of $\{x_t\}$ (as $t \to 0$) equals $q \in F(T)$. Therefore, $x_t \to q$ as $t \to 0$. This completes the proof. □

COROLLARY 3.3. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, K be a nonempty closed convex subset of E and $T : K \to C(E)$ be a nonexpansive mapping which is weakly inward on K. Suppose that $F(T) \neq \emptyset$ satisfying $T(y) = \{y\}$ for any fixed point $y \in F(T)$. Then, as $t \to 0$, the net $\{x_t\}$ defined by (3.1) converges strongly to a fixed point of T.*

REMARK 3.1. (1) Both Theorem 3.2 and Corollary 3.3 can be considered as an extension of Theorem 1 in [7] from uniformly convex Banach spaces to reflexive and strictly convex Banach spaces. At the same time, Theorem 3.2 also doesn't use the hypothesis for K as a nonexpansive retract of E.

(2) Theorem 3.2 extends Theorem 3.1 in [18] to the multivalued version and Corollary 3.3 extend also the main results of [6, 11, 21–24] to the multivalued version.

(3) Corollary 3.3 can be apply to all L^p spaces or l^p spaces for $1 < p < \infty$.

THEOREM 3.4. *Let E be a reflexive Banach space with weakly sequentially continuous duality mapping, K be a nonempty closed convex subset of E and $T : K \to C(E)$ be a nonexpansive mapping which is weakly inward on K. Suppose that $F(T) \neq \emptyset$ satisfying $T(y) = \{y\}$ for any fixed point $y \in F(T)$. Then, as $t \to 0$, the net $\{x_t\}$ defined by (3.1) converges strongly to a fixed point of T. In this case, letting $Pu = \lim_{t\to 0} x_t$, then P is unique sunny nonexpansive retraction from K to $F(T)$.*

Proof. Similarly to Theorem 3.2, we firstly show that the set $\{x_t : t \in (0,1)\}$ is sequentially compact. Indeed, since E is reflexive, the boundedness of the net $\{x_t\}$ implies that $\{x_t\}$ is weakly sequentially compact (see [20, Theorem 1.2.14]). Namely, there exists a weakly convergence subsequence $\{x_{t_n}\} \subseteq \{x_t\}$, where $\{t_n\}$ is a sequence in $(0,1)$ that converges to 0 as $n \to \infty$.

Now, we suppose $x_n := x_{t_n}$ and $x_n \rightharpoonup p \in K$. For this p, the compactness of Tp implies that there exists $z_n \in Tp$ such that

$$\|x_n - z_n\| = d(x_n, Tp), \quad z_n \to z \in Tp.$$

Assume that $z \neq p$. Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ by Proposition 3.1 (3), in Banach spaces with weakly sequentially continuous duality mappings satisfying Opial's condition (see [5, Theorem 5]), then we have

$$\begin{aligned} \limsup_{n\to\infty} \|x_n - z\| &\leqslant \limsup_{n\to\infty} (\|x_n - z_n\| + \|z_n - z\|) \\ &= \limsup_{n\to\infty} d(x_n, Tp) \\ &\leqslant \limsup_{n\to\infty} (d(x_n, Tx_n) + H(Tx_n, Tp)) \\ &\leqslant \limsup_{n\to\infty} \|x_n - p\| \\ &< \limsup_{n\to\infty} \|x_n - z\|, \end{aligned} \tag{3.6}$$

which is a contradiction. Hence $p = z \in Tp$. From Proposition 3.1 (2), interchanging p and y, then we obtain

$$\|x_n - p\|^2 \leqslant \langle u - p, j(x_n - p)\rangle .$$

Using the fact that j is weakly sequentially continuous, we get

$$x_n \to p \ \ (n \to \infty).$$

Thus we have proved that there exists a subsequence $\{x_{t_n}\}$ of $\{x_t : t \in (0.1)\}$ that converges to a fixed point p of T.

To prove that the entire net $\{x_t\}$ converges to p, suppose that there exists another subsequence $\{x_{s_k}\} \subset \{x_t\}$ such that $x_{s_k} \to q$ as $s_k \to 0$. Then we also have $q \in F(T)$.

Since the set $\{x_t\}$ is bounded and the duality map J is single-valued and weakly sequentially continuous from E to E^*, using the same argument as in (3.5), for any $y \in F(T)$, we get

$$\langle q - u, j(q - y)\rangle \ = \lim_{s_k\to 0} \langle x_{s_k} - u, j(x_{s_k} - y)\rangle \ \leqslant 0$$

and

$$\langle p - u, j(p - y)\rangle \ = \lim_{n\to\infty} \langle x_{t_n} - u, j(x_{t_n} - y)\rangle \ \leqslant 0.$$

Using similar methods to Theorem 3.2, we have $p = q$ and $x_t \to p$ as $t \to 0$. Furthermore, p is the unique solution in $F(T)$ satisfying the following variational inequality:

$$\langle p - u, j(p - y)\rangle \ \leqslant 0, \quad \forall y \in F(T).$$

Let $Pu = \lim_{t \to 0} x_t$ for any $u \in K$. Then we have

$$\langle Pu - u, j(Pu - y)\rangle \leqslant 0, \quad \forall y \in F(T).$$

It follows from Lemma 2.1 that P is unique sunny nonexpansive retraction from K to $F(T)$. This complete the proof. □

REMARK 3.2. Theorem 3.4 can be considered as the multivalued version of [26, Theorem 3.1] and [17, Theorem 2.2].

COROLLARY 3.5. *Let E be a reflexive Banach space with weakly sequentially continuous duality mapping, K be a nonempty closed convex subset of E and $T : K \to C(K)$ be a nonexpansive mapping. Suppose that $F(T) \neq \emptyset$ satisfying $T(y) = \{y\}$ for any fixed point $y \in F(T)$. Then $F(T)$ is a sunny nonexpansive retract of K. In this case, if x_t is defined by (3.1) and $Pu = \lim_{t \to 0} x_t$, then P is unique sunny nonexpansive retraction from K to $F(T)$.*

4. The strongly convergent theorems of $\{x_n\}$

LEMMA 4.1. *([13]) Let X be a complete metric space and $A, B \in C(X)$. Then, for any $a \in A$, there exists $b \in B$ such that*

$$d(a, b) \leqslant H(A, B).$$

Let K be a nonempty closed convex subset of Banach space E and $T : K \to C(K)$ be a multivalued nonexpansive mapping. Let $\alpha_n \in (0, 1)$ and $x_0 \in K$. For any given $u \in K$, let $y_0 \in Tx_0$ such that

$$x_1 = \alpha_0 u + (1 - \alpha_0) y_0.$$

By Lemma 4.1, we can choose $y_1 \in Tx_1$ such that

$$\|y_0 - y_1\| \leqslant H(Tx_0, Tx_1).$$

For the point y_1, let

$$x_2 = \alpha_1 u + (1 - \alpha_1) y_1.$$

Inductively, we can get the sequence $\{x_n\}$ as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \ \forall n \in N, \tag{4.1}$$

where, for each $n \in N$, $y_n \in Tx_n$ is such that

$$\|y_n - y_{n-1}\| \leqslant H(Tx_n, Tx_{n-1}).$$

Subsequently, we show the strong convergence of $\{x_n\}$.

THEOREM 4.2. *Let E be a reflexive Banach space with weakly sequentially continuous duality mapping, K be a nonempty closed convex subset of E and $T : K \to C(K)$ be a nonexpansive mapping. Suppose that $F(T) \neq \emptyset$ satisfying $T(y) = \{y\}$ for any fixed point $y \in F(T)$, $\{x_n\}$ is defined by (4.1) and $\alpha_n \in (0,1)$ satisfy the following conditions:*

(i) $\alpha_n \to 0$ *as* $n \to \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iii) *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

Then, as $n \to \infty$, the sequence $\{x_n\}$ converges strongly to Pu, where P is unique sunny nonexpansive retraction from K to $F(T)$.

Proof. First, we show that $\{x_n\}$ is bounded. Taking a point $p \in F(T)$ (noting $Tp = \{p\}$), then we have

$$\begin{aligned} \|x_{n+1} - p\| &\leqslant (1-\alpha_n)\|y_n - p\| + \alpha_n\|u - p\| \\ &= (1-\alpha_n)d(y_n, Tp) + \alpha_n\|u - p\| \\ &\leqslant (1-\alpha_n)H(Tx_n, Tp) + \alpha_n\|u - p\| \\ &\leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n\|u - p\| \\ &\leqslant \max\{\|x_n - p\|, \|u - p\|\} \\ &\ \ \vdots \\ &\leqslant \max\{\|x_0 - p\|, \|u - p\|\}. \end{aligned}$$

Thus $\{x_n\}$ is bounded and so is $\{y_n\}$ by (4.1) and the condition (i). Then we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \alpha_n\|u - y_n\| = 0. \tag{4.2}$$

Now, we claim that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.3}$$

Indeed, for some appropriate constant $M > 0$, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n u + (1-\alpha_n)y_n - (\alpha_{n-1}u + (1-\alpha_{n-1})y_{n-1})\| \\ &\leqslant (1-\alpha_n)\|y_n - y_{n-1}\| + \|(\alpha_n - \alpha_{n-1})(u - y_{n-1})\| \\ &\leqslant (1-\alpha_n)H(Tx_n, Tx_{n-1}) + |\alpha_n - \alpha_{n-1}|\|u - y_{n-1}\| \\ &= (1-\alpha_n)\|x_n - x_{n-1}\| + M|\alpha_n - \alpha_{n-1}|. \end{aligned}$$

By the conditions (ii) and (iii), we have

$$\sum_{n=0}^{\infty} \alpha_n = \infty,$$

$$\sum_{n=0}^{\infty} M|\alpha_n - \alpha_{n-1}| < +\infty,$$

or

$$\lim_{n\to\infty} M\left|1-\frac{\alpha_{n-1}}{\alpha_n}\right| = 0.$$

Therefore, it follows from Lemma 2.2 that (4.3) follows. Combining (4.2) and (4.3), we get

$$d(x_n, Tx_n) \leqslant \|x_n - y_n\| \to 0,$$

that is,

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{4.4}$$

From Corollary 3.5, we know that $F(T)$ is a sunny nonexpansive retract of K and P is the unique sunny nonexpansive retraction of K onto $F(T)$.

Next, We show that

$$\limsup_{n\to\infty}\langle u - Pu, j(x_{n+1} - Pu)\rangle \leqslant 0. \tag{4.5}$$

Indeed, we can take a subsequence $\{x_{n_k+1}\}$ of $\{x_{n+1}\}$ such that

$$\limsup_{n\to\infty}\langle u - Pu, j(x_{n+1} - Pu)\rangle = \lim_{n_k\to\infty}\langle u - Pu, j(x_{n_k+1} - Pu)\rangle.$$

We may assume that $x_{n_k} \rightharpoonup x^*$ by the reflexivity of E and the boundedness of $\{x_n\}$. Using the same technique as in (3.6) of Theorem 3.4 and (4.4), then we obtain that $x^* \in F(T)$. Hence, by Lemma 2.1 and the fact that the duality mapping J is weakly sequentially continuous from E to E^*, we obtain

$$\limsup_{n\to\infty}\langle u - Pu, j(x_{n+1} - Pu)\rangle = \langle u - Pu, j(x^* - Pu)\rangle \leqslant 0.$$

Finally, we show that $x_n \to Pu$ as $n \to \infty$. In fact, since

$$\|y_n - Pu\| = d(y_n, T(Pu)) \leqslant H(Tx_n, T(Pu)) \leqslant \|x_n - Pu\|,$$

we have

$$\begin{aligned}
&\|x_{n+1} - Pu\|^2 \\
&= (1-\alpha_n)\langle y_n - Pu, j(x_{n+1} - Pu)\rangle + \alpha_n\langle u - Pu, j(x_{n+1} - Pu)\rangle \\
&\leqslant (1-\alpha_n)\frac{\|y_n - Pu\|^2 + \|j(x_{n+1} - Pu)\|^2}{2} + \alpha_n\langle u - Pu, j(x_{n+1} - Pu)\rangle \\
&\leqslant (1-\alpha_n)\frac{\|x_n - Pu\|^2}{2} + \frac{\|x_{n+1} - Pu\|^2}{2} + \alpha_n\langle u - Pu, j(x_{n+1} - Pu)\rangle.
\end{aligned}$$

Therefore, it follows that

$$\|x_{n+1} - Pu\|^2 \leqslant (1-\alpha_n)\|x_n - Pu\|^2 + 2\alpha_n\langle u - Pu, j(x_{n+1} - Pu)\rangle. \tag{4.6}$$

By the condition (ii) and the inequality (4.5), if we apply Lemma 2.2 to (4.6), then we have

$$\lim_{n\to\infty}\|x_n - Pu\| = 0.$$

This completes the proof. □

REMARK 4.1. (1) Theorem 4.2 can be taken for the multivalued version of Theorem 2.4 in [17].

(2) The strong convergence of explicit iterates of multivalued nonexpansive mappings is attained in Theorem 4.2, which complements and develops some existence results. In particular, the implicit iterates in the literatures (see [7, 10, 15, 26]) are evolved to the explicit iterates. We don't know whether Theorem 4.2 still holds in a reflexive strictly convex and smooth Banach space or uniformly smooth Banach space.

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