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HALVING CLOSED CURVES IN NORMED PLANES AND RELATED INEQUALITIES

CHAN HE, HORST MARTINI AND SENLIN WU

Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. In an arbitrary normed plane we study the relation between the length of a closed curve and the length of its midpoint curve as well as the length of its image under the halving pair transformation. We show that the image curve under the halving pair transformation is convex provided the original curve is convex. We give a sufficient condition for the geometric dilation of a closed convex curve to be larger than a quarter of the perimeter of the unit circle. Moreover, we obtain several inequalities to show the relation between the halving distance and other quantities well known in convex geometry.

1. Introduction

Let C be a simple planar closed curve. A pair of points $p,q \in C$ is said to be a *halving pair* of C if the length of each part of C connecting p and q is one half of the perimeter of C. In the Euclidean plane, the property of halving pairs of simple planar closed curves plays an important role in recent investigations of the geometric dilation problem; see [3], [4], and [5]. Also, the relations between the halving distance (the distance between a halving pair) and some further important quantities of a closed curve yield many interesting results; see [3] and [5, Chapter 4]. In [10], the first attempt was made to extend the geometric dilation problem from the Euclidean plane to Minkowski planes (i.e., to real two-dimensional normed linear spaces). General lower bounds on the geometric dilation of closed planar curves in Minkowski planes were obtained by applying basic properties of halving pairs and the so called halving pair transformation. In the present paper we study further properties of halving pairs, the halving pair transformation and the halving distance in arbitrary Minkowski planes, deriving also related inequalities.

By X we denote a (*normed* or) *Minkowski plane* with *origin* o, *norm* $\|\cdot\|$, *unit disc* B_X, *unit circle* S_X, and a fixed orientation ω. We refer to [9], [7], [8], and [11] for more information about the geometry of Minkowski planes and spaces. Any homothet

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of S_X is said to be a *circle* in X. By $[p,q]$ we denote the *segment* (possibly degenerate) between two points $p,q \in X$, and by $\langle p,q \rangle$ the *line* passing through p and q. The convex hull of a set S is denoted by $\operatorname{conv}(S)$. Let $x,y \in X$. We say that x is *Birkhoff orthogonal* to y if $\|x+ty\| \geqslant \|x\|$ holds for any real number t, and this situation is denoted by $x \perp_B y$ (cf. [2] and [6]).

By a *curve* C in X we mean the range of a continuous function ϕ that maps a closed bounded interval $[\alpha,\beta]$ into X. The curve C defined by $\phi: [\alpha,\beta] \mapsto X$ is called *closed* if $[\alpha,\beta]$ is replaced by a Euclidean circle, say, and it is *simple* if it has no self-intersections. Furthermore, C is said to be *rectifiable* if the set of all Riemann sums

$$\left\{ \sum_{i=1}^{n} \|\phi(t_i) - \phi(t_{i-1})\| : \ (t_0, t_1, \cdots, t_n) \text{ is a partition of } [\alpha,\beta] \right\}$$

with respect to the norm $\|\cdot\|$ of X is bounded from above. If C is a rectifiable curve, then we denote by $|C|$ the *length of* C, i.e.,

$$|C| := \sup \left\{ \sum_{i=1}^{n} \|\phi(t_i) - \phi(t_{i-1})\| : \ (t_0, t_1, \cdots, t_n) \text{ is a partition of } [\alpha,\beta] \right\}.$$

Throughout this paper we consider simple, rectifiable, closed curves in an arbitrary Minkowski plane X. We shall frequently use the *arc-length parametrization* $c : [0,|C|) \to C$ of a rectifiable closed curve C, which is continuous, bijective, and has the property that $\|\dot{c}(t)\| = 1$ whenever the derivative exists. Two points $p = c(t)$ and $\hat{p} = c(t + |C|/2)$ on C that split C regarding its length into two equal parts form a *halving pair* of C, and the segment $[p,\hat{p}]$ is said to be a *halving chord*. For any $v \in S_X$, the v-*halving distance* in direction v, denoted by $h_C(v)$, is the length of the halving chord of C having direction v (note that this quantity is defined only for convex curves); the v-*length*, denoted by $l_C(v)$, is the maximum distance between pairs of points on C whose difference vector is of direction v. The *minimum width* w of a closed convex curve C is the minimum distance between two parallel supporting lines of $\operatorname{conv}(C)$. The *diameter* of C, denoted by $D(C)$, is the maximum of all possible v-lengths. The *inradius* r and *circumradius* R of C is the radius of the maximum inscribed circle and the minimum circumscribed circle of C, respectively. The *maximum halving distance* and *minimum halving distance* of C are defined by $H = H(C) = \max_{t \in [0,|C|)} \{\|c(t) - c(t + |C|/2)\|\}$ and $h = h(C) = \min_{t \in [0,|C|)} \{\|c(t) - c(t + |C|/2)\|\}$, respectively. The *midpoint curve* M of the curve C is formed by the midpoints of halving chords of C, and it is parameterized by

$$m(t) := \frac{1}{2}\left(c(t) + c(t + \frac{|C|}{2}) \right).$$

The image C^* of C under the *halving pair transformation* is given by the parametrization

$$c^*(t) := \frac{1}{2}\left(c(t) - c(t + \frac{|C|}{2}) \right).$$

The *geometric dilation* (or *detour*) $\delta_X(C)$ of C is defined by

$$\delta_X(C) := \sup_{p,q\in C, p\neq q} \frac{d_C(p,q)}{\|p-q\|},$$

where $d_C(p,q)$ is the minimum of the lengths of the two curve arcs of C connecting p and q. If C is convex, then $\delta_X(C) = |C|/2h$ (see [10]).

2. The halving pair transformation

Let C be a simple rectifiable closed curve. By definition, the halving pair transformation translates the midpoints of the halving chords to the origin, $c^*(t) = -c^*(t+|C|/2)$, and hence C^* is centrally symmetric. Moreover, $h(C^*) = h(C) = h$ and $H(C^*) = H(C) = H$.

First we would like to give an upper bound on $|M|$.

THEOREM 2.1. $|C| \geqslant \max\{2|M|, |C^*|\}$.

Proof. We deal only with the case when C is piecewise continuously differentiable, and the proof of this case can be extended to arbitrary rectifiable curves. By definitions and the triangle inequality we have

$$2|M| = \int_0^{|C|/2} \left\| \dot{c}(t) + \dot{c}\left(t+\frac{|C|}{2}\right)\right\| \mathrm{d}t \leqslant \int_0^{|C|/2} \left(\|\dot{c}(t)\| + \left\|\dot{c}\left(t+\frac{|C|}{2}\right)\right\|\right)\mathrm{d}t = |C|$$

and

$$|C^*| = \int_0^{|C|} \frac{1}{2} \left\| \dot{c}(t) - \dot{c}\left(t+\frac{|C|}{2}\right)\right\| \mathrm{d}t \leqslant \frac{1}{2} \int_0^{|C|/2} \left(\|\dot{c}(t)\| + \left\|\dot{c}\left(t+\frac{|C|}{2}\right)\right\|\right)\mathrm{d}t = |C|.$$

The proof is complete. $\square$

REMARK 2.2. Dumitrescu et al. [3] showed that the inequality $4|M|^2 + |C^*|^2 \leqslant |C|^2$ holds in the Euclidean plane, which means that the number $2|M|$ cannot be too large since we have the inequality $|C^*| \geqslant \pi h$. However, this is not true in general Minkowski planes.

Consider the closed curve C depicted in Figure 1, in the Minkowski plane $\mathbb{R}^2$ with norm $\|(\alpha,\beta)\| = |\alpha| + |\beta|$. Calculations show that $|C| = 6A + 8\varepsilon$, $h = A + 2\varepsilon$, and $H = A + 4\varepsilon$, where A is a constant positive number. Note that $|M|$ is not smaller than the perimeter of the triangle formed by m_1, m_2, and m_3, that is, $|M| \geqslant 3A$. By the symmetry of C^*, any two points p and $-p$ on C^* form a halving pair of distance $2\|p\| \geqslant h$. Hence C^* contains the disc $(h/2)B_X$, and then $|C^*| \geqslant (h/2)|S_X| = 4h = 4(A+2\varepsilon)$. Clearly, $2|M|$ tends to $|C|$ as ε tends to zero while $|C^*| > 4A$, and therefore $\sqrt{|C|^2 - |C^*|^2}/2$ is not an upper bound on $|M|$ in general Minkowski planes.

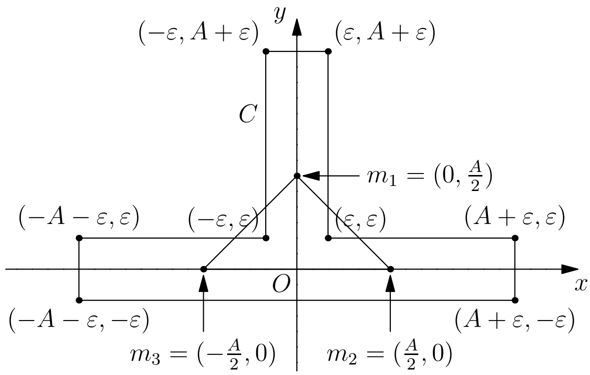

Figure 1: $2|M|$ *can be arbitrarily close to* $|C|$.

REMARK 2.3. It is also interesting to observe that $|C| = 2|M|$ may hold for some closed convex curves in a metric space on $\mathbb{R}^2$, where the metric is induced by a certain convex distance function (gauge) as in the following example, i.e., the corresponding metric is not centrally symmetric. Let C be a triangle in the metric space on $\mathbb{R}^2$ with unit circle S_X (see Figure 2), where a point is moving on C clockwise. Then the point on the midpoint curve M moves counter-clockwise, and simple calculations show that $|C| = 2|M|$.

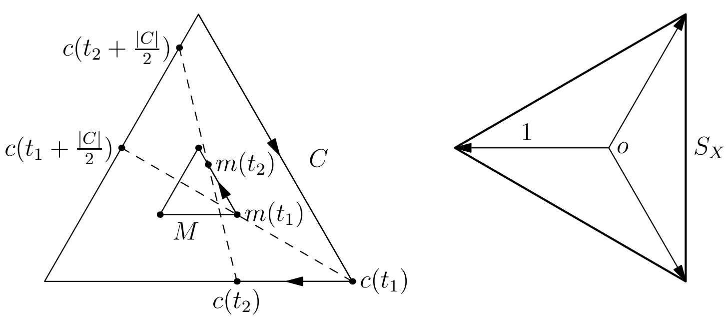

Figure 2: The case where $|C| = 2|M|$.

Ebbers-Baumann et al. [4] proved that, in the Euclidean plane, the image of a closed convex curve under the halving pair transformation is also convex. We show that this result still holds in general Minkowski planes.

LEMMA 2.4. *For any* $u, v \in S_X$ *and* $\lambda \in [0,1)$ *we have* $u \not\perp_B (u + \lambda v)$ *and* $v \not\perp_B (v + \lambda u)$. *Moreover,* $u \perp_B (u+v)$ *if and only if* $[u, -v] \subset S_X$.

Proof. The case $u = \pm v$ is trivial. Suppose that $u \neq \pm v$ and that there exists a number $\lambda_0 \in [0,1)$ such that $u \perp_B (u+\lambda_0 v)$. Then, by the definition of Birkhoff orthogonality, the inequality

$$\|u+t(u+\lambda_0 v)\| \geqslant \|u\| = 1$$

holds for any $t \in \mathbb{R}$. By setting $t=-1$ we have $|\lambda_0| \geqslant 1$, a contradiction.

Suppose that $u \perp_B (u+v)$. Then

$$\left\| u - \frac{1}{2}(u+v) \right\| = \frac{1}{2}\|u-v\| \geqslant 1,$$

which implies that $\|u-v\| = 2$. Thus $[u,-v] \subset S_X$. □

LEMMA 2.5. *Suppose that $C \subset X$ is a continuously differentiable, closed, convex curve, and p,q be two points on C such that a pair of parallel supporting lines of* conv(C) *contains p and q, respectively. Then $d_C(p,q) > \|p-q\|$.*

Proof. Let C_0 be that part of C connecting p and q which has minimum length, and l_p and l_q be the supporting lines of conv(C) at p and q, respectively (see Figure 3). Since C is continuously differentiable, there exists a point $q_0 \in C_0$ such that the supporting line of conv(C) at q_0 (which intersects l_p and l_q in p_1 and q_1, respectively) is parallel to the line $\langle p,q \rangle$.

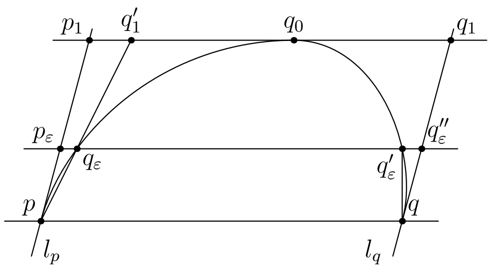

Figure 3: Proof of Lemma 2.5.

For any number $0 < \varepsilon < d_C(p,q_0)$, let $q_\varepsilon \in C_0$ be the point such that $d_C(p,q_\varepsilon) = \varepsilon$; p_ε, q''_ε, and q'_ε be the points where the line passing through q_ε parallel to $\langle p,q \rangle$ intersects l_p, l_q, and the arc on C_0 between q_0 and q, respectively. Since $d_C(p,q_\varepsilon) \geqslant \|p-q_\varepsilon\|$, $d_C(q_\varepsilon,q'_\varepsilon) \geqslant \|q_\varepsilon - q'_\varepsilon\|$, and $d_C(q'_\varepsilon,q) \geqslant \|q'_\varepsilon - q\|$, it suffices to show that $\|p-q_\varepsilon\| + \|q'_\varepsilon - q\| > \|p_\varepsilon - q_\varepsilon\| + \|q'_\varepsilon - q''_\varepsilon\|$ for some sufficiently small ε.

Suppose that the line $\langle p,q_\varepsilon \rangle$ intersects $[p_1,q_1]$ in q'_1. Then

$$\frac{\|p-p_\varepsilon\|}{\|p_\varepsilon - q_\varepsilon\|} = \frac{\|p-p_1\|}{\|p_1 - q'_1\|}.$$

Since $\|p-p_1\|$ is fixed and C is differentiable at p, we have $\lim_{\varepsilon\to 0}\|p_1-q_1'\|=0$, and therefore $\lim_{\varepsilon\to 0}\|p_\varepsilon-q_\varepsilon\|/\|p-p_\varepsilon\|=0$. Thus

$$\lim_{\varepsilon\to 0}\frac{\|p-q_\varepsilon\|}{\|p_\varepsilon-q_\varepsilon\|}\geqslant\lim_{\varepsilon\to 0}\frac{\|p-p_\varepsilon\|-\|p_\varepsilon-q_\varepsilon\|}{\|p_\varepsilon-q_\varepsilon\|}=\lim_{\varepsilon\to 0}\frac{\|p-p_\varepsilon\|}{\|p_\varepsilon-q_\varepsilon\|}-1=+\infty,$$

and then $\lim_{\varepsilon\to 0}\|p_\varepsilon-q_\varepsilon\|/\|p-q_\varepsilon\|=0$, which implies $\|p_\varepsilon-q_\varepsilon\|<\|p-q_\varepsilon\|$ for sufficiently small ε. In a similar way we can prove that $\|q_\varepsilon'-q\|>\|q_\varepsilon'-q_\varepsilon''\|$ when ε is sufficiently small. □

THEOREM 2.6. *If C is convex, then C^* is convex.*

Proof. First we assume that C is smooth. Then the derivative $\dot{c}(\cdot)$ is a continuous function mapping $[0,|C|)$ into the unit circle S_X. Due to convexity, the derivative vectors $\dot{c}(t)$ and $-\dot{c}(t+|C|/2)$ always turn into the same direction, say ω_0.

Note that $\dot{c}(t)-\dot{c}(t+|C|/2)=0$ cannot occur, since this would imply that $\dot{c}(\tau)=\dot{c}(t)=\dot{c}(t+|C|/2)$ holds for each τ in $[t,t+|C|/2]$ or $[t+|C|/2,t+|C|]$, due to convexity. Then C would contain a line segment of length $|C|/2$. On the other hand, it follows from the assumption that the supporting lines of conv(C) at $c(t)$ and $c(t+|C|/2)$ are parallel to each other. By Lemma 2.5 we have $\|c(t)-c(t+|C|/2)\|<|C|/2$, a contradiction.

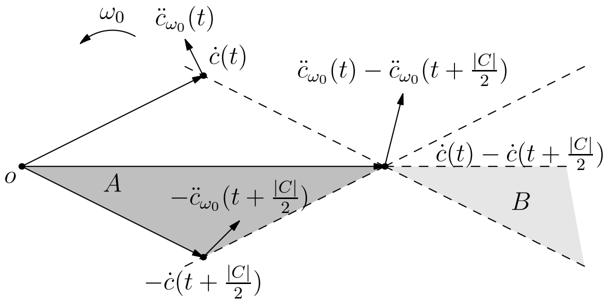

Figure 4: $\dot{c}(t)-\dot{c}(t+|C|/2)$ turns into the same direction as $\dot{c}(t)$ and $\dot{c}(t+|C|/2)$.

Furthermore, by Lemma 2.4 we have for any $\lambda\in(0,1)$

$$\dot{c}(t)\not\perp_B\left(\dot{c}(t)-\lambda\dot{c}\left(t+\frac{|C|}{2}\right)\right) \tag{2.1}$$

and

$$-\dot{c}\left(t+\frac{|C|}{2}\right)\not\perp_B\left(-\dot{c}\left(t+\frac{|C|}{2}\right)+\lambda\dot{c}(t)\right). \tag{2.2}$$

Denote by $\ddot{c}_{\omega_0}(t)$ the derivative of $\dot{c}(t)$ in direction ω_0, i.e., the one-side derivative turns $\dot{c}(t)$ in the direction ω_0 (see Figure 4). (2.1) and (2.2) imply that $\ddot{c}_{\omega_0}(t)+\dot{c}(t)-\dot{c}(t+|C|/2)$ and $-\ddot{c}_{\omega_0}(t+|C|/2)+\dot{c}(t)-\dot{c}(t+|C|/2)$ cannot lie in the domains A and B, respectively. Therefore, $\ddot{c}_{\omega_0}(t)$ and $-\ddot{c}_{\omega_0}(t+|C|/2)$ turn the vector $\dot{c}(t)-\dot{c}(t+|C|/2)$ into the direction ω_0. Hence C^* is convex.

This result can be extended to closed convex curves, approximating them by smooth closed convex curves. □

In [10] the lower bound $|S_X|/4$ for the geometric dilation in Minkowski planes was derived, and it was shown that a closed convex curve with the smallest geometric dilation is not necessarily a circle. In the following theorem we present a sufficient condition for the geometric dilation of a curve to be larger than $|S_X|/4$.

THEOREM 2.7. *Let C be a closed convex curve with $H/h>2$. Then $\delta_X(C)>|S_X|/4$.*

Proof. Suppose that $\delta_X(C)=|S_X|/4$. By Theorem 2.6 and the convexity of C we have

$$\delta_X(C)=\frac{|C|}{2h}\geqslant\frac{|C^*|}{2h}=\delta_X(C^*)\geqslant\frac{|S_X|}{4},$$

which yields $|C|=|C^*|$. As stated in Remark 2.2, C^* contains the disc $(h/2)B_X$. On the other hand, C^* has to connect some halving pair q and $-q$ having maximum halving distance H.

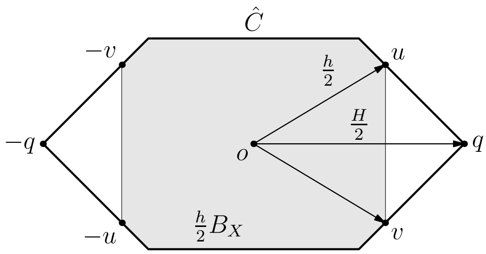

Figure 5: The curve $\hat{C}$ is the shortest curve containing the disc $(h/2)B_X$ and connecting q and $-q$.

Suppose that the supporting lines of $(h/2)B_X$ passing through q support $(h/2)B_X$ at u and v, respectively (see Figure 5: if one of the lines supports $(h/2)B_X$ at a segment, then we choose the point nearest to q on that segment). Thus the supporting lines of $(h/2)B_X$ passing through $-q$ support $(h/2)B_X$ at $-u$ and $-v$, respectively. Let $\hat{C}$ be the closed convex curve depicted in Figure 5. Then

$$\frac{h}{2}|S_X|\leqslant|\hat{C}|\leqslant|C^*|=\frac{h}{2}|S_X|,$$

which implies that $|\hat{C}|=(h/2)|S_X|$. It follows that

$$\|q-u\|+\|q-v\|=d_{\frac{h}{2}S_X}(u,v).$$

Thus $\|q-u\|+\|q-v\|=\|u-v\|$. Since $\|q\|=H/2>h$ and $\|u\|=\|v\|=h/2$, we have $\|q-u\|\geqslant\|q\|-\|u\|>h/2$ and $\|q-v\|>h/2$. Therefore $h\geqslant\|u-v\|=\|q-u\|+\|q-v\|>h$, a contradiction. □

3. On the halving distance

The relations between different geometric quantities of convex bodies yield interesting (geometric) inequalities. In this section we relate the minimum and maximun halving distance h and H to other geometric quantities, such as, for example, the minimum width w. The results in the following theorem can be derived immediately from the definitions of the corresponding quantities.

THEOREM 3.1. *Let $C\subset X$ be a closed convex curve. Then the following inequalities hold:*

1. $h\leqslant w$,
2. $H\leqslant|C|/2$,
3. $2r\leqslant w$,
4. $h\leqslant H\leqslant D$,
5. $H\leqslant 2R$.

LEMMA 3.2. *Let $C\subset X$ be a closed convex curve. Then there exists a point $p_0\in C$ such that* $\operatorname{conv}(C)$ *has parallel supporting lines at p_0 and $\hat{p_0}$.*

Proof. First we assume that C is a smooth curve. Note that

$$\int_0^{|C|/2}\dot{c}(t)\mathrm{d}t=c\left(\frac{|C|}{2}\right)-c(0)=-\int_0^{|C|/2}\dot{c}\left(t+\frac{|C|}{2}\right)\mathrm{d}t.$$

By the intermediate value theorem of integration, there exists a number $t_0\in(0,|C|/2)$ such that $\dot{c}(t_0)+\dot{c}(t_0+|C|/2)=0$. Let $p_0=c(t_0)$. Then $\operatorname{conv}(C)$ has parallel supporting lines at p_0 and $\hat{p_0}$.

Again this result can be generalized to closed convex curves, approximating them by smooth closed convex curves. □

THEOREM 3.3. *Let $C\subset X$ be a closed convex curve. Then $H\geqslant w$.*

Proof. By Lemma 3.2 there exists a point $p_0\in C$ such that $[p_0,\hat{p_0}]$ is a halving chord and $\operatorname{conv}(C)$ has parallel supporting lines at p_0 and $\hat{p_0}$. Then the distance between the supporting lines at p_0 and $\hat{p_0}$ is not smaller than the minimum width w of C. Since $(p_0,\hat{p_0})$ is a halving pair, it follows that $H\geqslant\|p_0-\hat{p_0}\|\geqslant w$. The inequality is tight, since circles attain the equality case. □

The following corollary follows from Theorem 3.1 and Theorem 3.3.

COROLLARY 3.4. *Let $C \subset X$ be a closed convex curve. Then $H \geqslant 2r$. This inequality is tight, because equality holds for circles.*

LEMMA 3.5. [1, Theorem 3] *If $C \subset X$ is a closed convex curve, then $w = \min_{v \in S_X} l_C(v)$.*

LEMMA 3.6. *Let $C \subset X$ be a closed convex curve. Then the inequality $h_C(v) > l_C(v)/2$ holds for every direction $v \in S_X$. This inequality cannot be improved.*

Proof. For any $v \in S_X$, let p and $\hat{p}$ be the halving pair in the direction v; $[q, \tilde{q}]$ be the longest chord of C in the direction v, i.e., $l_C(v) = \|q - \tilde{q}\|$. Without loss of generality, we can assume that $p - \hat{p}$ is a positive multiple of $q - \tilde{q}$. The following proof is similar to the proof of Lemma 4.12 in [5].

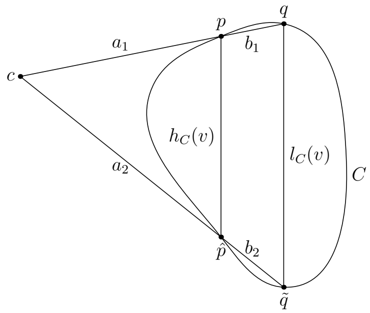

Figure 6: Proof of Lemma 3.6.

If $l_C(v) \leqslant h_C(v)$, then the proof is complete. If $l_C(v) > h_C(v)$, then the line $\langle p, q \rangle$ has to intersect the line $\langle \hat{p}, \tilde{q} \rangle$ at some point c which is seperated from the segment $[q, \tilde{q}]$ by the line $\langle p, \hat{p} \rangle$ (see Figure 6). Let $a_1 = \|c - p\|$, $a_2 = \|c - \hat{p}\|$, $b_1 = \|p - q\|$, and $b_2 = \|\hat{p} - \tilde{q}\|$. Since $[p, \hat{p}]$ is a halving chord and C is convex, we have $b_1 + b_2 + l_C(v) \leqslant |C|/2 \leqslant a_1 + a_2$. Note that both the chords $[p, \hat{p}]$ and $[q, \tilde{q}]$ have direction v. It follows that

$$\frac{a_1}{a_1 + b_1} = \frac{a_2}{a_2 + b_2} = \frac{h_C(v)}{l_C(v)},$$

and then

$$h_C(v) = l_C(v) \frac{a_1 + a_2}{a_1 + a_2 + b_1 + b_2} \geqslant l_C(v) \frac{b_1 + b_2 + l_C(v)}{2(b_1 + b_2) + l_C(v)} > \frac{1}{2} l_C(v).$$

The second inequality cannot become an equality, but the two numbers can come arbitrarily close to each other if $l_C(v)$, compared with $b_1 + b_2$, is small enough. □

COROLLARY 3.7. *Let $C \subset X$ be a closed convex curve. Then $h > w/2$, and this inequality is tight.*

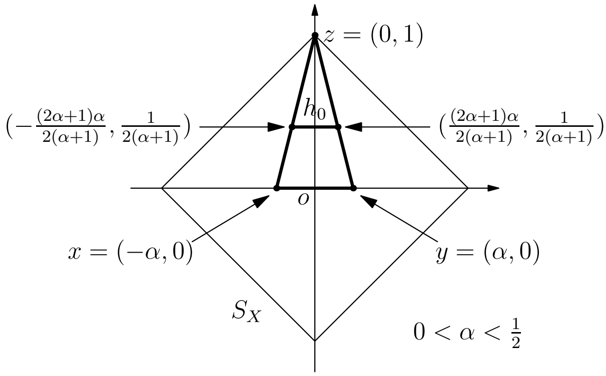

Figure 7: Proof of Corollary 3.7.

Proof. The first part of this result follows from Lemma 3.5 and Lemma 3.6.

In order to see that the bound is tight, we consider an isosceles triangle in $\mathbb{R}^2$ with the norm $\|(x_1,x_2)\| = |x_1|+|x_2|$, as shown in Figure 7. One can easily show that $w = 2\alpha$ and the halving distance in direction $(x-y)/\|x-y\|$ is $h_0 = (2\alpha+1)\alpha/(\alpha+1)$. Then $1/2 < h/w \leqslant h_0/w = (2\alpha+1)/(2(\alpha+1))$, which implies that h/w tends to $1/2$ when α tends to 0. $\square$

From Theorem 3.1 and Corollary 3.7, the relation between h and r can be derived immediately in the following corollary.

COROLLARY 3.8. *Let $C \subset X$ be a closed convex curve. Then $h > r$.*

In order to obtain the upper bound for h in terms of r, we need the following lemma.

LEMMA 3.9. *For any triangle in a Minkowski plane X, there exists a height (i.e., the distance from a vertex to the line containing its opposite side) which is not larger than three times the radius of the incircle of that triangle.*

Proof. Suppose that the vertices of the triangle are p_1, p_2, and p_3, and the incirle of the triangle is $c_0 + rS_X$ with radius r and center c_0. Let $c = (p_1+p_2+p_3)/3$, and $p_4 \in [p_1,p_3]$, $p_5 \in [p_1,p_2]$, and $p_6 \in [p_2,p_3]$ be points such that the lines $\langle c,p_4\rangle, \langle c,p_5\rangle$, and $\langle c,p_6\rangle$ are parallel to $\langle p_2,p_3\rangle, \langle p_1,p_3\rangle$, and $\langle p_1,p_2\rangle$, respectively (see Figure 8). Then

$$\frac{\|p_1-p_4\|}{\|p_1-p_3\|} = \frac{\|p_2-p_5\|}{\|p_2-p_1\|} = \frac{\|p_3-p_6\|}{\|p_3-p_2\|} = \frac{2}{3},$$

and the segments $[c,p_4],[c,p_5]$, and $[c,p_6]$ divide the triangle into three regions. The center c_0 of the incircle should lie in one of these three regions. Suppose, without loss of generality, that c_0 lies in the convex hull of the points c, p_4, p_1 and p_5. Let l be the line passing through c_0 parallel to $\langle p_2, p_3\rangle$, and d be the distance from p_1 to l. Then, since the distance between l and $\langle p_2, p_3\rangle$ is r, we have $d/(d+r) \leqslant 2/3$, which yields $d \leqslant 2r$. Hence the height from p_1 to the side $[p_2, p_3]$ is not larger than $3r$. □

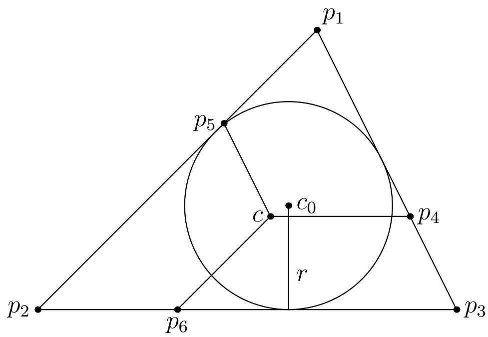

Figure 8: Proof of Lemma 3.9.

THEOREM 3.10. *Let $C \subset X$ be a closed convex curve. Then $h \leqslant 3r$.*

Proof. Suppose that the incirle of C is $c_0 + rS_X$. Then $c_0 + rS_X$ should touch C at more than one point. We consider the following two cases:

Case 1: $c_0 + rS_X$ touches C at exactly two points, say p and q.

In this case, there should be a pair of parallel supporting lines of conv(C) at p and q. Suppose the contrary, namely that any supporting line of conv(C) at p is not parallel to each supporting line of conv(C) at q. Let l_p and l_q be supporting lines of conv(C) at p and q, respectively, intersecting each other at a point c, and l be a line supporting $c_0 + rB_X$ at a point c_1 and parallel to $\langle p, q\rangle$, where c_1 is separated from c by $\langle p, q\rangle$. Note that l_p and l_q are also supporting lines of $c_0 + rB_X$ at p and q, respectively.

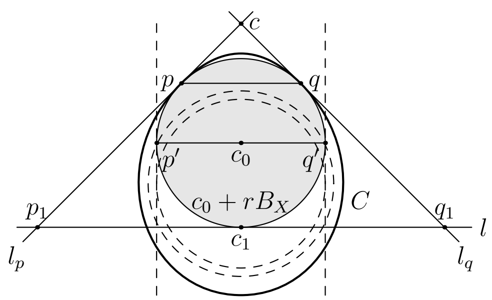

Figure 9: The chord $[p, q]$ is not a diameter of $c_0 + rS_X$.

If the chord $[p, q]$ is not a diameter of $c_0 + rS_X$, then there exist a diameter $[p', q']$ and $u \in S_X$ such that $p' - q'$ is a positive multiple of $p - q$ and $p' - q' \perp_B u$ (see

Figure 9). Since c_1, p', and q' are interior points of $\operatorname{conv}(C)$, there exists a number $\delta_1 > 0$ such that the points $c_1 + \delta_1 u$, $p' + \delta_1 u$, and $q' + \delta_1 u$ are still interior points of $\operatorname{conv}(C)$. Thus we can obtain a translate $c_0 + \delta_1 u + rS_X$ of the incirle which is contained in $\operatorname{conv}(C)$ and does not touch C. This is a contradiction.

Suppose that $[p,q]$ is a diameter of $c_0 + rS_X$. Suppose that l_p, l_q and the line passing through q parallel to l_p intersects l in p_1, q_1, and q', respectively. Let q_0 be the point on $[q',q_1]$ nearest to q' such that the line $\langle q,q_0\rangle$ supports $\operatorname{conv}(C)$; $p_0 \in [p_1,q_1]$ be a point such that the line $\langle p,p_0\rangle$ is parallel to $\langle q,q_0\rangle$ (see Figure 10). Then $\langle p,p_0\rangle$ does not support $\operatorname{conv}(C)$ because of the assumption. Let $p_2 \in [p_1,p_0]$ be a point sufficiently close to p_0 such that $p_2 \neq p_0$ and the line $\langle p,p_2\rangle$ still does not support $\operatorname{conv}(C)$, and q_2 be the point on l such that $\langle q,q_2\rangle$ is parallel to $\langle p,p_2\rangle$. Then $q_2 \in [q',q_0]$, $q_2 \neq q_0$, and $\langle q,q_2\rangle$ is not a supporting line of $\operatorname{conv}(C)$. We note that l_p, l_q, $\langle p,p_0\rangle$, $\langle q,q_0\rangle$, $\langle p,p_2\rangle$, and $\langle q,q_2\rangle$ are all supporting lines of $c_0 + rB_X$. Since c_1 is a interior point of $\operatorname{conv}(C)$, there exists a number $\delta_2 > 0$ such that the points $c_1 + \delta_2(p_2 - p)$, $p + \delta_2(p_2 - p)$, and $q + \delta_2(p_2 - p)$ are still interior points of $\operatorname{conv}(C)$. Thus the translate $c_0 + \delta_2(p_2 - p) + rS_X$ of the incirle is contained in $\operatorname{conv}(C)$ and does not touch C, a contradiction to the fact that $c_0 + rS_X$ is the incircle of C.

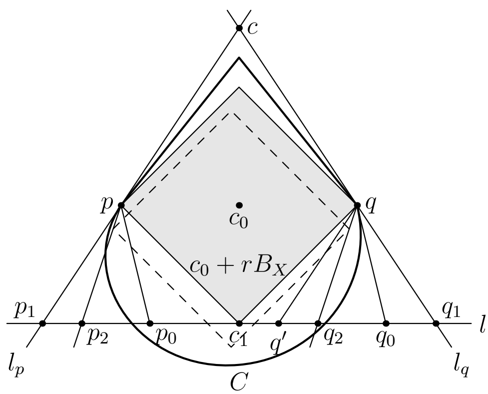

Figure 10: The chord $[p,q]$ is a diameter of $c_0 + rS_X$.

The minimum width w of C cannot be larger than the distance between the parallel supporting lines of $\operatorname{conv}(C)$ at p and q, which is the minimum width of the incircle $c_0 + rS_X$, that is, $w \leqslant 2r$. By Theorem 3.1, it follows that $h \leqslant w < 3r$.

Case 2: The set $c_0 + rS_X$ touches C at more than two points. Then there exist at least three such points that are not collinear. Otherwise, we could obtain a translate of $c_0 + rS_X$ which is contained in $\operatorname{conv}(C)$ and does not touch C, in a similar way as in Case 1, a contradiction.

Let q_1, q_2, and q_3 be three points in the intersection of $c_0 + rS_X$ and C with supporting lines l_1, l_2, and l_3, respectively. Suppose that any two of these lines are not parallel to each other, and $\{p_1\} = l_1 \cap l_3$, $\{p_2\} = l_1 \cap l_2$, and $\{p_3\} = l_2 \cap l_3$. Thus the triangle formed by p_1, p_2, and p_3 contains the curve C, and then the minimum width of C is not larger than the minimum height of the triangle. By Lemma 3.9 we have $w \leqslant 3r$, and then $h \leqslant w \leqslant 3r$. □

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Chan He, Senlin Wu
Department of Mathematics
Harbin University of Science and Technology
150080 Harbin
China

Horst Martini
Faculty of Mathematics
University of Technology Chemnitz
09107 Chemnitz
Germany
e-mail: horst.martini@mathematik.tu-chemnitz.de