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A BEST POSSIBLE INEQUALITY FOR CURVATURE–LIKE TENSOR FIELDS

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Abstract. We give an inequality for curvature-like tensor fields and apply this to Lagrangian submanifolds of complex space forms and to centroaffine hypersurfaces. In both settings we investigate the equality case and give a classification theorem if equality is attained at every point of the submanifold. We also provide an example showing that this inequality is best possible, in a sense explained in the paper.

1. Introduction

In differential geometry of submanifolds theorems which relate intrinsic and extrinsic curvatures play an important role. A major achievement in this research area was obtained by B.Y. Chen in [3] in 1993. In this paper he introduced a new curvature invariant, which was thereafter named the δ curvature of Chen, given by the following definition

$$\delta(p) = \tau(p) - (\inf K)(p), \tag{1}$$

where τ is the scalar curvature and $(\inf K)(p)$ is the infimum of the sectional curvatures at the point p. In this paper he also proved an inequality relating δ and the mean curvature H for submanifolds in real space forms. Immersions of submanifolds for which equality is attained at every point in this inequality were later called ideal immersions and they were intensively studied by many geometers. For an overview we refer to [4].

Similar inequalities were obtained for Lagrangian submanifolds of complex space forms and for centroaffine hypersurfaces in $\mathbb{R}^{n+1}$, see for instance [6], [12], [13], [10] and [7]. Recently this inequality for the Lagrangian case was improved by T. Oprea in [15] using optimization techniques. In this article we prove a similar inequality for curvature-like tensor fields and give at the same time an alternative algebraic proof for Oprea's result. From this general inequality we can find Oprea's inequality in the Lagrangian case and an inequality for centroaffine hypersurfaces in $\mathbb{R}^{n+1}$ which improves an inequality from [12]. The inequality of Chen and this improvement reduce to the same inequality if the mean curvature of the Lagrangian submanifold, respectively

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the Tchebychev form of the centroaffine hypersurface, vanishes. We also show that if equality is attained in the improved inequality then these quantities do indeed vanish if the dimension is greater than or equal to 4. For the case that the dimension is 3, we give some classification theorems.

2. Preliminaries

Let (M,g) be any Riemannian n-manifold and let T be a $(0,4)$-tensor field on M. We say that T is curvature-like if T satisfies the following symmetry properties

$$\begin{aligned} T(X,Y,Z,W) &= -T(Y,X,Z,W), \\ T(X,Y,Z,W) &= -T(X,Y,W,Z), \\ T(X,Y,Z,W) &+ T(X,Z,W,Y) + T(X,W,Y,Z) = 0, \end{aligned}$$

for all tangent vector fields X,Y,Z and W of M. If T is a curvature-like tensor field, then we can talk about the sectional curvature $K_T(\pi) = T(X,Y,Y,X)$ associated with a 2-plane section $\pi \subset T_pM$, $p \in M$ spanned by the orthonormal vectors X and Y, and about the scalar curvature

$$\tau_T(p) = \sum_{i<j} T(e_i,e_j,e_j,e_i), \tag{2}$$

where $\{e_1,e_2,\dots,e_n\}$ is an orthonormal basis of T_pM.

As in [7], we can define for every curvature-like tensor field T the curvature invariant δ_T by

$$\delta_T(p) = \tau_T(p) - \inf_{\pi\subset T_pM}\{K_T(\pi)\}. \tag{3}$$

Now let (B,g) be any Riemannian vector bundle over M and let μ be a B-valued symmetric (1,2)-tensor field. If T is a (0,4)-tensor field on M such that

$$T(X,Y,Z,W) = g(\mu(Y,Z),\mu(X,W)) - g(\mu(X,Z),\mu(Y,W)), \tag{4}$$

for all tangent vector fields $X,Y,Z,W,$ then one easily checks that T is curvature-like. Equation (4) is said to be an algebraic Gauss equation. A typical example is given for a submanifold M of Euclidean space, if B is the normal bundle, μ the second fundamental form and T the curvature tensor.

For the rest of the article we take the vector bundle B to be the tangent bundle TM of M and we suppose that $g(\mu(X,Y),Z)$ is totally symmetric.

3. An inequality for δ

In [15] Oprea gave a proof for an improved inequality for the δ-invariant of B.Y. Chen for Lagrangian submanifolds of complex space forms using optimization techniques on Riemannian submanifolds. We generalize this to curvature-like tensor fields and give an algebraic proof.

THEOREM 1. *Let* (M^n, g) *be a Riemannian manifold of dimension* $n \geqslant 3$, T *a curvature-like tensor field and* μ *a symmetric (1,2)-tensor field which takes values in* TM. *Suppose that* T *and* μ *are related by* (4) *and that* $g(\mu(X,Y),Z)$ *is totally symmetric. Then we have*

$$\delta_T(p) \leqslant \frac{2n-3}{2(2n+3)} g(\text{trace}\,\mu, \text{trace}\,\mu), \tag{5}$$

where trace $\mu = \sum_{i=1}^n \mu(e_i, e_i)$.

The equality case of inequality (5) *holds at a point* $p \in M$ *if and only if there exists an orthonormal basis* $\{e_1, e_2, \ldots, e_n\}$ *at* p *such that with respect to this basis,* μ *takes the following form*

$$\begin{array}{lll} \mu(e_1,e_1) = ae_1 + 3\lambda e_3, & \mu(e_1,e_3) = 3\lambda e_1, & \mu(e_3,e_j) = 4\lambda e_j, \\ \mu(e_2,e_2) = -ae_1 + 3\lambda e_3, & \mu(e_2,e_3) = 3\lambda e_2, & \mu(e_j,e_k) = 4\lambda e_3 \delta_{jk}, \\ \mu(e_1,e_2) = -ae_2, & \mu(e_3,e_3) = 12\lambda e_3, & \mu(e_1,e_j) = \mu(e_2,e_j) = 0, \end{array}$$

for some numbers a *and* λ *and* $j,k \in \{4,\ldots,n\}$.

The inequality (5) is best possible, in the sense that the constant $\frac{2n-3}{2(2n+3)}$ cannot be improved. We will prove this by constructing an example where equality is attained at one point and trace $\mu \neq 0$ at that point.

REMARK 1. In [7] an inequality similar to (5) is proved with constant $\frac{n-2}{2(n-1)}$. In fact, the second author wants to point out a misprint in the more general inequality (3.6) in Theorem 1 of [7]: the factor n^2 should be removed.

Proof. Take a point $p \in M$ and an orthonormal frame $\{e_1, \ldots, e_n\}$ in T_pM such that the plane spanned by e_1 and e_2 minimizes the sectional curvature at the point p. Then we have from (4) that

$$\tau_T(p) = \sum_{r=1}^n \sum_{1 \leqslant i<j \leqslant n} \mu_{ii}^r \mu_{jj}^r - \sum_{r=1}^n \sum_{1 \leqslant i<j \leqslant n} (\mu_{ij}^r)^2, \tag{6}$$

$$T(e_1,e_2,e_2,e_1) = \sum_{r=1}^n \mu_{11}^r \mu_{22}^r - \sum_{r=1}^n (\mu_{12}^r)^2, \tag{7}$$

where $\mu_{ij}^r = g(\mu(e_i,e_j),e_r)$. Thus we find that

$$\delta_T(p) = \sum_{r=1}^n \left(\sum_{3 \leqslant j \leqslant n} (\mu_{11}^r + \mu_{22}^r)\mu_{jj}^r + \sum_{3 \leqslant i<j \leqslant n} \mu_{ii}^r \mu_{jj}^r - \sum_{3 \leqslant j \leqslant n} (\mu_{1j}^r)^2 - \sum_{2 \leqslant i<j \leqslant n} (\mu_{ij}^r)^2 \right).$$

Using now that $g(\mu(X,Y),Z)$ is totally symmetric, we have the following inequality

$$\begin{aligned} \delta_T(p) \leqslant \sum_{r=1}^n & \left(\sum_{3 \leqslant j \leqslant n} (\mu_{11}^r + \mu_{22}^r)\mu_{jj}^r + \sum_{3 \leqslant i<j \leqslant n} \mu_{ii}^r \mu_{jj}^r \right) \\ & - \sum_{3 \leqslant j \leqslant n} (\mu_{11}^j)^2 - \sum_{3 \leqslant j \leqslant n} (\mu_{jj}^1)^2 - \sum_{i,j \in \{2,3,\ldots,n\}, i \neq j} (\mu_{jj}^i)^2. \end{aligned} \tag{8}$$

We first show that for $r \in \{1,2\}$ we have

$$\sum_{3\leqslant j\leqslant n}(\mu_{11}^r+\mu_{22}^r)\mu_{jj}^r+\sum_{3\leqslant i<j\leqslant r}\mu_{ii}^r\mu_{jj}^r-\sum_{3\leqslant j\leqslant n}(\mu_{jj}^r)^2 \leqslant \frac{1}{2}\frac{n-2}{n+1}(\mu_{11}^r+\mu_{22}^r+\cdots+\mu_{nn}^r)^2. \tag{9}$$

This is equivalent to

$$-\sum_{j=3}^{n}\left((\mu_{11}^r+\mu_{22}^r)-3\mu_{jj}^r\right)^2-3\sum_{3\leqslant i<j\leqslant n}(\mu_{ii}^r-\mu_{jj}^r)^2 \leqslant 0. \tag{10}$$

Evidently we have that this inequality holds, moreover we see that we have equality if and only if $\mu_{11}^r+\mu_{22}^r=3\mu_{jj}^r$ for every j in $\{3,\ldots,n\}$.

Since $\frac{n-2}{n+1}<\frac{2n-3}{2n+3}$ for $n\geqslant 3$ we also have that

$$\sum_{3\leqslant j\leqslant n}(\mu_{11}^r+\mu_{22}^r)\mu_{jj}^r+\sum_{3\leqslant i<j\leqslant r}\mu_{ii}^r\mu_{jj}^r-\sum_{3\leqslant j\leqslant n}(\mu_{jj}^r)^2 \leqslant \frac{1}{2}\frac{2n-3}{2n+3}(\mu_{11}^r+\mu_{22}^r+\cdots+\mu_{nn}^r)^2, \tag{11}$$

with equality if and only if $\mu_{11}^r+\mu_{22}^r=0$ and $\mu_{jj}^r=0$ for every $j\in\{3,\ldots,n\}$.

Finally we prove that for $r\in\{3,\ldots,n\}$, we have

$$\sum_{3\leqslant j\leqslant n}(\mu_{11}^r+\mu_{22}^r)\mu_{jj}^r+\sum_{3\leqslant i<j\leqslant n}\mu_{ii}^r\mu_{jj}^r-\sum_{1\leqslant j\leqslant n,\ j\neq r}(\mu_{jj}^r)^2 \leqslant \frac{1}{2}\frac{2n-3}{2n+3}(\mu_{11}^r+\mu_{22}^r+\cdots+\mu_{nn}^r)^2. \tag{12}$$

This is equivalent to

$$\begin{aligned}12&\sum_{3\leqslant j\leqslant n}(\mu_{11}^r+\mu_{22}^r)\mu_{jj}^r+12\sum_{3\leqslant i<j\leqslant n}\mu_{ii}^r\mu_{jj}^r\\&-3(n+1)\sum_{3\leqslant j\leqslant n,\ j\neq r}(\mu_{jj}^r)^2-3(2n+1)(\mu_{11}^r)^2-3(2n+1)(\mu_{22}^r)^2\\&-(2n-3)(\mu_{rr}^r)^2-2(2n-3)\mu_{11}^r\mu_{22}^r \leqslant 0.\end{aligned} \tag{13}$$

Now remark that

$$\begin{aligned}-2&\sum_{3\leqslant j\leqslant n,\ j\neq r}(\mu_{rr}^r-3\mu_{jj}^r)^2-3\left(\mu_{rr}^r-2(\mu_{11}^r+\mu_{22}^r)\right)^2\\&=-(2n-3)(\mu_{rr}^r)^2+12\mu_{rr}^r(\mu_{11}^r+\mu_{22}^r)+12\sum_{3\leqslant j\leqslant n,\ j\neq r}\mu_{rr}^r\mu_{jj}^r\\&\quad-18\sum_{3\leqslant j\leqslant n,\ j\neq r}(\mu_{jj}^r)^2-12\left(\mu_{11}^r+\mu_{22}^r\right)^2.\end{aligned} \tag{14}$$

Using (14), inequality (13) becomes

$$12\sum_{3\leqslant j\leqslant n, j\neq r}(\mu_{11}^r+\mu_{22}^r)\mu_{jj}^r+12\sum_{3\leqslant i<j\leqslant n,\ i,j\neq r}\mu_{ii}^r\mu_{jj}^r-3(2n-5)\sum_{3\leqslant j\leqslant n,\ j\neq r}(\mu_{jj}^r)^2$$
$$-3(2n-3)(\mu_{11}^r)^2-3(2n-3)(\mu_{22}^r)^2-2(2n-15)\mu_{11}^r\mu_{22}^r$$
$$-2\sum_{3\leqslant j\leqslant n,\ j\neq r}(\mu_{rr}^r-3\mu_{jj}^r)^2-3\left(\mu_{rr}^r-2(\mu_{11}^r+\mu_{22}^r)\right)^2\leqslant 0. \tag{15}$$

We also note that

$$-\sum_{3\leqslant j\leqslant n,\ j\neq r}\left(2(\mu_{11}^r+\mu_{22}^r)-3\mu_{jj}^r\right)^2=-4(n-3)(\mu_{11}^r+\mu_{22}^r)^2$$
$$+12\sum_{3\leqslant j\leqslant n,\ j\neq r}(\mu_{11}^r+\mu_{22}^r)\mu_{jj}^r-4\sum_{3\leqslant j\leqslant n,\ j\neq r}(\mu_{jj}^r)^2. \tag{16}$$

Finally from (16) we find that (15) is equivalent with

$$-\sum_{3\leqslant j\leqslant n,\ j\neq r}\left(2(\mu_{11}^r+\mu_{22}^r)-3\mu_{jj}^r\right)^2-(2n+3)(\mu_{11}^r-\mu_{22}^r)^2$$
$$-6\left(\sum_{3\leqslant i<j\leqslant n,\ i,j\neq r}(\mu_{ii}^r-\mu_{jj}^r)^2\right)-2\sum_{3\leqslant j\leqslant n,\ j\neq r}\left(\mu_{rr}^r-3\mu_{jj}^r\right)^2$$
$$-3\left(\mu_{rr}^r-2(\mu_{11}^r+\mu_{22}^r)\right)^2\leqslant 0. \tag{17}$$

This proves inequality (12). Moreover we see from (17) that we have equality if and only if

$$\mu_{11}^r=\mu_{22}^r=3\lambda^r, \tag{18}$$
$$\mu_{jj}^r=4\lambda^r\quad\text{for } j\in\{3,\dots,n\}, j\neq r, \tag{19}$$
$$\mu_{rr}^r=12\lambda^r, \tag{20}$$

for a number λ^r.

From (9) and (12) we get inequality (5). Combining the two equality cases described above with the fact that $g(\mu(X,Y),Z)$ is totally symmetric, by choosing e_1 and e_2 such that $\mu_{12}^1=0$ and e_3 such that trace μ is parallel with e_3, we get the equality case of the theorem.

□

4. Lagrangian submanifolds of complex space forms

For the rest of this section we denote by $\widetilde{M}^n(4c)$ a complex space form of constant holomorphic sectional curvature $4c$ and real dimension $2n$. Recall that for $\widetilde{M}^n(4c)$ the Riemann curvature tensor $\widetilde{R}$ is given by

$$\widetilde{R}(X,Y)Z=c\left(g(X,Z)Y-g(Y,Z)X+g(JX,Z)JY-g(JY,Z)JX+2g(JX,Y)JZ\right) \tag{21}$$

for all vector fields X, Y and Z on $\widetilde{M}$.

We call an n-dimensional submanifold M of $\widetilde{M}^n(4c)$ a Lagrangian submanifold if $J(T_pM) = T_p^{\perp}M$ for every $p \in M$.

Now we can use the setting of the previous sections by taking

$$T(X,Y,Z,W) = g(R(X,Y)Z,W) + c(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)) \tag{22}$$

with R the Riemannian curvature tensor on M, and $\mu(X,Y) = Jh(X,Y)$ with h the second fundamental form of the immersion of M into $\widetilde{M}^n(4c)$. Then we see that

$$\delta_T = \delta - \frac{(n-2)(n+1)}{2}c,$$

where $\delta = \delta_R$ is defined as in (1). Thus we immediately get the following inequality for Lagrangian submanifolds in complex space forms.

THEOREM 2. *Let (M^n, g) be a Lagrangian submanifold of a complex space form $\widetilde{M}^n(4c)$. Then we have*

$$\delta(p) \leqslant \frac{(n-2)(n+1)}{2}c + \frac{n^2(2n-3)}{2(2n+3)}\|H\|^2, \tag{23}$$

where $H = \frac{1}{n}\text{trace}\, h$ is the mean curvature vector.

The equality case of inequality (23) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ at p such that with respect to this basis, h takes the following form

$$\begin{aligned} &h(e_1,e_1) = aJe_1 + 3\lambda Je_3, && h(e_1,e_3) = 3\lambda Je_1, && h(e_3,e_j) = 4\lambda Je_j,\\ &h(e_2,e_2) = -aJe_1 + 3\lambda Je_3, && h(e_2,e_3) = 3\lambda Je_2, && h(e_j,e_k) = 4\lambda Je_3\delta_{jk},\\ &h(e_1,e_2) = -aJe_2, && h(e_3,e_3) = 12\lambda Je_3, && h(e_1,e_j) = h(e_2,e_j) = 0, \end{aligned}$$

for some numbers a and λ and $j,k \in \{4,\ldots,n\}$.

This inequality generalizes an inequality of B. Y. Chen in [6] and was proven in a different way in [15].

We now investigate the equality case. Remark that if the submanifold is minimal, then the inequality reduces to the inequality of Chen and submanifolds attaining equality in this inequality were already intensively studied in for instance [8] and [9]. We first show that this is always the case if $n \geqslant 4$.

THEOREM 3. *Let M^n be a Lagrangian submanifold of a complex space form $\widetilde{M}^n(4c)$ attaining equality in (23) at every point. If $n \geqslant 4$, then M is minimal.*

REMARK 2. This theorem was already proven in [1] for $c > 0$.

To prove this theorem, we take the same approach as in [1] with a slightly different argument for $c \leqslant 0$. So now assume that $n \geqslant 4$ and that M has no minimal points, i.e.

λ is nowhere zero. In this case Je_3 is a multiple of the mean curvature vector implying that λ is a globally defined differentiable function. In accordance to [1] we denote by T the vector field corresponding to e_3, which is also a globally defined differentiable vector field, and by $\mathscr{D}_1$ the distribution spanned by T.

At every point, the linear symmetric operator A_{JT} has three distinct eigenvalues of multiplicity respectively 1, 2 and $n-3$ (where the first eigenspace is spanned by T). Again, in accordance with [1] we let $\mathscr{D}_2$ be the 2-dimensional distribution and $\mathscr{D}_3$ the orthogonal $(n-3)$-dimensional distribution corresponding to the two other eigenspaces.

Let $\{V_1,V_2\}$ and $\{W_1,W_2,\dots,W_{n-3}\}$ be a smooth orthonormal basis of $\mathscr{D}_2$ respectively $\mathscr{D}_3$.

As we now suppose that M attains equality in (23), it follows that there exist smooth local functions b and $\bar{c}$ such that the second fundamental form has the following form:

$$\begin{aligned}
&h(V_1,V_1)=3\lambda JT+bJV_1+\bar{c}JV_2, && h(V_i,T)=3\lambda JV_i,\\
&h(V_2,V_2)=3\lambda JT-bJV_1-\bar{c}JV_2, && h(T,W_p)=4\lambda JW_p,\\
&h(V_1,V_2)=\bar{c}JV_1-bJV_2, && h(V_i,W_p)=0,\\
&h(T,T)=12\lambda JT && h(W_p,W_q)=\delta_{pq}4\lambda JT.
\end{aligned}$$

From [1] with $\lambda_1=12\lambda$, $\lambda_2=3\lambda$ and $\lambda_3=4\lambda$ we get the following lemma where for the rest of this section $V,\widetilde{V},V^*$ and $W,\widetilde{W},W^*$ are vector fields belonging to $\mathscr{D}_2$ respectively $\mathscr{D}_3$ and σ denotes cyclic summation over respectively $V,\widetilde{V},V^*$ and $W,\widetilde{W},W^*$.

LEMMA 1. *We have*

$$2V(\lambda)=\lambda\langle\nabla_T T,V\rangle, \tag{24}$$

$$6\lambda\langle\nabla_V\widetilde{V},T\rangle+3T(\lambda)\langle V,\widetilde{V}\rangle-\langle h(V,\widetilde{V}),J\nabla_T T\rangle=0, \tag{25}$$

$$4\lambda\langle\nabla_V W,T\rangle-\lambda\langle\nabla_T V,W\rangle=0, \tag{26}$$

$$3W(\lambda)=\lambda\langle\nabla_T T,W\rangle, \tag{27}$$

$$4\lambda\langle\nabla_W\widetilde{W},T\rangle+4T(\lambda)\langle W,\widetilde{W}\rangle-\langle h(W,\widetilde{W}),J\nabla_T T\rangle=0, \tag{28}$$

$$6\lambda\langle\nabla_W V,T\rangle-\lambda\langle\nabla_T V,W\rangle=0, \tag{29}$$

$$\lambda\langle\nabla_V\widetilde{V},W\rangle=\langle h(V,\widetilde{V}),J\nabla_T W\rangle, \tag{30}$$

$$\lambda\langle\nabla_V\widetilde{V},W\rangle=-3W(\lambda)\langle V,\widetilde{V}\rangle+\langle h(V,\widetilde{V}),J\nabla_W T\rangle, \tag{31}$$

$$-\lambda\langle\nabla_W\widetilde{W},V\rangle=\langle h(W,\widetilde{W}),J\nabla_T V\rangle, \tag{32}$$

$$-\lambda\langle\nabla_W\widetilde{W},V\rangle=-3V(\lambda)\langle W,\widetilde{W}\rangle+\langle h(W,\widetilde{W}),J\nabla_V T\rangle, \tag{33}$$

$$\begin{aligned}T\left(\langle h(V,\widetilde{V}),JV^*\rangle\right)&-3V(\lambda)\langle\widetilde{V},V^*\rangle\\ &=\sigma\left(\langle h(V,\widetilde{V}),J\nabla_T V^*\rangle\right)-\langle h(\widetilde{V},V^*),J\nabla_V T\rangle,\end{aligned} \tag{34}$$

$$\begin{aligned}T\left(\langle h(W,\widetilde{W}),JW^*\rangle\right)&-4W(\lambda)\langle\widetilde{W},W^*\rangle\\ &=\sigma\left(\langle h(W,\widetilde{W}),J\nabla_T W^*\rangle\right)-\langle h(\widetilde{W},W^*),J\nabla_W T\rangle,\end{aligned} \tag{35}$$

$$\langle h(V,V_1), J\nabla_W W_1\rangle = \langle h(W,W_1), J\nabla_V V_1\rangle. \tag{36}$$

We now determine the connection coefficients of M as in [1]. We introduce the following notations in which $i,j,k \in \{1,2\}$ and $p,q,r \in \{1,\dots,n-3\}$. We let

(i) T_i denote the $\mathscr{D}_3$ component of $\nabla_T V_i$,

(ii) $\bar{V}$ denote the $\mathscr{D}_2$ component of grad(12λ),

(iii) $\bar{W}$ denote the $\mathscr{D}_3$ component of grad(12λ).

Then we have

$$\nabla_T T = \frac{1}{6\lambda}\bar{V} + \frac{1}{4\lambda}\bar{W}, \tag{37}$$

$$\langle \nabla_{V_i} V_j, T\rangle = -\frac{T(\lambda)}{2\lambda}\delta_{ij} + \frac{1}{36\lambda^2}\langle h(V_i,V_j), J\bar{V}\rangle, \tag{38}$$

$$\langle \nabla_{V_i} V_j, W_p\rangle = -\frac{9}{\lambda}W_p(\lambda)\delta_{ij} - \frac{1}{\lambda}\sum_{k=1}^{2}\langle T_k, W_p\rangle\langle h(V_i,V_j)JV_k\rangle, \tag{39}$$

$$\langle \nabla_{W_p} W_q, T\rangle = -\frac{T(\lambda)}{\lambda}\delta_{pq} + \frac{1}{16\lambda^2}\langle h(W_p,W_q), J\bar{W}\rangle, \tag{40}$$

$$\langle \nabla_{W_p} W_q, V_i\rangle = \frac{8}{\lambda}V_i(\lambda)\delta_{pq} - \frac{1}{\lambda}\sum_{k=1}^{n-3}\langle T_i, W_r\rangle\langle h(W_p,W_q)JW_r\rangle, \tag{41}$$

$$\langle \nabla_{V_i} W_p, T\rangle = \frac{1}{4}\langle T_i, W_p\rangle, \tag{42}$$

$$\langle \nabla_{W_p} V_i, T\rangle = \frac{1}{6}\langle T_i, W_p\rangle. \tag{43}$$

From this we have also the following lemma.

LEMMA 2. *We have*

$$3W_p(\lambda)\delta_{ij} = \delta_{ij}9W_p(\lambda) - \sum_{k=1}^{2}\frac{5}{6}\langle T_k, W_p\rangle\langle h(V_i,V_j), JV_k\rangle, \tag{44}$$

$$4V_i(\lambda)\delta_{pq} = \delta_{pq}8V_i(\lambda) - \sum_{r=1}^{n-3}\frac{5}{4}\langle T_i, W_r\rangle\langle h(W_p,W_q), JW_r\rangle, \tag{45}$$

$$\delta_{ij}\delta_{pq}T(\lambda) - \frac{15}{16\lambda}\delta_{ij}\langle h(W_p,W_q), J\bar{W}\rangle - \frac{5}{9\lambda}\delta_{pq}\langle h(V_i,V_j), J\bar{V}\rangle = 0. \tag{46}$$

With all this we are able to prove Theorem 3.

Proof of Theorem 3. Assume that there exists a point x and hence a neighbourhood of x on which M is not minimal. We construct a smooth local orthonormal frame $\{T, V_1, V_2, W_1, \dots, W_n\}$, on a (possibly smaller) neighbourhood of x as described above.

We first show, as in [1], that λ has to be a constant. First note that it follows from (34) by taking first $\widetilde{V} = V^* = V_1$ and then $\widetilde{V} = V^* = V_2$ and adding these two equations that

$$\begin{aligned} 6V(\lambda) &= -6\lambda\langle T, \nabla_T V\rangle - 2\langle h(V,V_1), J\nabla_T V_1\rangle - 2\langle h(V,V_2), J\nabla_T V_2\rangle \\ &= 12V(\lambda) - 2\langle h(V,V_1), JT\rangle\langle T, \nabla_T V_1\rangle - 2\langle h(V,V_2), JT\rangle\langle T, \nabla_T V_2\rangle \\ &\quad -2\langle h(V,V_1), JV_2\rangle\left(\langle V_2, \nabla_T V_1\rangle + \langle \nabla_T V_2, V_1\rangle\right) \\ &= 12V(\lambda) + 6\lambda\langle \nabla_T T, V\rangle = 24V(\lambda). \end{aligned}$$

Hence $V(\lambda) = 0$. Similarly if follows from (35) that $W(\lambda) = 0$. Consequently, using (37), we have $\bar{V} = \bar{W} = \nabla_T T = 0$. The third equation of Lemma 2 then implies that $T(\lambda) = 0$.

Suppose that $b^2 + \bar{c}^2 \neq 0$. As λ is a constant and $\nabla_T T = 0$, the first equation of Lemma 2 implies that $\langle h(V_i,V_j), J\nabla_T W_p\rangle = 0$. As $b^2 + \bar{c}^2 \neq 0$, the above equation implies that $\nabla_T W_p$ has no $\mathscr{D}_2$ component. Consequently for all i, T_i vanishes. From (38), (39), (40) and (41) we then find that

$$\nabla_{\mathscr{D}_2}\mathscr{D}_2 \subset \mathscr{D}_2, \quad \nabla_{\mathscr{D}_3}\mathscr{D}_3 \subset \mathscr{D}_3. \tag{47}$$

Note that from (38), (39), (40) and (41) we get the same conclusion if $b^2 + \bar{c}^2 = 0$ on an open set. Therefore continuity shows that we have the above condition everywhere.

Using (26), (29) and (47) we get that

$$\begin{aligned} \langle R(V,W)W,V\rangle &= \langle \nabla_V\nabla_W W, V\rangle - \langle \nabla_W\nabla_V W, V\rangle - \langle \nabla_{\nabla_V W - \nabla_W V} W, V\rangle \\ &= -\langle \nabla_V W, T\rangle\langle \nabla_W T, V\rangle - \nabla_V W, T\rangle\langle \nabla_T W, V\rangle + \langle \nabla_W V, T\rangle\langle \nabla_T W, V\rangle \\ &= \left(\frac{1}{4}\cdot\frac{1}{6} + \frac{1}{4} - \frac{1}{6}\right)\langle \nabla_T V, W\rangle^2 \\ &= \frac{1}{8}\langle \nabla_T V, W\rangle^2. \end{aligned}$$

On the other hand by the Gauss equation we have

$$\langle R(V,W)W,V\rangle = c + 12\lambda^2. \tag{48}$$

In view of the dimensions we can always find a $W \in \mathscr{D}_3$ and a $V \in \mathscr{D}_2$ such that $\langle \nabla_T V, W\rangle = 0$. Hence we must have

$$c + 12\lambda^2 = 0 \tag{49}$$

$$\langle \nabla_T V, W\rangle = 0 \quad \text{for arbitrary } V \text{ and } W. \tag{50}$$

If $c > 0$ we immediately get a contradiction from (49).

If $c = 0$, we also get a contradiction by remarking that $H = 0$ is equivalent to $\lambda = 0$.

For the case that $c < 0$ we need an extra argument. From (49) we get that $c = -12\lambda^2$. Using this and the equation of Gauss we also find that

$$\langle R(T,V)V,T\rangle = 15\lambda^2. \tag{51}$$

We also see that from (50) it follows that $\nabla_T V \in \mathscr{D}_2$. Together with (47) and $\nabla_T T = 0$ we have with the definition of the curvature tensor that

$$\langle R(T,V)V,T\rangle = 0. \tag{52}$$

Thus again we find that $\lambda = 0$ which gives a contradiction. $\square$

For the case $n = 3$ and $c > 0$ we have the following classification theorem from [2].

THEOREM 4. *Let M be a 3-dimensional non-minimal Lagrangian submanifold of $\mathbb{C}P^3$ which attains equality at every point in (23). Then there is a minimal Lagrangian surface W in $\mathbb{C}P^2$ such that M can be locally written as $[E_0]$ where*

$$E_0 = \frac{e^{\frac{it}{3}}}{\sqrt{1+b_1^2+9\lambda^2}}(0,W) + \frac{-b_1+3i\lambda}{\sqrt{1+b_1^2+9\lambda^2}}(e^{it},0,0,0), \tag{53}$$

where b_1 and λ are solutions of the following system of ordinary differential equations:

$$\frac{db_1}{dt} = -\frac{1+27\lambda^2+b_1^2}{9\lambda}, \quad 3\frac{d\lambda}{dt} = \frac{2}{3}b_1. \tag{54}$$

Conversely any 3-dimensional Lagrangian submanifold obtained in this way is non-minimal and attains equality at each point in (23).

5. Centroaffine hypersurfaces

First we recall some basic facts about centroaffine hypersurfaces. For more details see for instance [14].

Let M^n be an n-dimensional C^∞-manifold and let $f : M \to \mathbb{R}^{n+1}$ be a non-degenerate hypersurface whose position vector is nowhere tangent to M. Then f is a transversal field along itself and we call $\xi = -f$ the centroaffine normal. Following Nomizu, we call f together with this normalization a centroaffine hypersurface.

The centroaffine structure equations are given by

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi, \tag{55}$$

$$D_X \xi = -f_*(X), \tag{56}$$

where D denotes the canonical flat connection of $\mathbb{R}^{n+1}$, ∇ is a torsion-free connection on M, called the induced centroaffine connection, and h is a non-degenerate symmetric

(0,2)-tensor field, called the centroaffine metric. The corresponding equations of Gauss and Codazzi are given by

$$R(X,Y)Z = h(Y,Z)X - h(X,Z)Y, \tag{57}$$

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z). \tag{58}$$

We will assume that the centroaffine hypersurface is definite, i.e., h is definite. If h is negative definite, we shall replace $\xi = -f$ by $\xi = f$ for the affine normal. In this way, the second fundamental form h is always positive definite. In both cases, (55) and (58) hold whereas (56) and (57) change sign. In case $\xi = -f$, we call M positive definite; in case $\xi = f$, we call M negative definite.

Denote by $\widehat{\nabla}$ the Levi-Civita connection of h and by $\widehat{R}$ and $\widehat{\tau}$ the curvature tensor and the scalar curvature of h, respectively. The difference tensor K is then defined by

$$K_X Y = K(X,Y) = \nabla_X Y - \widehat{\nabla}_X Y, \tag{59}$$

which is a symmetric (1,2)-tensor field. Furthermore we have

$$(\nabla_X h)(Y,Z) = -2h(K_X Y,Z). \tag{60}$$

Thus, for each X, K_X is self-adjoint with respect to h.

The Tchebychev form T and the Tchebychev vector field $T^\sharp$ of M are defined by

$$T(X) = \frac{1}{n}\text{trace } K_X, \tag{61}$$

$$h(T^\sharp, X) = T(X). \tag{62}$$

If $T = 0$ and if we consider M as a hypersurface of the equiaffine space, then M is a so-called proper affine hypersphere centered at the origin, in the sense of [14]. If the difference tensor K vanishes, then M is a quadric, centered at the origin, in particular an ellipsoid if M is positive definite and a two-sheeted hyperboloid if M is negative definite.

It is well known in centroaffine geometry that

$$h(K_X Y,Z) = h(Y,K_X Z), \tag{63}$$

$$\widehat{R}(X,Y)Z = K_Y K_X Z - K_X K_Y Z + \varepsilon(h(Y,Z)X - h(X,Z)Y), \tag{64}$$

$$(\widehat{\nabla}K)(X,Y,Z) = (\widehat{\nabla}K)(Y,Z,X) = (\widehat{\nabla}K)(Z,X,Y), \tag{65}$$

where $\varepsilon = 1$ or -1 according to M being positive definite or negative definite. Now let us take $\mu = K$ and

$$T(X,Y,Z,W) = -\widehat{R}(X,Y,Z,W) + \varepsilon\,(h(Y,Z)h(X,W) - h(X,Z)h(Y,W)).$$

In order to formulate everything in terms of the curvature tensor $\widehat{R}$, we take

$$\delta^\sharp(p) = \widehat{\tau} - \sup_{\pi\subset T_pM}\{\widehat{K}(\pi)\}, \tag{66}$$

where $\widehat{K}(\pi)$ is the sectional curvature of the plane π. Then we get the following inequality from Theorem 1.

THEOREM 5. *Let M be a definite centroaffine hypersurface in $\mathbb{R}^{n+1}$. Then we have*

$$\delta^\sharp(p) \geqslant \frac{1}{2}\varepsilon(n^2-n-2)+\frac{n^2}{2}\frac{2n-3}{2n+3}h(T^\sharp,T^\sharp), \tag{67}$$

where $\varepsilon = 1$ or -1, according to M being positive or negative definite.

The equality case of inequality (67) holds at a point $p \in M$ if and only if there exists an h-orthonormal basis $\{e_1,e_2,\ldots,e_n\}$ at p such that with respect to this basis, K takes the following form

$$\begin{array}{lll} K(e_1,e_1)=ae_1+3\lambda e_3, & K(e_1,e_3)=3\lambda e_1, & K(e_3,e_j)=4\lambda e_j, \\ K(e_2,e_2)=-ae_1+3\lambda e_3, & K(e_2,e_3)=3\lambda e_2, & K(e_j,e_k)=4\lambda e_3\delta_{jk}, \\ K(e_1,e_2)=-ae_2, & K(e_3,e_3)=12\lambda e_3, & K(e_1,e_j)=K(e_2,e_j)=0, \end{array}$$

for some numbers a and λ, and $j,k \in \{4,\ldots,n\}$.

In order to show that the inequality (5) is the best possible, we give the following example. Consider the graph hypersurface F given by

$$F:\mathbb{R}^n \to \mathbb{R}^{n+1}:(x_1,\ldots,x_n)\mapsto(x_1,\ldots,x_n,f(x_1,\ldots,x_n)), \tag{68}$$

where

$$f(x_1,\ldots,x_n)=1+\frac{\varepsilon}{2}\sum_{i=1}^n x_i^2-\frac{\varepsilon a}{3}x_1^3-3\varepsilon\lambda(x_1^2+x_2^2)x_3+a\varepsilon x_1x_2^2-4\varepsilon\lambda x_3^3-4\varepsilon\lambda x_3\sum_{j=4}^n x_j^2,$$

with a and $\lambda \neq 0$ real numbers.

We can easily compute that at the origin we have

$$(\nabla h)\left(\frac{\partial F}{\partial x_i},\frac{\partial F}{\partial x_j},\frac{\partial F}{\partial x_k}\right)(0,0,\ldots,0)=\frac{\partial^3 f}{\partial x_i\partial x_j\partial x_k}(0,0,\ldots,0). \tag{69}$$

By computing the difference tensor K with (60), we see that Theorem 5 implies that F is attaining equality in (67). Thus this is an example of a centroaffine hypersurface attaining equality in (67) and for which the Tchebychev vector field doesn't vanish. Hence the constant in the inequality (5) cannot be improved.

We have the following theorem.

THEOREM 6. *Let M^n be a centroaffine hypersurface of $\mathbb{R}^{n+1}$ attaining equality in (67) at every point. If $n \geqslant 4$, then the Tchebychev vector field T vanishes.*

The proof of this theorem is analogous to the proof of Theorem 3.

For the case $n=3$, with $T \neq 0$, we have a classification theorem.

THEOREM 7. *Let M^3 be a centroaffine hypersurface of $\mathbb{R}^4$ which attains equality at every point in (67). Then M is locally given by one of the following immersions*

(1)
$$f=\frac{(-b_1+3\lambda)e^{-3t}}{\sqrt{-9\lambda^2+b_1^2+\varepsilon}}V+\frac{e^{-t}}{\sqrt{-9\lambda^2+b_1^2+\varepsilon}}W,$$

where V is a constant vector along the hypersurface, W is a proper affine sphere in a 3-dimensional totally geodesic subspace of $\mathbb{R}^4$ not containing V, and λ and b_1 are solutions of the following system of ordinary differential equations

$$\frac{d\lambda}{dt}=-\frac{2}{3}b_1, \tag{70}$$

$$\frac{db_1}{dt}=\frac{\varepsilon+b_1^2-27\lambda^2}{3\lambda}, \tag{71}$$

and $-9\lambda^2+b_1^2+\varepsilon\neq 0$.

(2)
$$f(t,u,v)=(tu,tv,tg(u,v)+\gamma_2(t),t),$$

where
$$\gamma_2(t)=\left(\int_{t_0}^{t}-\frac{1}{u}\sqrt{\frac{c}{2u^2}+d}\,du\right)t, \tag{72}$$

for c and d in $\mathbb{R}$, and where $(u,v)\mapsto(u,v,g(u,v))$ defines an improper affine sphere with affine normal $(0,0,1)$.

Proof. Since we have equality in (67) in a neighborhood of p, we can take an orthonormal basis $\{e_1,e_2,e_3\}$ such that

$$\begin{aligned}
K(e_1,e_1)&=12\lambda e_1, & K(e_2,e_2)&=3\lambda e_1+ae_2,\\
K(e_1,e_2)&=3\lambda e_2, & K(e_2,e_3)&=-ae_3,\\
K(e_1,e_3)&=3\lambda e_3, & K(e_3,e_3)&=3\lambda e_1-ae_2
\end{aligned}$$

with a and λ functions on a neighborhood of p. Here we used e_1,e_2 respectively e_3 instead of e_3,e_1 and e_2 in Theorem 5 in order to get the same notations as in [11]. Now we define the functions $a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2$ and c_3 by the Levi-Civita connection $\widehat{\nabla}$ in the following way

$$\begin{aligned}
\widehat{\nabla}_{e_1}e_1&=a_1e_2+a_2e_3, & \widehat{\nabla}_{e_1}e_2&=-a_1e_1+a_3e_3, & \widehat{\nabla}_{e_1}e_3&=-a_2e_1-a_3e_2,\\
\widehat{\nabla}_{e_2}e_1&=b_1e_2+b_2e_3, & \widehat{\nabla}_{e_2}e_2&=-b_1e_1+b_3e_3, & \widehat{\nabla}_{e_2}e_3&=-b_2e_1-b_3e_2,\\
\widehat{\nabla}_{e_3}e_1&=c_1e_2+c_2e_3, & \widehat{\nabla}_{e_3}e_2&=-c_1e_1+c_3e_3, & \widehat{\nabla}_{e_3}e_3&=-c_2e_1-c_3e_2.
\end{aligned}$$

Now using the Codazzi equation (65) we find analogously as in [11] the following equations

$$\lambda(c_1-b_2)=4\lambda(c_1-b_2)=0, \tag{73}$$

$$a(b_2+c_1)=6a_2\lambda=-2a(3a_3-b_2), \tag{74}$$

$$a(b_1-c_2)=-6a_1\lambda, \tag{75}$$

$$6b_2\lambda = aa_2, \tag{76}$$
$$6\lambda(c_2 - b_1) = 2aa_1, \tag{77}$$
$$2e_2(\lambda) = a_1\lambda, \tag{78}$$
$$2e_3(\lambda) = a_2\lambda, \tag{79}$$
$$3e_1(\lambda) = 6b_1\lambda + aa_1, \tag{80}$$
$$3e_2(\lambda) = a(c_2 - b_1), \tag{81}$$
$$3e_3(\lambda) = a(b_2 + c_1), \tag{82}$$
$$e_1(a) = 3a_1\lambda - c_2 a, \tag{83}$$
$$e_2(a) = 3(b_1 - c_2)\lambda - 3c_3 a, \tag{84}$$
$$e_3(a) = 3(c_1 - 3b_2)\lambda + 3b_3 a. \tag{85}$$

If we take λ_1 and λ_2 in [11] equal to 12λ and 3λ, we find from [11] that $a_1 = a_2 = 0$, $c_1 = b_2 = 0$ and $b_1 = c_2$ since $\lambda \neq 0$ because we suppose that $T \neq 0$.

Using this, (73)–(85) and the Gauss equation we get the following equations.

$$aa_3 = 0, \tag{86}$$
$$e_2(\lambda) = e_3(\lambda) = 0, \tag{87}$$
$$e_1(\lambda) = 2b_1\lambda, \tag{88}$$
$$e_1(a) = -b_1 a, \tag{89}$$
$$e_2(a) = -3ac_3, \tag{90}$$
$$e_3(a) = 3ab_3, \tag{91}$$
$$e_1(b_1) = -b_1^2 + 27\lambda^2 - \varepsilon, \tag{92}$$
$$e_1(b_3) - e_2(a_3) - a_3c_3 + b_1b_3 = 0, \tag{93}$$
$$-e_1(c_3) + e_3(a_3) - a_3b_3 - b_1c_3 = 0, \tag{94}$$
$$e_2(b_1) = 0, \tag{95}$$
$$-b_1^2 - e_2(c_3) + e_3(b_3) - b_3^2 - c_3^2 = 2a^2 - 9\lambda^2 + \varepsilon, \tag{96}$$
$$e_3(b_1) = 0. \tag{97}$$

We divide the proof into two cases.

Case 1: $\varepsilon + b_1^2 - 9\lambda^2 \neq 0$.

First consider the following system of differential equations.

$$\begin{cases} e_1(\theta) = -3\lambda, \\ e_2(\theta) = 0, \\ e_3(\theta) = 0. \end{cases} \tag{98}$$

From (87) we see that the compatibility equations for this system are satisfied. Thus we can find a solution θ of (98).

Now consider the maps into $\mathbb{R}^4$ given by

$$V = \frac{e^{3\theta}}{\sqrt{\varepsilon + b_1^2 - 9\lambda^2}} \left(-(b_1 + 3\lambda) f + e_1\right), \tag{99}$$

$$W = \frac{e^{\theta}}{\sqrt{\varepsilon + b_1^2 - 9\lambda^2}} \left(\varepsilon f - (-b_1 + 3\lambda) e_1\right). \tag{100}$$

It is easy to see that $D_{e_1}V = D_{e_2}V = D_{e_3}V = 0$. Thus V is a constant vector along the hypersurface.

For W we have that $D_{e_1}W = 0$. Thus W describes a surface. Moreover we have

$$W_*(e_2) = D_{e_2}W = e^{\theta}\sqrt{\varepsilon + b_1^2 - 9\lambda^2}\, e_2, \tag{101}$$

$$W_*(e_3) = D_{e_3}W = e^{\theta}\sqrt{\varepsilon + b_1^2 - 9\lambda^2}\, e_3, \tag{102}$$

and

$$D_{e_2}(D_{e_2}W) = -(\varepsilon + b_1^2 - 9\lambda^2)W + aD_{e_2}W + b_3 D_{e_3}W, \tag{103}$$

$$D_{e_2}(D_{e_3}W) = -b_3 D_{e_2}W - aD_{e_3}W, \tag{104}$$

$$D_{e_3}(D_{e_2}W) = (c_3 - a)D_{e_3}W, \tag{105}$$

$$D_{e_3}(D_{e_3}W) = -(\varepsilon + b_1^2 - 9\lambda^2)W - (a + c_3)D_{e_2}W. \tag{106}$$

From this we can easily compute that W is an immersion of a surface with centroaffine metric $\tilde{h}$ given by

$$\tilde{h}(e_2, e_2) = \varepsilon + b_1^2 - 9\lambda^2, \tag{107}$$

$$\tilde{h}(e_2, e_3) = 0, \tag{108}$$

$$\tilde{h}(e_3, e_3) = \varepsilon + b_1^2 - 9\lambda^2. \tag{109}$$

Together with (60) we find that

$$\tilde{h}(\operatorname{tr} K, e_2) = \tilde{h}(\operatorname{tr} K, e_3) = 0. \tag{110}$$

Thus the Tchebychev form of W vanishes and W is a proper affine sphere. Remark also that V and $W, W_*(e_2), W_*(e_3)$ are linearly independent.

Moreover the immersion f is given by

$$f = \frac{(-b_1 + 3\lambda)e^{-3\theta}}{\sqrt{-9\lambda^2 + b_1^2 + \varepsilon}} V + \frac{e^{-\theta}}{\sqrt{-9\lambda^2 + b_1^2 + \varepsilon}} W. \tag{111}$$

By choosing a coordinate t in the direction of e_1 such that $e_1(t) = -3\lambda$, we may assume, after applying a translation if necessary, that $\theta = t$. From (87), (88), (92), (95) and (97) we now get the ordinary differential equations (70) and (71) relating the functions λ and b_1.This proves the first part of the theorem.

Case 2: $\varepsilon + b_1^2 - 9\lambda^2 = 0$.

In this case consider the following system of differential equations.

$$\begin{cases} e_1(\beta) = -\beta(-b_1+3\lambda), \\ e_2(\beta) = 0, \\ e_3(\beta) = 0. \end{cases} \tag{112}$$

From (87), (95) and (97) we see that the compatibility equations for this system are satisfied. Thus we can find a non-zero solution β of (112).

Now consider the map into $\mathbb{R}^4$ defined by

$$W = \beta\,(\varepsilon f - (-b_1+3\lambda)e_1). \tag{113}$$

Then it is easy to see that $D_{e_1}W = D_{e_2}W = D_{e_3}W = 0$ and thus W is a constant vector along the hypersurface.

Now remark that

$$D_{e_2}e_2 = -\frac{1}{\beta}W + ae_2 + b_3e_3, \tag{114}$$

$$D_{e_2}e_3 = -b_3e_2 - ae_3, \tag{115}$$

$$D_{e_3}e_2 = (c_3 - a)e_3, \tag{116}$$

$$D_{e_3}e_3 = -\frac{1}{\beta}W - ae_2 - c_3e_3. \tag{117}$$

From this and the fact that W is a constant vector along the hypersurface, we see that an integral manifold of the distribution spanned by e_2 and e_3 is contained as an improper affine sphere (i.e. a hypersurface for which the affine normal is constant) in a 3-dimensional affine subspace of $\mathbb{R}^4$ with affine normal a multiple of W.

Choosing now a coordinate t in the direction of e_1 and using that $\varepsilon + b_1^2 - 9\lambda^2 = 0$ we can rewrite (113) as

$$\frac{\partial}{\partial t}f = (b_1+3\lambda)f - \frac{\varepsilon(b_1+3\lambda)}{\beta}W. \tag{118}$$

With an appropriate affine transformation we can assume that $W = (0,0,1,0)$. Then we can assume also that for an initial value t_0 we have

$$f(t_0,u,v) = (u,v,g(u,v),1) \tag{119}$$

where $(u,v) \mapsto (u,v,g(u,v))$ defines an improper affine sphere with affine normal $(0,0,1)$. Remark that we can take the last component 1 by applying an affine transformation since it cannot be 0. Otherwise (118) would imply that the hypersurface f lies in an $\mathbb{R}^3$ through the origin and this contradicts with the assumption that the hypersurface is non-degenerate.

From (118) and (119) it follows that there is a curve $\gamma = (\gamma_1,\gamma_2)$ such that the immersion f is given by

$$f(t,u,v) = (\gamma_1(t)u, \gamma_1(t)v, \gamma_1(t)g(u,v) + \gamma_2(t), \gamma_1(t)). \tag{120}$$

After a suitable reparametrization we can suppose that

$$f(t,u,v) = (tu, tv, tg(u,v) + \gamma_2(t), t). \tag{121}$$

Then we can calculate that

$$\begin{aligned} f_t &= (u, v, g + \gamma_2{}', 1), && (122)\\ f_u &= (t, 0, tg_u, 0), && (123)\\ f_v &= (0, t, tg_v, 0), && (124)\end{aligned}$$

and

$$\begin{aligned} f_{uu} &= tg_{uu}(0,0,1,0), & f_{ut} &= \tfrac{1}{t} f_u,\\ f_{vv} &= tg_{vv}(0,0,1,0), & f_{vt} &= \tfrac{1}{t} f_v,\\ f_{uv} &= tg_{uv}(0,0,1,0), & f_{tt} &= \gamma_2{}''(0,0,1,0).\end{aligned}$$

Since $(0,0,1,0) = \frac{f - tf_t}{\gamma_2 - t\gamma_2{}'}$ we get

$$\begin{aligned} f_{uu} &= \tfrac{tg_{uu}}{\gamma_2 - t\gamma_2{}'} f - \tfrac{t^2 g_{uu}}{\gamma_2 - t\gamma_2{}'} f_t, & f_{ut} &= \tfrac{1}{t} f_u,\\ f_{vv} &= \tfrac{tg_{vv}}{\gamma_2 - t\gamma_2{}'} f - \tfrac{t^2 g_{vv}}{\gamma_2 - t\gamma_2{}'} f_t, & f_{vt} &= \tfrac{1}{t} f_v,\\ f_{uv} &= \tfrac{tg_{uv}}{\gamma_2 - t\gamma_2{}'} f - \tfrac{t^2 g_{uv}}{\gamma_2 - t\gamma_2{}'} f_t, & f_{tt} &= \tfrac{\gamma_2{}''}{\gamma_2 - t\gamma_2{}'} f - \tfrac{t\gamma_2{}''}{\gamma_2 - t\gamma_2{}'} f_t.\end{aligned}$$

So we find that

$$\begin{aligned} h(\tfrac{\partial}{\partial u}, \tfrac{\partial}{\partial u}) &= -\varepsilon \tfrac{tg_{uu}}{\gamma_2 - t\gamma_2{}'}, & h(\tfrac{\partial}{\partial u}, \tfrac{\partial}{\partial t}) &= 0,\\ h(\tfrac{\partial}{\partial v}, \tfrac{\partial}{\partial v}) &= -\varepsilon \tfrac{tg_{vv}}{\gamma_2 - t\gamma_2{}'}, & h(\tfrac{\partial}{\partial v}, \tfrac{\partial}{\partial t}) &= 0,\\ h(\tfrac{\partial}{\partial u}, \tfrac{\partial}{\partial v}) &= -\varepsilon \tfrac{tg_{uv}}{\gamma_2 - t\gamma_2{}'}, & h(\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial t}) &= -\varepsilon \tfrac{\gamma_2{}''}{\gamma_2 - t\gamma_2{}'}.\end{aligned}$$

Now we must check that there exists a function λ on M such that

$$\frac{h\left(K\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right), \frac{\partial}{\partial t}\right)}{\|\frac{\partial}{\partial t}\|^3} = 12\lambda, \tag{125}$$

$$\frac{h\left(K\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial u}\right), \frac{\partial}{\partial u}\right)}{\|\frac{\partial}{\partial t}\|\|\frac{\partial}{\partial u}\|} = 3\lambda \frac{h(\frac{\partial}{\partial u}, \frac{\partial}{\partial u})}{\|\frac{\partial}{\partial u}\|}, \tag{126}$$

$$\frac{h\left(K\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial v}\right), \frac{\partial}{\partial v}\right)}{\|\frac{\partial}{\partial t}\|\|\frac{\partial}{\partial v}\|} = 3\lambda \frac{h(\frac{\partial}{\partial v}, \frac{\partial}{\partial v})}{\|\frac{\partial}{\partial v}\|}, \tag{127}$$

$$\frac{h\left(K\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial u}\right), \frac{\partial}{\partial v}\right)}{\|\frac{\partial}{\partial t}\|\|\frac{\partial}{\partial u}\|} = 3\lambda \frac{h(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})}{\|\frac{\partial}{\partial u}\|}. \tag{128}$$

From equations (126), (127) and (128) we find the same expression for λ. Combining this with equation (125) gives the following condition on the curve γ_2

$$t\gamma_2'''\gamma_2 - t^2\gamma_2'''\gamma_2' - t^2(\gamma_2'')^2 + 4\gamma_2''\gamma_2 - 4t\gamma_2'\gamma_2'' = 0. \tag{129}$$

By making the substitution $\phi(t) = \gamma_2(t) - t\gamma_2'(t)$, equation (129) becomes

$$-t^3\phi'(t)\phi(t) = c, \tag{130}$$

for some $c \in \mathbb{R}$. Solving (130) gives $\phi(t) = \sqrt{\frac{c}{2t^2} + d}$ with $d \in \mathbb{R}$. So we find that

$$\gamma_2(t) = \left(\int_{t_0}^{t} -\frac{1}{u}\sqrt{\frac{c}{2u^2} + d}\, du\right) t. \tag{131}$$

This gives case (2) of the theorem. □

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